Hypergraph framework for Spekkens contextuality applied to Kochen-Specker scenarios

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Outline

Contextuality à la Spekkens

Kochen-Specker contextuality à la CSW

Hypergraph-theoretic ingredients

Beyond CSW

Takeaway
Contextuality à la Spekkens\textsuperscript{1}

Schematic of a prepare-and-measure scenario and its two descriptions
A prepare-and-measure scenario

\[ m \in V_M \]

\[ M \in M \]

\[ s \in V_S \]

\[ P_{(S,s)} \]

\[ S \in S \]
Two descriptions: Operational vs. Ontological

- **Operational:**
  \[ p(m, s|M, S) \in [0, 1], \]  
  where \( p(m, s|M, S) = p(m|M, S, s)p(s|S). \)

- **Ontological:**
  \[ p(m, s|M, S) = \sum_{\lambda \in \Lambda} \xi(m|M, \lambda)\mu(\lambda, s|S), \]  
  where \( \mu(\lambda, s|S) = \mu(\lambda|S, s)p(s|S). \)
Features of the operational theory necessary to define noncontextuality
Operational equivalences

Preparations

▶ Source events:
\[[s|S] \simeq [s'|S'], \text{ i.e.,} \]
\[p(m, s|M, S) = p(m, s'|M, S') \quad \forall [m|M]. \quad (3)\]

▶ Source settings:
\[[\top|S] \simeq [\top|S'], \text{ i.e.,} \]
\[\sum_{s \in V_S} p(m, s|M, S) = \sum_{s' \in V_{S'}} p(m, s'|M, S') \quad \forall [m|M]. \quad (4)\]
Measurements

Measurement events are operationally equivalent 
([m|M] \sim [m'|M']) if no source event can distinguish them, i.e.,

\forall [s|S] : p(m, s|M, S) = p(m', s|M', S), \hspace{1cm} (5)

e.g., when the same projector appears in two different measurement bases.
What is a ‘context’?

Any distinction between operationally equivalent procedures.
Examples

**Preparation contexts:** Different realizations of a given quantum state, e.g., different convex decompositions,

\[
\frac{I}{2} = \frac{1}{2} |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| = \frac{1}{2} |+\rangle\langle +| + \frac{1}{2} |-\rangle\langle -|,
\]

or different purifications,

\[
\rho_A = \text{Tr}_B |\psi\rangle\langle \psi|_{AB} = \text{Tr}_C |\phi\rangle\langle \phi|_{AC}, etc.
\]
Measurement contexts: Different realizations of a given POVM or a POVM element, e.g., same projector appearing in different measurement bases, joint measurability contexts for a given POVM, or even different ways of implementing a fair coin flip measurement.²

²Mazurek et. al., Nature Communications 7:11780 (2016).
Noncontextuality
Noncontextuality: identity of indiscernibles

If there exists no operational way to distinguish two things, then they must be physically identical.\(^3\)

- Measurement noncontextuality:

\[
[m|M] \simeq [m'|M'] \Rightarrow \xi(m|M, \lambda) = \xi(m'|M', \lambda) \quad \forall \lambda \in \Lambda
\]

- Preparation noncontextuality:

\[
[s|S] \simeq [s'|S'] \Rightarrow \mu(\lambda, s|S) = \mu(\lambda, s'|S') \quad \forall \lambda \in \Lambda, \\
[\top|S] \simeq [\top|S'] \Rightarrow \mu(\lambda|S) = \mu(\lambda|S') \quad \forall \lambda \in \Lambda.
\]

\(^3\)Equivalently: if two things are non-identical, or physically distinct, then there must exist an operational way to distinguish them.
Kochen-Specker (KS) noncontextuality

KS-noncontextuality
⇔ Measurement noncontextuality and Outcome determinism

\[\xi(m|M, \lambda) \in \{0, 1\} \quad \forall \lambda \in \Lambda.\]

\(^4\)Applied to measurement contexts of the type arising from joint measurability. Outcome determinism: for any \([m|M]\),
\[\xi(m|M, \lambda) \in \{0, 1\} \quad \forall \lambda \in \Lambda.\]
Kochen-Specker theorem: logical proof

Kochen-Specker theorem: statistical proof

Kochen-Specker contextuality à la CSW

\footnote{Cabello et al., PRL 112, 040401 (2014).}
Contextuality scenario, $\Gamma$

A hypergraph $\Gamma$ where the nodes of the hypergraph $\nu \in V(\Gamma)$ denote measurement outcomes and hyperedges denote measurements $e \in E(\Gamma) \subseteq 2^{V(\Gamma)}$ such that $\bigcup_{e \in E(\Gamma)} = V(\Gamma)$.

We will further assume that no hyperedge is a strict subset of another in $\Gamma$, following Acín et al. (AFLS), Comm. Math. Phys. 334(2), 533-628 (2015).


Figure: $\Gamma$ for KCBS.
Orthogonality graph of $\Gamma$, i.e., $O(\Gamma)$

Vertices of $O(\Gamma)$ are given by $V(O(\Gamma)) \equiv V(\Gamma)$, and the edges of $O(\Gamma)$ are given by

$$E(O(\Gamma)) \equiv \{\{v, v'\}| v, v' \in e \text{ for some } e \in E(\Gamma)\}.$$
Probabilistic models on $\Gamma$

A *probabilistic model* on $\Gamma$ is given by $p : V(\Gamma) \to [0, 1]$ such that \( \sum_{v \in e} p(v) = 1 \) for all $e \in E(\Gamma)$. The set of all probabilistic models on $\Gamma$ is denoted $\mathcal{G}(\Gamma)$. Relevant subsets of $\mathcal{G}(\Gamma)$:

- **KS-noncontextual, $\mathcal{C}(\Gamma)$**: a convex mixture of $p : V(\Gamma) \to \{0, 1\}$, $\sum_{v \in e} p(v) = 1 \ \forall e \in E(\Gamma)$.

- **Consistently exclusive, $\mathcal{CE}^1(\Gamma)$**: $p : V(\Gamma) \to [0, 1]$, such that $\sum_{v \in c} p(v) \leq 1$ for all cliques $c$ in $O(\Gamma)$.

Clearly,

$$\mathcal{C}(\Gamma) \subseteq \mathcal{CE}^1(\Gamma) \subseteq \mathcal{G}(\Gamma).$$
Exclusivity graph, $G$: a subgraph of $O(\Gamma)$

\[ R([s|S]) \equiv \sum_{v \in V(G)} w_v p(v|S, s), \quad (6) \]

where $w_v > 0$ for all $v \in V(G)$ and $p(v|S, s)$ is a probabilistic model induced by source event $[s|S]$ on measurements events in $\Gamma$. 
CSW bounds

\[ R([s|S]) \equiv \sum_{v \in V(G)} w_v p(v|S, s) \]

\[ KS \leq \alpha(G, w) \]
\[ Q \leq \theta(G, w) \]
\[ E^1 \leq \alpha^*(G, w), \]

\( \text{KCBS}^8 : w_v = 1 \text{ for all } v \in V(G), \)
\( \alpha = 2, \theta = \sqrt{5}, \text{ and } \alpha^* = 5/2. \)

Missing ingredients?

- Measurement noncontextuality alone yields a trivial upper bound $\alpha^*(G, w)$. (Remember: no outcome determinism.)
- Need to invoke preparation noncontextuality.
- We do this next.
Hypergraph-theoretic ingredients
The contextuality scenario $\Gamma_G$

Turn maximal cliques in $G$ into hyperedges and add an extra ("nondetection") vertex to each hyperedge.

We can now take $p(v|S,s)$ to be a probabilistic model on $\Gamma_G$ rather than the full scenario $\Gamma$ and retain the same probabilities on $G$. 
Weighted max-predictability, $\beta(\Gamma_G, q)$

$$\beta(\Gamma_G, q) \equiv \max_{p \in \mathcal{G}(\Gamma_G)_{\text{ind}}} \sum_{e \in E(\Gamma_G)} q_e \zeta(M_e, p), \quad (7)$$

where $q_e \geq 0$ for all $e \in E(\Gamma_G)$, $\sum_{e \in E(\Gamma_G)} q_e = 1$, and

$$\zeta(M_e, p) \equiv \max_{v \in e} p(v) \quad (8)$$

is the maximum probability assigned to a vertex in $e \in E(\Gamma_G)$ by an indeterministic probabilistic model $p \in \mathcal{G}(\Gamma_G)$. 
Source hypergraph

Hypergraph $\Sigma_G$

$[T|S_e] \simeq [T|S_{e'}] \quad \forall e, e' \in E(\Sigma_G)$
Hypergraph $\Sigma_G$

$[\uparrow|S_e] \simeq [\uparrow|S_{e'}]$ \quad \forall e, e' \in E(\Sigma_G)$

Hypergraph $\Gamma_G$
Source-measurement correlations: Corr

\[
\text{Corr} \equiv \sum_{e \in E(\Gamma_G)} q_e \sum_{m_e, s_e} \delta_{m_e, s_e} p(m_e, s_e | M_e, S_e), \tag{9}
\]

where \( \{q_e\}_{e \in E(\Gamma_G)} \) is a probability distribution, i.e., \( q_e \geq 0 \) for all \( e \in E(\Gamma_G) \) and \( \sum_{e \in E(\Gamma_G)} q_e = 1. \)

\footnote{Such that \( \beta(\Gamma_G, q) < 1 \) holds.}
Beyond CSW:
Hypergraph framework for Spekkens contextuality
General form of the noise-robust noncontextuality inequality: KS-colourable case \(^{10,11}\)

\[
R([s_{e^*} = 0|S_{e^*}]) \leq \alpha(G, w) + \frac{\alpha_*(G, w) - \alpha(G, w)}{p_*} \frac{1 - \text{Corr}}{1 - \beta(\Gamma_G, q)}.
\]

Here, \(p_* \equiv p(s_{e^*} = 0|S_{e^*}) = p(v^0_{e^*})\) and all the measurement events in \(G\) are evaluated on the source event \([s_{e^*} = 0|S_{e^*}]\) to compute \(R([s_{e^*} = 0|S_{e^*}])\).

For the KCBS scenario: \(\alpha(G, w) = 2\), \(\alpha_*(G, w) = 5/2\), and \(\beta(\Gamma_G, q) = 1/2\). We then have

\[
R \leq 2 + \frac{1 - \text{Corr}}{p_*}
\]


The framework presented so far applies to KS-colourable contextuality scenarios where statistical proofs of the KS theorem apply. In particular, it covers contextuality scenarios $\Gamma$ (hence also $\Gamma_G$) such that

- $\mathcal{C}(\Gamma) \neq \emptyset$,
- $\mathcal{CE}^1(\Gamma) = \mathcal{G}(\Gamma)$. 

Scope of this generalization of CSW
Hypergraph framework for KS-uncolourable scenarios

- For $\Gamma$ such that $C(\Gamma) = \emptyset$, we obtain a framework (cf. arXiv:1805.02083) based entirely on the hypergraph invariant $\beta(\Gamma_G, q)$.
- Its basic ingredients are still the contextuality scenario $\Gamma$ and the corresponding source events hypergraph.
Recall

\[ \beta(\Gamma_G, q) \equiv \max_{p \in \mathcal{G}(\Gamma_G)_{\text{ind}}} \sum_{e \in E(\Gamma_G)} q_e \zeta(M_e, p), \quad (10) \]

where \( q_e \geq 0 \) for all \( e \in E(\Gamma_G) \), \( \sum_{e \in E(\Gamma_G)} q_e = 1 \), and

\[ \zeta(M_e, p) \equiv \max_{v \in e} p(v) \quad (11) \]

is the maximum probability assigned to a vertex in \( e \in E(\Gamma_G) \) by an indeterministic probabilistic model \( p \in \mathcal{G}(\Gamma_G) \).
Recall

\[
\text{Corr} \equiv \sum_{e \in E(\Gamma)} q_e \sum_{m_e, s_e} \delta_{m_e,s_e} p(m_e, s_e| M_e, S_e), \tag{12}
\]

where \( \{q_e\}_{e \in E(\Gamma)} \) is a probability distribution, i.e., \( q_e \geq 0 \) for all \( e \in E(\Gamma_G) \) and \( \sum_{e \in E(\Gamma)} q_e = 1. \)\(^{12}\)

\(^{12}\)Such that \( \beta(\Gamma, q) < 1 \) holds.
General form of the noise-robust noncontextuality inequality: KS-uncolourable case

\[ \text{Corr} \leq \beta(\Gamma, q). \]  

(13)
Example: 18 ray

\[ \text{Corr} \leq \frac{5}{6}, \quad (14) \]

where \( q_{e_i} = \frac{1}{9} \) for all \( i \in [9] \).
Properties of $\beta(\Gamma, q)$ from structure of the KS-uncolourable hypergraph

- See arXiv:1805.02083 for a study of $\beta(\Gamma, q)$ for various KS-uncolourable hypergraphs.
- It presents a framework for identifying subsets of contexts (i.e., the supports of $\{q_e\}_{e \in E(\Gamma)}$) which admit a nontrivial bound on $\text{Corr}$ given by $\beta(\Gamma, q)$.
- It applies the framework to a family of KS-uncolourable hypergraphs: those where each vertex appears in two hyperedges.
## Comparision of KS vs. Spekkens

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<th>Traditional Bell-KS approaches</th>
<th>Spekkens' approach</th>
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| **Type of context**       | 1) ONB contexts  
2) Compatibility contexts | Includes more types of contexts, for both preps and mmts. |
| **Assumptions**           | MNC and OD (or at least Factorizability) | MNC and PNC (and resp. convex mixtures etc.) |
| **Quantity of interest**  | Mmt-mmt correlations for a fixed input state | Also includes source-mmt correlations |
| **Type of inequalities**  | Constraints on mmt-mmt corr from the classical marginal problem | More refined approach: tradeoff b/w mmt-mmt corr and source-mmt corr |
| **KS-uncolourability proofs** | Logical contradiction, no ineqs on mmt-mmt corr needed. | Robust inequality bounding source-mmt corr. No mmt-mmt corr needed. |
1. We have obtained two complementary hypergraph-based frameworks for KS-colourable and KS-uncolourable scenarios.

2. Together, they complete the project of turning KS-type proofs of contextuality into noise-robust noncontextuality inequalities applicable to noisy measurements and preparations.

3. Open questions:
   - applications of these frameworks to quantum information?
   - hypergraph-theoretic properties of $\beta(\Gamma, q)$ vis-à-vis the structure of $\Gamma$, possible relevance to information theory?