Minimal Distance to Approximating Noncontextual System as a Measure of Contextuality: Comparison to Contextuality-by-Default

Janne V. Kujala <jvk@iki.fi>

Faculty of Information Technology, Univeristy of Jyväskylä

Purdue Winer Memorial Lectures 2018

Alice-Bob System with 2×2 Settings

- Alice's setting $i \in \{1, 2\}$
- Bob's setting $j \in \{1, 2\}$
- Alice observes $A_{ij} \in \{-1, +1\}$
- Bob ovserves $B_{ij} \in \{-1, +1\}$

Alice-Bob System with 2×2 Settings

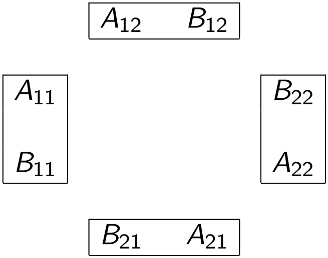
- Alice's setting $i \in \{1, 2\}$
- Bob's setting $j \in \{1, 2\}$
- Alice observes $A_{ij} \in \{-1, +1\}$
- Bob ovserves $B_{ij} \in \{-1, +1\}$

How this corresponds to general CbD notation:

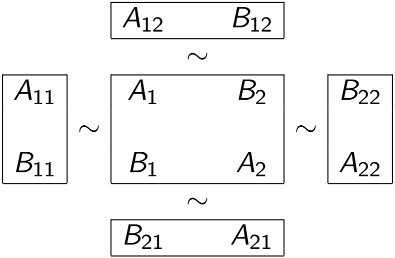
contexts 11, 12, 21, 22,
contents $1\cdot$, $2\cdot$ (for Alice) and $\cdot 1$, $\cdot 2$ (for Bob):

	$1\cdot$	$2\cdot$	$\cdot 1$	$\cdot 2$
11	$R^{11}_{1\cdot} = A_{11}$		$R^{11}_{\cdot 1} = B_{11}$	
12	$R^{12}_{1\cdot} = A_{12}$			$R^{12}_{\cdot 2} = B_{12}$
21		$R^{21}_{2\cdot} = A_{21}$	$R^{21}_{\cdot 1} = B_{21}$	
22		$R^{22}_{2\cdot} = A_{22}$		$R^{22}_{\cdot 2} = B_{22}$

Traditional Understading of Noncontextuality



Traditional Understading of Noncontextuality



Traditional Understanding of Noncontextuality

Definition

Alice-Bob system is noncontextual (in traditional sense) if there exist jointly distributed (A_1, A_2, B_1, B_2) such that $(A_i, B_j) \sim (A_{ij}, B_{ij})$ for all $i, j \in \{1, 2\}$.

Traditional Understanding of Noncontextuality

Definition

Alice-Bob system is noncontextual (in traditional sense) if there exist jointly distributed (A_1, A_2, B_1, B_2) such that $(A_i, B_j) \sim (A_{ij}, B_{ij})$ for all $i, j \in \{1, 2\}$.

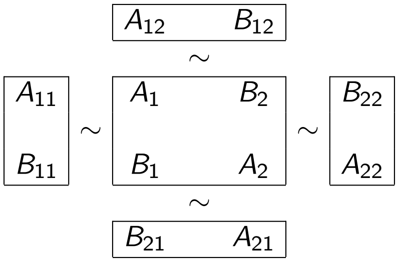
Definition

The Negative Probability (NP) measure of contextuality given by the minimum possible negative probability mass

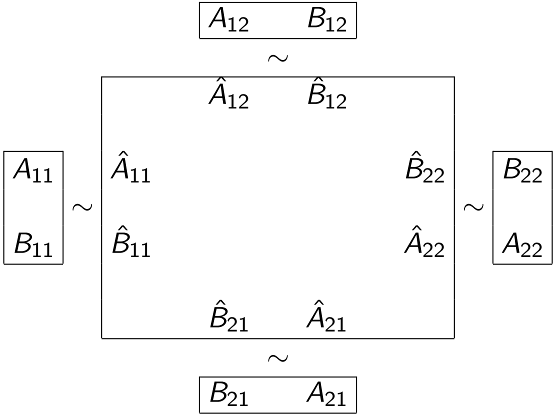
$$\begin{aligned}\Delta^{\mathrm{NP}} &= -\sum_{a_1,a_2,b_1,b_2\in\{-1,1\}} \min\{0, p(a_1, a_2, b_1, b_2)\} \\ &= -1 + \sum_{a_1,a_2,b_1,b_2\in\{-1,1\}} |p(a_1, a_2, b_1, b_2)|\,.\end{aligned}$$

over all negative probability joints $p(a_1, a_2, b_1, b_2)$ of (A_1, A_2, B_1, B_2) satisfying $(A_i, B_j) \sim (A_{ij}, B_{ij})$ for all $i, j \in \{1, 2\}$ (known to exist provided that $\langle A_{i1} \rangle = \langle A_{i2} \rangle$ and $\langle B_{1j} \rangle = \langle B_{2j} \rangle$ for all $i, j \in \{1, 2\}$).

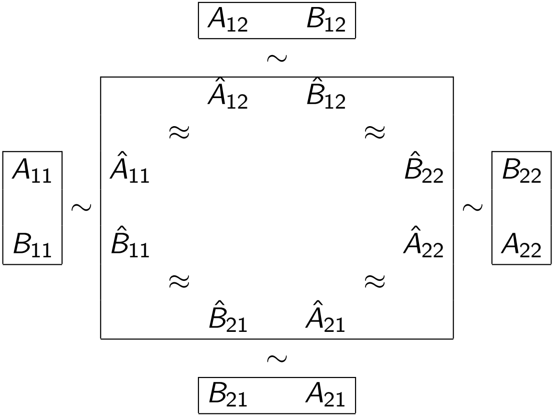
Traditional Understading of Noncontextuality



Contextuality-by-Default



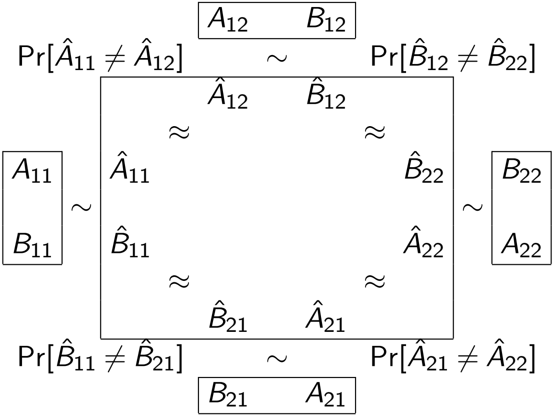
Contextuality-by-Default



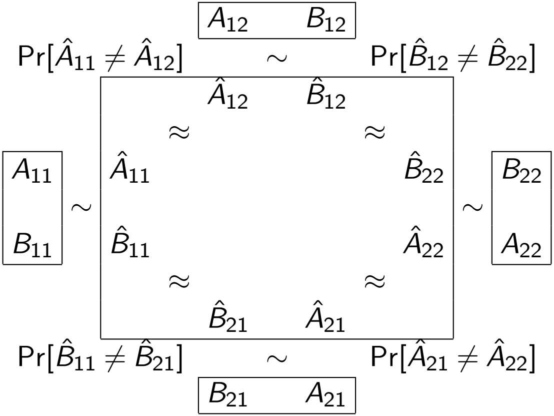
Contextuality-by-Default: measure of contextuality

$$
\begin{array}{ccccccc}
 & & & \boxed{A_{12} \quad B_{12}} & & & \\
 & & & \sim & & & \\
 & & \hat{A}_{12} & & \hat{B}_{12} & & \\
 & \approx & & & & \approx & \\
\boxed{\begin{array}{c} A_{11} \\ \\ B_{11} \end{array}} \sim & \begin{array}{c} \hat{A}_{11} \\ \\ \hat{B}_{11} \end{array} & & & & \begin{array}{c} \hat{B}_{22} \\ \\ \hat{A}_{22} \end{array} & \sim \boxed{\begin{array}{c} B_{22} \\ \\ A_{22} \end{array}} \\
 & \approx & & & & \approx & \\
 & & \hat{B}_{21} & & \hat{A}_{21} & & \\
 & & & \sim & & & \\
 & & & \boxed{B_{21} \quad A_{21}} & & &
\end{array}
$$

Contextuality-by-Default: measure of contextuality



Contextuality-by-Default: measure of contextuality



$$\Delta^{\mathrm{CbD}} = \sum_{i\in\{1,2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j\in\{1,2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}]$$

Contextuality-by-Default: measure of contextuality

Definition

Alice-Bob system is *CbD-noncontextual* if there exists a coupling $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j\in\{1,2\}}$ of the bunches $\{(A_{ij}, B_{ij})\}_{i,j\in\{1,2\}}$ such that

$$\Delta^{\mathrm{CbD}} = \sum_{i\in\{1,2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j\in\{1,2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}] \tag{1}$$

equals

$$\Delta_0^{\mathrm{CbD}} = \sum_{i\in\{1,2\}} \frac{1}{2} \left|\langle A_{i1}\rangle - \langle A_{i2}\rangle\right| + \sum_{j\in\{1,2\}} \frac{1}{2} \left|\langle B_{1j}\rangle - \langle B_{2j}\rangle\right|, \tag{2}$$

the minimum possible value of Δ^{CbD} allowed by $\langle A_{ij}\rangle, \langle B_{ij}\rangle$, $i,j \in \{1,2\}$.

Contextuality-by-Default: measure of contextuality

Definition

Alice-Bob system is *CbD-noncontextual* if there exists a coupling $\{(\hat{A}_{ij}, \hat{B}_{ij})\}_{i,j\in\{1,2\}}$ of the bunches $\{(A_{ij}, B_{ij})\}_{i,j\in\{1,2\}}$ such that

$$\Delta^{\text{CbD}} = \sum_{i\in\{1,2\}} \Pr[\hat{A}_{i1} \neq \hat{A}_{i2}] + \sum_{j\in\{1,2\}} \Pr[\hat{B}_{1j} \neq \hat{B}_{2j}] \tag{1}$$

equals

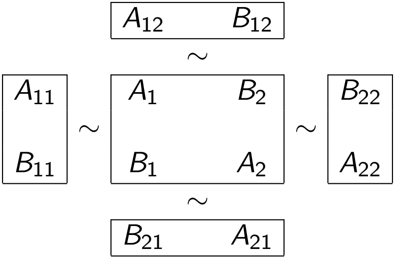
$$\Delta_0^{\text{CbD}} = \sum_{i\in\{1,2\}} \frac{1}{2} |\langle A_{i1}\rangle - \langle A_{i2}\rangle| + \sum_{j\in\{1,2\}} \frac{1}{2} |\langle B_{1j}\rangle - \langle B_{2j}\rangle| , \tag{2}$$

the minimum possible value of Δ^{CbD} allowed by $\langle A_{ij}\rangle, \langle B_{ij}\rangle$, $i, j \in \{1, 2\}$.

Definition

The CbD-measure of contextuality is given by the minimum possible value of the difference $\Delta^{\text{CbD}} - \Delta_0^{\text{CbD}}$ over all possible couplings.

Optimal Approximation (non-)contextuality



Optimal Approximation (non-)contextuality

$$
\begin{array}{ccccc}
 & & \boxed{A_{12} \quad\quad B_{12}} & & \\
 & & \approx & & \\
\boxed{\begin{array}{c} A_{11} \\ \\ B_{11} \end{array}} & \approx & \boxed{\begin{array}{cc} A_1 & B_2 \\ & \\ B_1 & A_2 \end{array}} & \approx & \boxed{\begin{array}{c} B_{22} \\ \\ A_{22} \end{array}} \\
 & & \approx & & \\
 & & \boxed{B_{21} \quad\quad A_{21}} & &
\end{array}
$$

Optimal Approximation (non-)contextuality

$$\boxed{\begin{array}{cc} A_{12} & B_{12} \end{array}}$$

$$\sim$$

$$\boxed{\begin{array}{cc} \hat{A}_{12} & \hat{B}_{12} \\ \approx & \approx \\ \hat{A}_{1} & \hat{B}_{2} \end{array}}$$

$$\sim$$

$$\boxed{\begin{array}{c} A_{11} \\ \\ B_{11} \end{array}} \sim \boxed{\begin{array}{ccc} \hat{A}_{11} & \approx & \hat{A}_{1} \\ \\ \hat{B}_{11} & \approx & \hat{B}_{1} \end{array}} \sim \boxed{\begin{array}{cc} A_{1} & B_{2} \\ \\ B_{1} & A_{2} \end{array}} \sim \boxed{\begin{array}{ccc} \hat{B}_{2} & \approx & \hat{B}_{22} \\ \\ \hat{A}_{2} & \approx & \hat{A}_{22} \end{array}} \sim \boxed{\begin{array}{c} B_{22} \\ \\ A_{22} \end{array}}$$

$$\sim$$

$$\boxed{\begin{array}{cc} \hat{B}_{1} & \hat{A}_{2} \\ \approx & \approx \\ \hat{B}_{21} & \hat{A}_{21} \end{array}}$$

$$\sim$$

$$\boxed{\begin{array}{cc} B_{21} & A_{21} \end{array}}$$

Optimal Approximation (non-)contextuality

$$\boxed{A_{12} \quad B_{12}}$$

$$\sim$$

$$\Pr[\hat{A}_{12} \neq \hat{A}_2] \quad \boxed{\begin{matrix} \hat{A}_{12} & \hat{B}_{12} \\ \approx & \approx \\ \hat{A}_1 & \hat{B}_2 \end{matrix}} \quad \Pr[\hat{B}_{12} \neq \hat{B}_2]$$

$$\Pr[\hat{A}_{11} \neq \hat{A}_1] \qquad \sim \qquad \Pr[\hat{B}_{22} \neq \hat{B}_2]$$

$$\boxed{\begin{matrix} A_{11} \\ \\ B_{11} \end{matrix}} \sim \boxed{\begin{matrix} \hat{A}_{11} & \approx & \hat{A}_1 \\ & & \\ \hat{B}_{11} & \approx & \hat{B}_1 \end{matrix}} \sim \boxed{\begin{matrix} A_1 & B_2 \\ & \\ B_1 & A_2 \end{matrix}} \sim \boxed{\begin{matrix} \hat{B}_2 & \approx & \hat{B}_{22} \\ & & \\ \hat{A}_2 & \approx & \hat{A}_{22} \end{matrix}} \sim \boxed{\begin{matrix} B_{22} \\ \\ A_{22} \end{matrix}}$$

$$\Pr[\hat{B}_{11} \neq \hat{B}_1] \qquad \sim \qquad \Pr[\hat{A}_{22} \neq \hat{A}_2]$$

$$\Pr[\hat{B}_{11} \neq \hat{B}_1] \quad \boxed{\begin{matrix} \hat{B}_1 & \hat{A}_2 \\ \approx & \approx \\ \hat{B}_{21} & \hat{A}_{21} \end{matrix}} \quad \Pr[\hat{A}_{21} \neq \hat{A}_2]$$

$$\sim$$

$$\boxed{B_{21} \quad A_{21}}$$

Definition

Alice–Bob system is *optimal approximation (OA) -noncontextual*, if there exist jointly distributed (A_1, A_2, B_1, B_2) such that each pair (A_i, B_j) can be coupled with the observable pair (A_{ij}, B_{ij}) by a coupling

$$((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij})) \tag{3}$$

such that over the four such couplings, the sum

$$\Delta = \sum_{i,j \in \{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}_i] + \Pr[\hat{B}_{ij} \neq \hat{B}_j] \right) \tag{4}$$

equals

$$\Delta_0 = \sum_{i \in \{1,2\}} \frac{1}{2} \left| \langle A_{i1} \rangle - \langle A_{i2} \rangle \right| + \sum_{j \in \{1,2\}} \frac{1}{2} \left| \langle B_{1j} \rangle - \langle B_{2j} \rangle \right|. \tag{5}$$

Definition

Alice–Bob system is *optimal approximation (OA) -noncontextual*, if there exist jointly distributed (A_1, A_2, B_1, B_2) such that each pair (A_i, B_j) can be coupled with the observable pair (A_{ij}, B_{ij}) by a coupling

$$((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij})) \tag{3}$$

such that over the four such couplings, the sum

$$\Delta = \sum_{i,j \in \{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}_i] + \Pr[\hat{B}_{ij} \neq \hat{B}_j] \right) \tag{4}$$

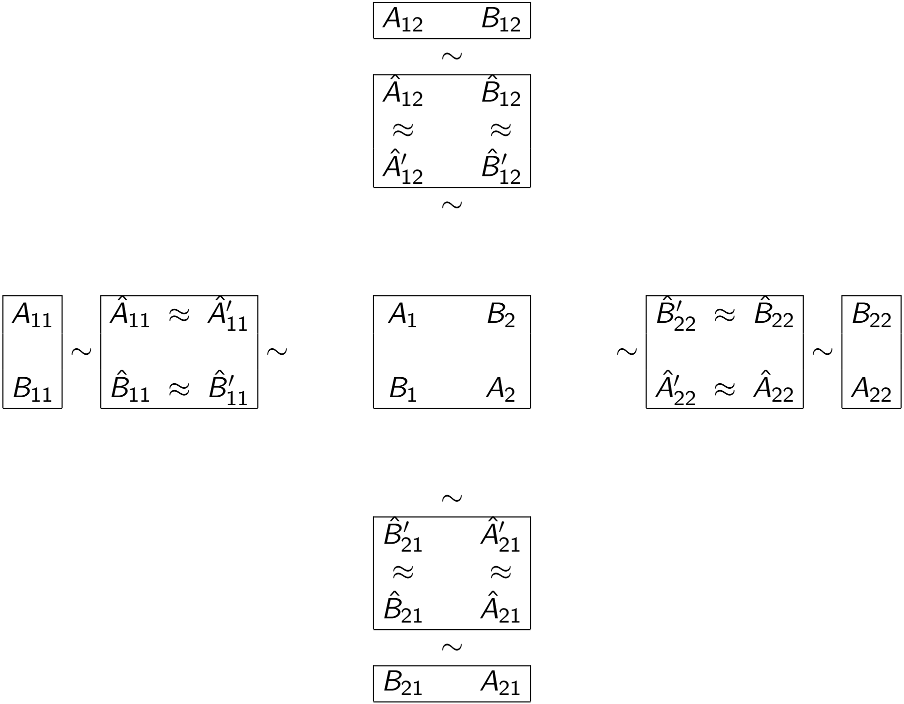
equals

$$\Delta_0 = \sum_{i \in \{1,2\}} \frac{1}{2} \left| \langle A_{i1} \rangle - \langle A_{i2} \rangle \right| + \sum_{j \in \{1,2\}} \frac{1}{2} \left| \langle B_{1j} \rangle - \langle B_{2j} \rangle \right|. \tag{5}$$

Definition

The OA measure of contextuality is given by the minimum possible value of $\Delta - \Delta_0$ over the possible couplings and choices of (A_1, A_2, B_1, B_2).

$$
\begin{array}{ccccccccc}
 & & & & \boxed{\begin{array}{cc} A_{12} & B_{12} \end{array}} & & & & \\
 & & & & \sim & & & & \\
 & & & & \boxed{\begin{array}{cc} \hat{A}_{12} & \hat{B}_{12} \\ \approx & \approx \\ \hat{A}'_{12} & \hat{B}'_{12} \end{array}} & & & & \\
 & & & & \sim & & & & \\
\boxed{\begin{array}{c} A_{11} \\ \\ B_{11} \end{array}} & \sim & \boxed{\begin{array}{ccc} \hat{A}_{11} & \approx & \hat{A}'_{11} \\ \\ \hat{B}_{11} & \approx & \hat{B}'_{11} \end{array}} & \sim & \boxed{\begin{array}{cc} A_1 & B_2 \\ \\ B_1 & A_2 \end{array}} & \sim & \boxed{\begin{array}{ccc} \hat{B}'_{22} & \approx & \hat{B}_{22} \\ \\ \hat{A}'_{22} & \approx & \hat{A}_{22} \end{array}} & \sim & \boxed{\begin{array}{c} B_{22} \\ \\ A_{22} \end{array}} \\
 & & & & \sim & & & & \\
 & & & & \boxed{\begin{array}{cc} \hat{B}'_{21} & \hat{A}'_{21} \\ \approx & \approx \\ \hat{B}_{21} & \hat{A}_{21} \end{array}} & & & & \\
 & & & & \sim & & & & \\
 & & & & \boxed{\begin{array}{cc} B_{21} & A_{21} \end{array}} & & & &
\end{array}
$$



A_{12} B_{12}
$\sim$
$\hat{A}_{12}$ $\hat{B}_{12}$
$\approx$ $\approx$
$\hat{A}'_{12}$ $\hat{B}'_{12}$
$\sim$
A_{11} B_{11}
$\sim$
$\hat{A}_{11} \approx \hat{A}'_{11}$
$\hat{B}_{11} \approx \hat{B}'_{11}$
$\sim$
A_1 B_2
B_1 A_2
$\sim$
$\hat{B}'_{22} \approx \hat{B}_{22}$
$\hat{A}'_{22} \approx \hat{A}_{22}$
$\sim$
B_{22} A_{22}
$\sim$
$\hat{B}'_{21}$ $\hat{A}'_{21}$
$\approx$ $\approx$
$\hat{B}_{21}$ $\hat{A}_{21}$
$\sim$
B_{21} A_{21}

$$\boxed{A_{12} \qquad B_{12}}$$

$$\sim$$

$$\boxed{\begin{matrix} \hat{A}_{12} & \hat{B}_{12} \\ \approx & \approx \\ \hat{A}'_{12} & \hat{B}'_{12} \end{matrix}}$$

$$\sim$$

$$\boxed{A'_{12} \qquad B'_{12}}$$

$$\sim \qquad\qquad\qquad\qquad \sim$$

$$\boxed{\begin{matrix} A_{11} \\ \\ B_{11} \end{matrix}} \sim \boxed{\begin{matrix} \hat{A}_{11} & \approx & \hat{A}'_{11} \\ \\ \hat{B}_{11} & \approx & \hat{B}'_{11} \end{matrix}} \sim \boxed{\begin{matrix} A'_{11} \\ \\ B'_{11} \end{matrix}} \qquad\qquad\qquad\qquad \boxed{\begin{matrix} B'_{22} \\ \\ A'_{22} \end{matrix}} \sim \boxed{\begin{matrix} \hat{B}'_{22} & \approx & \hat{B}_{22} \\ \\ \hat{A}'_{22} & \approx & \hat{A}_{22} \end{matrix}} \sim \boxed{\begin{matrix} B_{22} \\ \\ A_{22} \end{matrix}}$$

$$\sim \qquad\qquad\qquad\qquad \sim$$

$$\boxed{B'_{21} \qquad A'_{21}}$$

$$\sim$$

$$\boxed{\begin{matrix} \hat{B}'_{21} & \hat{A}'_{21} \\ \approx & \approx \\ \hat{B}_{21} & \hat{A}_{21} \end{matrix}}$$

$$\sim$$

$$\boxed{B_{21} \qquad A_{21}}$$

Optimal approximation

Definition

A set of bivariate random variables $\{(A'_{ij}, B'_{ij})\}_{i,j\in\{1,2\}}$ that is consistently connected, i.e.,

$$A'_{ij} \sim A'_{ij'} \text{ and } B'_{ij} \sim B'_{i'j} \text{ for all } i, i', j, j' \in \{1,2\},$$

is said to *approximate optimally* the system $\{(A_{ij}, B_{ij})\}_{i,j\in\{1,2\}}$ if there exists couplings $((\hat{A}_{ij}, \hat{B}_{ij}), (\hat{A}'_{ij}, \hat{B}'_{ij}))$ for all $i, j \in \{1,2\}$ such that

$$\Delta' = \sum_{i,j\in\{1,2\}} \left(\Pr[\hat{A}_{ij} \neq \hat{A}'_{ij}] + \Pr[\hat{B}_{ij} \neq \hat{B}'_{ij}] \right)$$

equals

$$\Delta_0 = \sum_{i\in\{1,2\}} \frac{1}{2} |\langle A_{i1}\rangle - \langle A_{i2}\rangle| + \sum_{j\in\{1,2\}} \frac{1}{2} |\langle B_{1j}\rangle - \langle B_{2j}\rangle| .$$

Optimal approximation

Fact

A system is OA-noncontextual if and only if it is approximated optimally by a system that is noncontextual in the traditional sense.

Optimal approximation

Fact

A system is OA-noncontextual if and only if it is approximated optimally by a system that is noncontextual in the traditional sense.

Definition

Consider all negative probability joints of (A_1, A_2, B_1, B_2) having proper marginals for all (A_i, B_j) with $\Delta' = \Delta_0$ where we define $(A'_{ij}, B'_{ij}) = (A_i, B_j)$. The degree of OA-NP contextuality in the system is then defined as the minimum possible total negative probability mass among all such optimally approximating negative probability joints.

Optimal approximation

Example

Optimal approximation can also be applied to a specific model predicting a consistently connected set of jointly distributed pairs (A'_{ij}, B'_{ij}), for example, to the quantum model

$$\langle A'_{ij} B'_{ij} \rangle = -\cos(\alpha_i - \beta_j), \quad \langle A'_{ij} \rangle = \langle B'_{ij} \rangle = 1/2, \quad i,j \in \{1,2\}$$

of the EPR experiment for photons. Thus, if the observations deviate somewhat from consistent connectedness, the above approach still allows one to test whether the observations are as close to the prediction as possible ignoring the contextual changes in the marginals. This allows, for example, $\langle A_{11} \rangle$ and $\langle A_{12} \rangle$ to deviate from the predicted value $1/2$ in some cases (but only if they deviate to different directions).

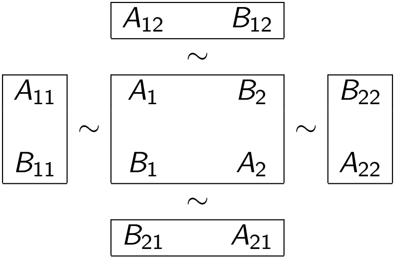
Results

<table>
<tr><th rowspan="2">Type of system</th><th colspan="4">Measure of contextuality</th></tr>
<tr><th>NP</th><th>OA-NP</th><th>OA</th><th>CbD</th></tr>
<tr><td>Traditional noncontextual</td><td colspan="4">$\mu_{\text{NP}} = \mu_{\text{OA-NP}} = \mu_{\text{OA}} = \mu_{\text{CbD}} = 0$</td></tr>
<tr><td>Non-signaling & traditional contextual</td><td colspan="2">$\mu_{\text{NP}} = \mu_{\text{OA-NP}} > 0$</td><td>$\mu_{\text{OA}} > 0$</td><td>$\mu_{\text{CbD}} > 0$</td></tr>
<tr><td>Signaling & consistently connected</td><td>$\nexists \mu_{\text{NP}}$</td><td>$\mu_{\text{OA-NP}} = \infty$</td><td>$\mu_{\text{OA}} > 0$</td><td>$\mu_{\text{CbD}} > 0$</td></tr>
<tr><td>Inconsistently connected</td><td>$\nexists \mu_{\text{NP}}$</td><td>$\mu_{\text{OA-NP}}$</td><td>μ_{OA}</td><td>μ_{CbD}</td></tr>
<tr><td>OA-noncontextual</td><td></td><td colspan="2">$\mu_{\text{OA-NP}} = \mu_{\text{OA}} = 0$</td><td>$\mu_{\text{CbD}}$</td></tr>
<tr><td>OA-contextual</td><td></td><td>$\mu_{\text{OA-NP}} > 0$</td><td>$\mu_{\text{OA}} > 0$</td><td>μ_{CbD}</td></tr>
<tr><td>2 contexts per content</td><td></td><td>$\mu_{\text{OA-NP}}$</td><td colspan="2">$\mu_{\text{OA}} = \mu_{\text{CbD}}$</td></tr>
<tr><td>Non-signaling Alice–Bob, $2+2$ contents</td><td colspan="4">$0 \leq \mu_{\text{NP}} = \mu_{\text{OA-NP}} = \mu_{\text{OA}} = \mu_{\text{CbD}} \leq 1$</td></tr>
</table>

Results

Type of system	Measure of contextuality: NP	OA-NP	OA	CbD
Traditional noncontextual	$\mu_{\text{NP}} = \mu_{\text{OA-NP}} = \mu_{\text{OA}} = \mu_{\text{CbD}} = 0$			
Non-signaling & traditional contextual	$\mu_{\text{NP}} = \mu_{\text{OA-NP}} > 0$		$\mu_{\text{OA}} > 0$	$\mu_{\text{CbD}} > 0$
Signaling & consistently connected	$\nexists \mu_{\text{NP}}$	$\mu_{\text{OA-NP}} = \infty$	$\mu_{\text{OA}} > 0$	$\mu_{\text{CbD}} > 0$
Inconsistently connected	$\nexists \mu_{\text{NP}}$	$\mu_{\text{OA-NP}}$	μ_{OA}	μ_{CbD}
OA-noncontextual		$\mu_{\text{OA-NP}} = \mu_{\text{OA}} = 0$		μ_{CbD}
OA-contextual		$\mu_{\text{OA-NP}} > 0$	$\mu_{\text{OA}} > 0$	μ_{CbD}
2 contexts per content		$\mu_{\text{OA-NP}}$	$\mu_{\text{OA}} = \mu_{\text{CbD}}$	
Non-signaling Alice–Bob, 2+2 contents	$0 \leq \mu_{\text{NP}} = \mu_{\text{OA-NP}} = \mu_{\text{OA}} = \mu_{\text{CbD}} \leq 1$			

Computation of the NP measure



Computation of the NP measure

The negative probability joint of (A_1, A_2, B_1, B_2) is represented by the 16 signed variables

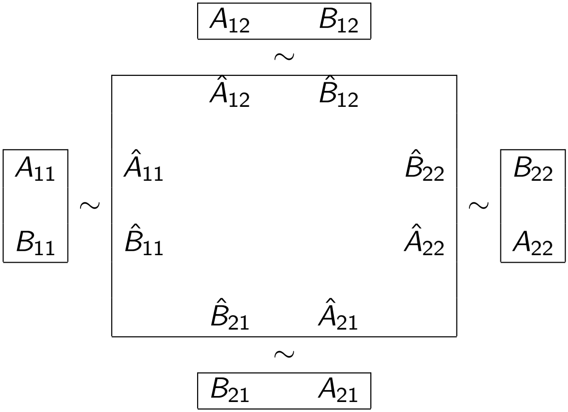
$$p_{a_1 a_2 b_1 b_2} = \Pr[A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2]. \tag{6}$$

For each $i, j \in \{1, 2\}$ there are 4 linear equations constraining the marginal of (A_i, B_j) of (6) to the observed joint. To minimize the total negative probability mass, the signed variables (6) have to be represented by their negative and positive parts,

$$p_{a_1 a_2 b_1 b_2} = p^{+}_{a_1 a_2 b_1 b_2} - p^{-}_{a_1 a_2 b_1 b_2}, \qquad p^{+}_{a_1 a_2 b_1 b_2}, p^{-}_{a_1 a_2 b_1 b_2} \geq 0.$$

The total negative probability mass is then obtained as the linear expression $\sum_{a_1, a_2, b_1, b_2 \in \{+1, -1\}} p^{-}_{a_1 a_2 b_1 b_2}$ and this expression can be minimized using linear programming given the $2 \cdot 16 = 32$ nonnegative variables and 16 equation constraints.

Computation of the CbD measure



Computation of the CbD measure

In the CbD approach, the coupling

$$((\hat{A}_{11},\hat{B}_{11}),(\hat{A}_{12},\hat{B}_{12}),(\hat{A}_{21},\hat{B}_{21}),(\hat{A}_{22},\hat{B}_{22}))$$

is represented by the $2^8 = 256$ nonnegative variables

$$p_{a_{11}b_{11}a_{12}b_{12}a_{21}b_{21}a_{22}b_{22}} = \Pr[\hat{A}_{ij} = a_{ij}, \hat{B}_{ij} = b_{ij} : i,j \in \{1,2\}],$$

representing the joint probabilities. For each $i,j \in \{1,2\}$, there are 4 linear equations constraining the distribution of $(\hat{A}_{ij},\hat{B}_{ij})$ to that of the observed pair (A_{ij},B_{ij}). The expression Δ^{CbD} can be evaluated directly from the values of these 256 variables and so the degree of contextuality is obtained by linear programming, minimizing this expression given the constraints (and subtracting Δ_0^{CbD})

Computation of the OA measure

$$
\begin{array}{c}
\boxed{\begin{array}{cc} A_{12} & B_{12} \end{array}} \\
\sim \\
\boxed{\begin{array}{cc} \hat{A}_{12} & \hat{B}_{12} \\ \approx & \approx \\ \hat{A}'_{12} & \hat{B}'_{12} \end{array}} \\
\sim
\end{array}
$$

$$
\boxed{\begin{array}{c} A_{11} \\ \\ B_{11} \end{array}}
\sim
\boxed{\begin{array}{ccc} \hat{A}_{11} & \approx & \hat{A}'_{11} \\ \\ \hat{B}_{11} & \approx & \hat{B}'_{11} \end{array}}
\sim
\boxed{\begin{array}{cc} A_1 & B_2 \\ \\ B_1 & A_2 \end{array}}
\sim
\boxed{\begin{array}{ccc} \hat{B}'_{22} & \approx & \hat{B}_{22} \\ \\ \hat{A}'_{22} & \approx & \hat{A}_{22} \end{array}}
\sim
\boxed{\begin{array}{c} B_{22} \\ \\ A_{22} \end{array}}
$$

$$
\begin{array}{c}
\sim \\
\boxed{\begin{array}{cc} \hat{B}'_{21} & \hat{A}'_{21} \\ \approx & \approx \\ \hat{B}_{21} & \hat{A}_{21} \end{array}} \\
\sim \\
\boxed{\begin{array}{cc} B_{21} & A_{21} \end{array}}
\end{array}
$$

Computation of the OA measure

The approximating noncontextual system is represented by the 16 nonnegative variables

$$p_{a_1 a_2 b_1 b_2} = \Pr[A_1 = a_1, A_2 = a_2, B_1 = b_1, B_2 = b_2] \tag{7}$$

and for each $i, j \in \{1, 2\}$, the coupling $((\hat{A}_i, \hat{B}_j), (\hat{A}_{ij}, \hat{B}_{ij}))$ of $((A_i, B_j), (A_{ij}, B_{ij}))$ is represented by the 16 nonnegative variables

$$p^{ij}_{a_{ij} b_{ij} a_i b_i} = \Pr[\hat{A}_{ij} = a_{ij}, \hat{B}_{ij} = b_{ij}, \hat{A}_i = a_i, \hat{B}_i = b_i]. \tag{8}$$

For each $i, j \in \{1, 2\}$ there are 4 linear equations constraining the marginal of $(\hat{A}_{ij}, \hat{B}_{ij})$ of (8) to the observed joint and another 4 linear equations constraining the marginal of $(\hat{A}_i, \hat{B}_j)$ in (8) to agree with the margial (A_i, B_j) in (7). The degree of contextuality is obtained by minimizing Δ under these constraints.

Computational Complexity: LP Problem Size

System:

- m settings for Alice, n settings for Bob (± 1 outcomes).

LP problem:

- $Mq = p$ subject to $q \geq 0$, where
- M matrix determined by m, n
- p vector of probabilities of possible joint outcomes in each context (padded by the same number of 0's in the OA approach).

	CbD	NP	OA
Nonnegative variables (columns of M)	2^{mn}	2^{m+n+1}	$2^{m+n} + 16mn$
Equation constraints (rows of M)	$4mn$	$4mn$	$8mn$
Inequality constraints	0	0	0

Conclusion

- OA-measure computationally more efficient than the CbD-measure

Conclusion

- OA-measure computationally more efficient than the CbD-measure
- OA-measure equivalent to CbD for *systems with 2 contexts per content*

Conclusion

- OA-measure computationally more efficient than the CbD-measure
- OA-measure equivalent to CbD for *systems with 2 contexts per content*
    - CbD handles more general systems by considering connections in pairs

Conclusion

- OA-measure computationally more efficient than the CbD-measure
- OA-measure equivalent to CbD for *systems with 2 contexts per content*
  - CbD handles more general systems by considering connections in pairs
  - applying the same splitting of connections into pairs in OA yields a variant of the OA-measure that is always equivalent to CbD

Conclusion

- OA-measure computationally more efficient than the CbD-measure
- OA-measure equivalent to CbD for *systems with 2 contexts per content*
  - CbD handles more general systems by considering connections in pairs
  - applying the same splitting of connections into pairs in OA yields a variant of the OA-measure that is always equivalent to CbD
  - OA-measure becomes a *computational variant* of CbD