Measures and Measurements

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Starting Points

- We should not run away from Probability Theory (agree with Ehtibar)
- Quantum Theory is a Generalisation of Probability Theory
- (Quantum) Contextuality appears as a failure of a Global Probability Space
- Let us define "local" Probability Spaces and "glue them together"

Quick Review on Probability Theory

A Measurable Space is a pair (Ω, Σ)

Ω A set, called Sample Space

Σ A sigma-algebra of subsets of the Sample Space

Quick Review on Probability Theory

A Probability Space is a triple (Ω, Σ, μ)

Ω A set, called Sample Space

Σ A sigma-algebra of subsets of the Sample Space

μ A probability measure on Σ

Remind: sigma-Algebra

A family of subsets of Ω such that

$$\emptyset \in \Sigma$$

$$A \in \Sigma \Longrightarrow \Omega \backslash A \in \Sigma$$

$$A_i \in \Sigma, i \in \mathbb{N} \Longrightarrow \bigcup_i A_i \in \Sigma \wedge \bigcap_i A_i \in \Sigma$$

Remind: Probability Measure

Kolmogorov

$$\mu : \Sigma \longrightarrow \mathbb{R}$$

$$\mu(A) \geq 0$$

$$\mu(\Sigma) = 1$$

$$A_i \cap A_j = \varnothing \, \forall i, j \Longrightarrow \mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

(countable disjoint union)

Small Detour: My understanding of Kolmogorov’s “ontology”

- Sigma is the Event-Space, where “observables” live
- Omega is the “Underlying Reality”, from where all “observables” are determined

The Problem

- What if not all observables can be jointly defined???
- What if Compatibility Conditions should be imposed to the theory?

The Solution

- Just like a manifold is obtained glueing together “pieces” of vector spaces, we can define a Probability Space for each context and glue them together!

The Solution

- More precisely, we will build two fibre bundles where the fibres are:
  - Measurable Spaces
  - Probability Spaces

The Basis: Contextuality Scenarios

A Pair $(\mathscr{X}, \mathscr{C})$

$\mathscr{X}$ A Set of possible Measurements

$\mathscr{C}$ A Compatibility Cover, i.e. $\mathscr{C} \subseteq \mathscr{P}(\mathscr{X})$

s.t. $C \in \mathscr{C} \wedge C' \subset C \Longrightarrow C' \in \mathscr{C}$

A Basic Concept: Measurement

A Measurement, $\mathscr{M}$, is characterised by the set of its possible outcomes

A Realisation of $\mathscr{M}$ is given by a Measurable Space, (Ω, Σ), with a partition of Ω, subordinated to $\mathscr{M}$

A Probability Measure for $\mathscr{M}$ is given by a Probability Measure on Σ

Compatibility

Compatible Measurements can be Realised in the same Measurable Space

Thm: Measurements $\mathscr{M}$ and $\mathscr{N}$ are compatible iff there is the joint measurement $\mathscr{M} \wedge \mathscr{N}$

Attaching the Fibres

Given a Contextuality Scenario, $(\mathscr{X}, \mathscr{C})$, for each maximal context, C, one attaches a Measurable Space (Ω^C, Σ^C).

Rmk: Up to this point, we have Contextuality-by-Default, as defined by Dzhafarov

Digression

- Up to this moment, the contexts are isolated! There is no precise meaning in saying one measurement belongs to two (or more) different contexts
- How to fix it? How to include Kochen-Specker contextuality in this framework?

Glueing Contexts

For each context, $\mathscr{M}$ will have a different realisation

In (Ω, Σ), with partition $\left\{A_m\right\}_{m\in\mathscr{M}}$

In (Ω', Σ'), with partition $\left\{A'_m\right\}_{m\in\mathscr{M}}$

This defines a bijection for such sets:

$$A_m \leftrightarrow A'_m$$

which plays the rôle of transition functions
in this fibre bundle.

Empirical Models

- Up to now, our fibres are Measurable Spaces
- Another fibre bundle over the contextuality scenario has Probability Spaces as fibres
  - This we call (following Abramsky) an Empirical Model

Empirical Models

Given a Contextuality Scenario, $(\mathscr{X}, \mathscr{C})$, for each maximal context, C, one attaches a Probability Space $(\Omega^C, \Sigma^C, \mu^C)$.

New interpretation to Non-Disturbance condition!

Non-Disturbance

$$\mathscr{M} \in C \cap C' \Longrightarrow \mu^{C}\left(A_m\right) = \mu^{C'}\left(A'_m\right)$$

In words, this is the condition for the Empirical Model to be defined on the Fibre Bundle we built by identifying the same measurements in different contexts.

In other words, Non-Disturbing Empirical Model defines a Probability Bundle

Trivial Fibre Bundles

- A Fibre Bundle is called trivial when it can be identified with $B \times F$, where B is the basis and F is the fibre
- A Probability Bundle is trivial when all the probability measures can be defined on the same measurable space

Classification

- An Empirical Model is noncontextual when it can be described using one probability space
- An Empirical Model is quantum when it can be described using one state space and Born's rule

Fine-Abramsky-Brandenburger Thm

An Empirical Model is noncontextual iff its Probability Bundle is trivial

A Lesson from Bundles

- If the basis is topologically trivial, all fibre bundles are trivial
- This stresses the importance of Contextuality Scenarios
- And connects topology of the Scenario with the possible manifestations of contextuality

Other Lesson from Bundles: Extensions

We have just interpreted non-contextuality as the possibility of extending a given empirical model to a trivial probability bundle

What about other extensions?

Subscenarios

Given a Scenario $(\mathscr{X}, \mathscr{C})$, we call $(\mathscr{X}', \mathscr{C}')$ a Subscenario if $\mathscr{X}' \subseteq \mathscr{X}$ and $\mathscr{C}' \subseteq \mathscr{C}$

Special Case (Induced Subscenario): For a chosen $\mathscr{X}' \subseteq \mathscr{X}$, take all $C \in \mathscr{C}$ which are made of elements of $\mathscr{X}'$

Special Family (Nested Subscenarios): Fixed $\mathscr{X}$, nested Compatibility Covers gives Nested Subscenarios

Thank you!