Miscellaneous Comments on (Quantum) Contextuality

Ehtibar N. Dzhafarov

Purdue University

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  - because the term comes from QM historically,
  - and because in QM we have the traditional examples of contextual systems.
- Otherwise, contextuality is not about QM specifically.
  - (One could even say, it has nothing to do with QM.)
- Contextuality is all about random variables, more specifically, about identities of random variables.

What are random variables, mathematically?

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- Mathematically, a random variable is (identified by) a measurable mapping $D \overset{measurable}{\longrightarrow} C$,
  - with D in (D, Σ_D, μ) and C in (C, Σ_C),
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- Mathematically, elements of a set $\mathbb{X}$ of random variables are jointly distributed, JD$(\mathbb{X})$, iff they have the same domain probability space (D, Σ_D, μ).
- Ergo: if $\mathbb{A}$ and $\mathbb{B}$ are sets of random variables with $\mathbb{A} \cap \mathbb{B} \neq \emptyset$,

$$\boxed{\text{JD}\left(\mathbb{A}\right) \& \text{JD}\left(\mathbb{B}\right) \Rightarrow \text{JD}\left(\mathbb{A} \cup \mathbb{B}\right)}$$ ("agglutinativity").

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  - in particular, random variables A and B generated by Hermitian operators $\overline{A}, \overline{B}$ with $[\overline{A}, \overline{B}] \neq 0$ may very well be measured "together" (e.g., in succession).

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  - and if $[\overline{A}, \overline{B}] = 0$, the A and B they generate may very well be recorded in different, mutually exclusive contexts.

A system of random variables: contents and contexts

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$\star$	$\star$			$c = 1$
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		$\star$	$\star$	$c = 3$
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A system of random variables: contents and contexts

(R_1^1, R_2^1)

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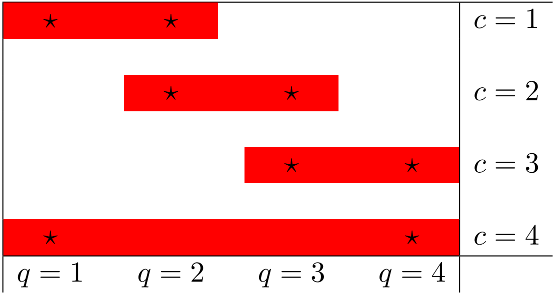

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Content-sharing variables are the same?

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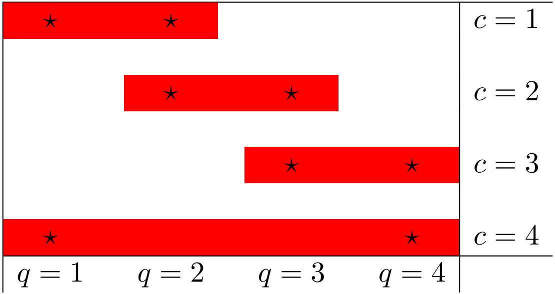

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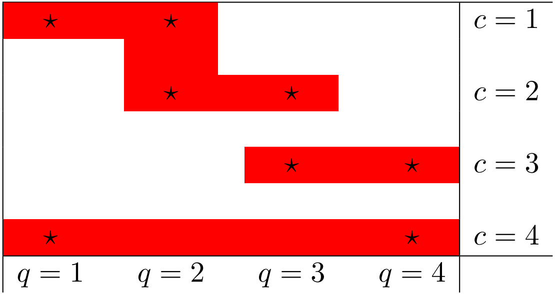

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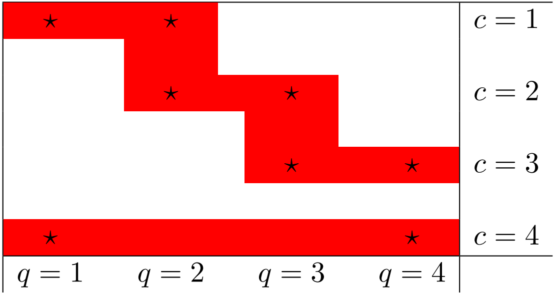

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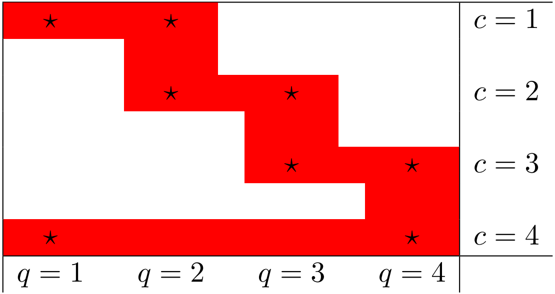

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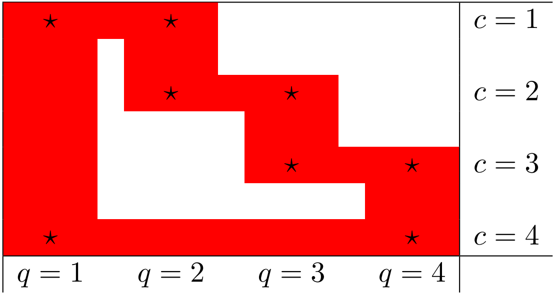

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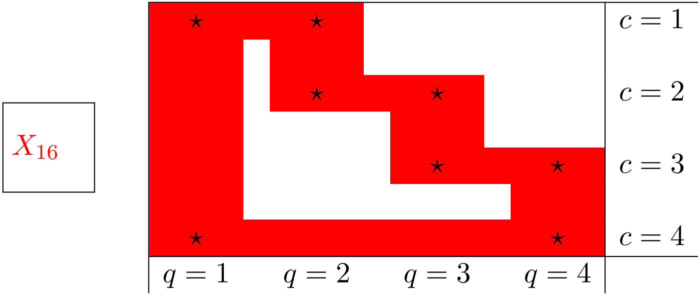
Contextuality of a system of random variables



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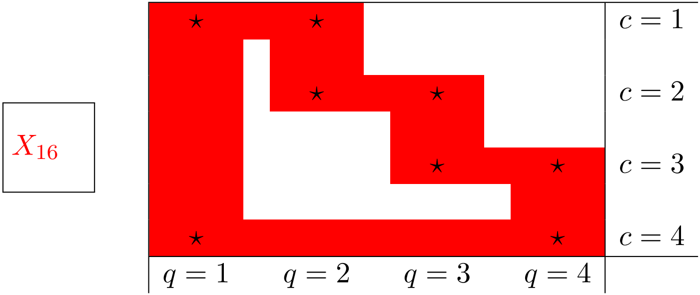

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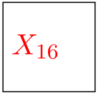

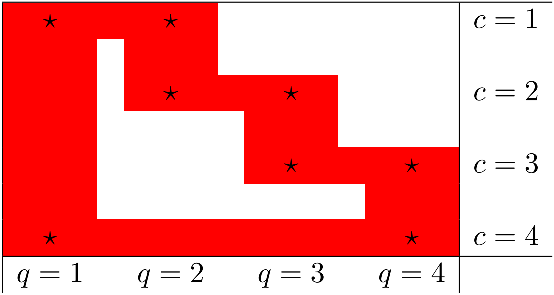

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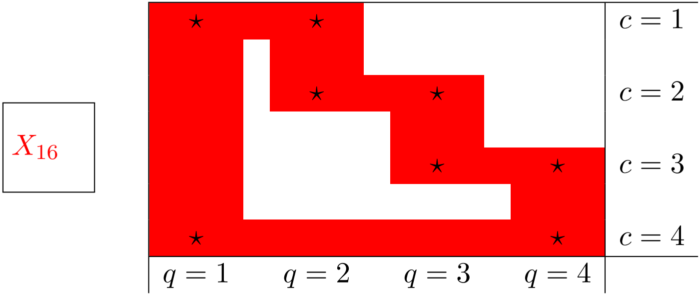

- Each $R_i^j = f_i(X_{16}) \Rightarrow$ 9 linear equations with 16 unknown probabilities $\Rightarrow$ may or may not have a (nonnegative) solution.
- If no solution exists, we have to reject the assumption (the only one made) that the content-sharing variables are the same.

Kochen-Specker system (Cabello-Estebaranz-Alcaine)

c_9													⋆		⋆		⋆	⋆
c_8						⋆						⋆				⋆		⋆
c_7				⋆						⋆						⋆	⋆	
c_6									⋆		⋆			⋆	⋆			
c_5		⋆			⋆								⋆	⋆				
c_4							⋆	⋆			⋆	⋆						
c_3			⋆					⋆	⋆	⋆								
c_2	⋆				⋆	⋆	⋆											
c_1	⋆	⋆	⋆	⋆														
	q_{0001}	q_{0010}	q_{1100}	$q_{1\bar{1}00}$	q_{0100}	q_{1010}	$q_{10\bar{1}0}$	$q_{1\bar{1}1\bar{1}}$	$q_{1\bar{1}\bar{1}1}$	q_{0011}	q_{1111}	$q_{010\bar{1}}$	q_{1001}	$q_{100\bar{1}}$	$q_{01\bar{1}0}$	$q_{11\bar{1}1}$	$q_{111\bar{1}}$	$q_{\bar{1}111}$

- We have to reject the assumption that the content-sharing variables are the same, because it is incompatible with any assignment of values that complies with the within-context distributions.

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5. we refute the assumption at step 3 and conclude: (at least some of the) random variables measuring the same thing in different contexts are not the same, even if they have the same distribution.

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- In practice, some of us prefer to "conclude" things more imaginative, such as JD (A, B) and JD (B, C) and JD (A, C) but not JD (A, B, C) — which is impossible in probability theory.

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- There is a much better way of conceptualizing the situation.

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$$T$$
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$$H$$
$$H$$
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$$T$$
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- Do not ask why and how the G's and F's create different variables: this is true by construction, there are no causes involved.
- There are, however, other questions one can meaningfully ask, the answers to which will cover every aspect of traditional contextuality analysis (and more).

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(R_4^4, R_1^4) (R_3^3, R_4^3)

Contextuality-by-Default: Couplings

$\left(S_1^1, S_2^1, S_2^2, S_3^2, S_3^3, S_4^3, S_4^4, S_1^4\right)$

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$\left(\qquad S_2^2, S_3^2 \qquad \right)$

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⋆	⋆			$c = 1$
	⋆	⋆		$c = 2$
		⋆	⋆	$c = 3$
⋆			⋆	$c = 4$
$q = 1$	$q = 2$	$q = 3$	$q = 4$	

$\left(R_4^4, R_1^4\right)$ $\left(R_3^3, R_4^3\right)$

Generally, there is an infinity of such couplings. In CbD, we are interested in the existence among them of certain, "special" couplings.

Maximal couplings for content-sharing pairs

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	$+$	$-$
A	p	$1-p$
B	q	$1-q$

Maximal couplings for content-sharing pairs

	$+$	$-$
A	p	$1-p$
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	$B' = +$	$B' = -$	
$A' = +$	$\min(p,q)$	$p - \min(p,q)$	p
$A' = -$	$q - \min(p,q)$	$\min(1-p, 1-q)$	$1-p$
	q	$1-q$	

$$equivalently,$$

$$A' = B'$$

with max possible probability

Contextuality-by-Default: Logic of contextuality analysi

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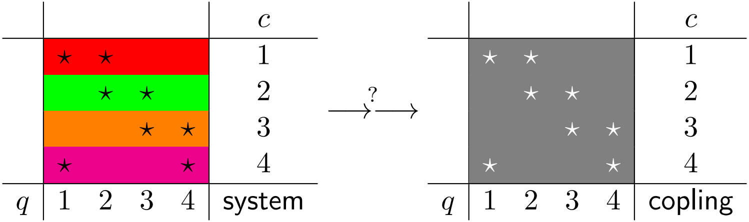
					c
	$\star$	$\star$			1
		$\star$	$\star$		2
			$\star$	$\star$	3
	$\star$			$\star$	4
q	1	2	3	4	system

- Given: A system of dichotomous random variables (each random variable is uniquely identified by its content and its context).

Contextuality-by-Default: Logic of contextuality analysis



- Given: A system of dichotomous random variables (each random variable is uniquely identified by its content and its context).
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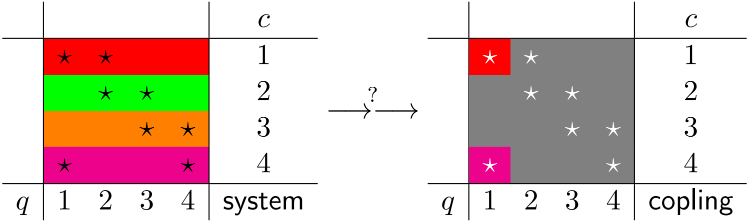

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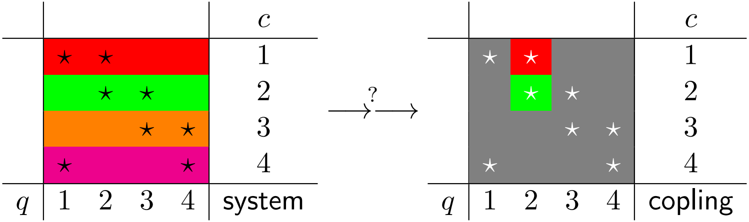

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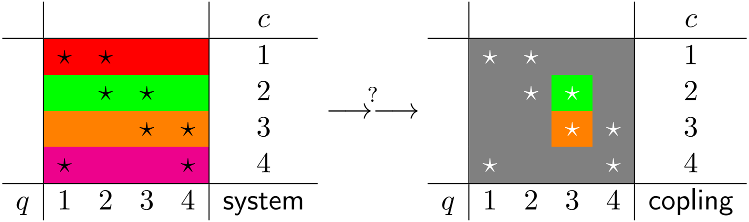

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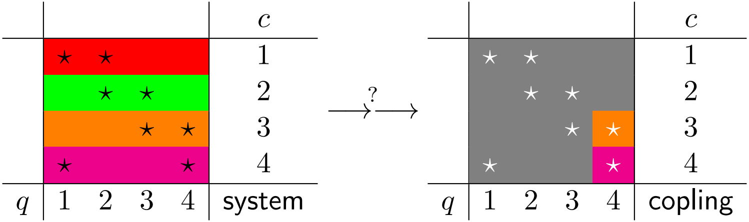

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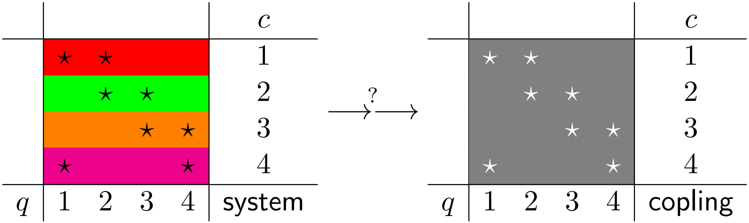

- Given: A system of dichotomous random variables (each random variable is uniquely identified by its content and its context).
- Question: Does it have a coupling which contains the maximal coupling of every pair of content-sharing random variables?
- If it does (does not), the system is noncontextual (resp., contextual).

Intuition behind the notion of a contextual system

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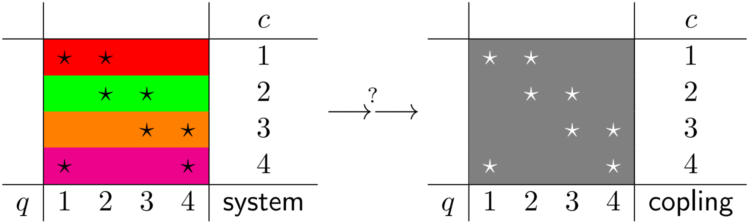

- The maximal coupling of two content-sharing random variables shows how similar they would be in isolation from other random variables in the system.

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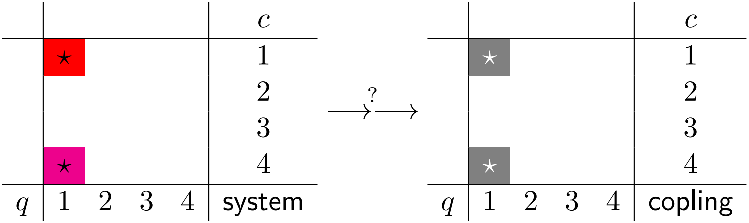

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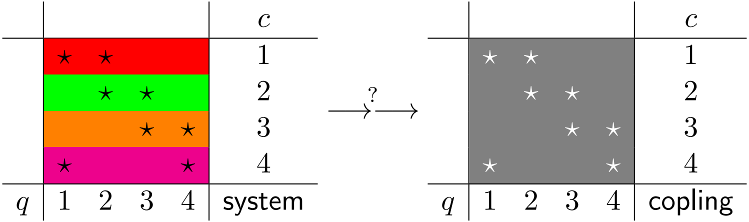

- The maximal coupling of two content-sharing random variables shows how similar they would be in isolation from other random variables in the system.
- In contextual systems, the within-context joint distributions make the content-sharing random variables more dissimilar than they would be in isolation.

Special case: Consistently connected systems and identity couplings

- In QM, traditional interest is in *consistently connected* systems.

	$+$	$-$
A	p	$1-p$
B	q	$1-p$

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	$+$	$-$
A	p	$1-p$
B	q	$1-p$

	$B'=+$	$B'=-$	
$A'=+$	p	0	p
$A'=-$	0	$1-p$	$1-p$
	p	$1-p$	

equivalently, $A'=B'$ with probability 1

Why to deal with inconsistently connected systems?

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- Many reasons, including experimental design biases, "signaling in time" (in Leggett-Garg-type systems), and the fact that in behavioral sciences consistently connected systems seem to never exist.

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$\star$	$\star$			$c_{\circ\circ}$
	$\star$	$\star$		$c_{\times\circ}$
		$\star$	$\star$	$c_{\times\times}$
$\star$			$\star$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

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$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

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		$\star$	$\star$	$c_{\times\times}$
$\star$			$\star$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

Why to deal with inconsistently connected systems?

$\star$	$\star$			$c_{\circ\circ}$	
	$\star$	$\star$		$c_{\times\circ}$	
		$\star$	$\star$	$c_{\times\times}$	
$\star$			$\star$	$c_{\circ\times}$	
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$		

	$R_{\cdot\circ}^{\circ\circ}=+1$	$R_{\cdot\circ}^{\circ\circ}=-1$
$R_{\circ\cdot}^{\circ\circ}=+1$	r	p
$R_{\circ\cdot}^{\circ\circ}=-1$	q	$1-p-q-r$

	$R_{\cdot\circ}^{\times\circ}=+1$	$R_{\cdot\circ}^{\times\circ}=-1$
$R_{\times\cdot}^{\times\circ}=+1$	0	0
$R_{\times\cdot}^{\times\circ}=-1$	q'	$1-q'$

	$R_{\cdot\times}^{\times\times}=+1$	$R_{\cdot\times}^{\times\times}=-1$
$R_{\times\cdot}^{\times\times}=+1$	0	0
$R_{\times\cdot}^{\times\times}=-1$	0	1

	$R_{\cdot\times}^{\circ\times}=+1$	$R_{\cdot\times}^{\circ\times}=-1$
$R_{\circ\cdot}^{\circ\times}=+1$	0	p'
$R_{\circ\cdot}^{\circ\times}=-1$	0	$1-p'$

Why to deal with inconsistently connected systems?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

Counterfactual values?

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Can we assign values to the same-content variables in other contexts?

Counterfactual values?

			★	★	★	$c_{\times\times\times}$
★			★	★		$c_{\circ\times\times}$
	★		★		★	$c_{\times\circ\times}$
		★		★	★	$c_{\times\times\circ}$
☆		☆		☆		$c_{\circ\times\circ}$
★	★		★			$c_{\circ\circ\times}$
	★	★			★	$c_{\times\circ\circ}$
★	★	★				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

Counterfactual values?

$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	
			★	★	★	$c_{\times\times\times}$
★			★	★		$c_{\circ\times\times}$
	★		★		★	$c_{\times\circ\times}$
		★		★	★	$c_{\times\times\circ}$
★		★		★		$c_{\circ\times\circ}$
★	★		★			$c_{\circ\circ\times}$
	★	★			★	$c_{\times\circ\circ}$
★	★	★				$c_{\circ\circ\circ}$

- Equivalently: Can we assign values to all variables in all contexts?

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Equivalently: Can we construct a coupling of the system?

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Equivalently: Can we construct a coupling of the system? Yes, of course (in the absence of constraints).

Counterfactual values?

			★	★	★	$c_{\times\times\times}$
★			★	★		$c_{\circ\times\times}$
	★		★		★	$c_{\times\circ\times}$
		★		★	★	$c_{\times\times\circ}$
☆		☆		☆		$c_{\circ\times\circ}$
★	★		★			$c_{\circ\circ\times}$
	★	★			★	$c_{\times\circ\circ}$
★	★	★				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- A meaningful version: Can we [assign values to / construct a coupling of] the system *subject to certain constraints* imposed on the same-content variables?

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Can we assign values so that any two same-content variables coincide with maximal probability?

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Can we assign values so that any two same-content variables coincide with maximal probability? Only if the system is noncontextual.

Counterfactual values?

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Can we assign values so that any two same-content variables always coincide?

Counterfactual values?

			★	★	★	$c_{\times\times\times}$
★			★	★		$c_{\circ\times\times}$
	★		★		★	$c_{\times\circ\times}$
		★		★	★	$c_{\times\times\circ}$
☆		☆		☆		$c_{\circ\times\circ}$
★	★		★			$c_{\circ\circ\times}$
	★	★			★	$c_{\times\circ\circ}$
★	★	★				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

- Can we assign values so that any two same-content variables always coincide? Only if the system is noncontextual and consistently connected (non-signaling).

Distributional differences and the language of direct (causal) influences

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- That's why they can be considered manifestations of what we call *direct influences*.
- Direct influences are *information-transferring*: by observing that a distribution is different, one can guess that the variable is in another context.
- Difference in the identity of content-sharing random variables due to direct influences (= differences in their distributions) is measured by maximal couplings.

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- The contextual differences in the identity are *non-causal*: nothing causes them by transferring information.
- The identity of a random variable can be defined by the elements of contexts that occur simultaneously with its recording, or even in the future.
  - This may be the main reason why contextuality is not a physical concept.

Contextuality-by-Default: Measures of contextuality

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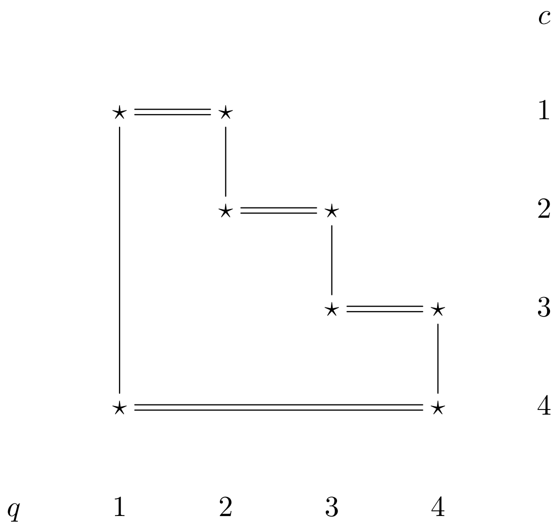

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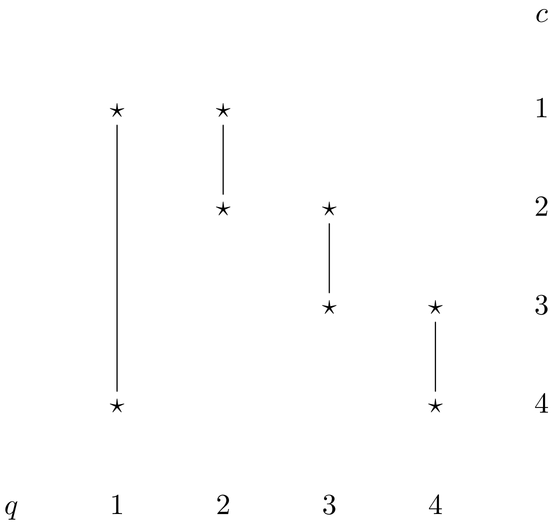

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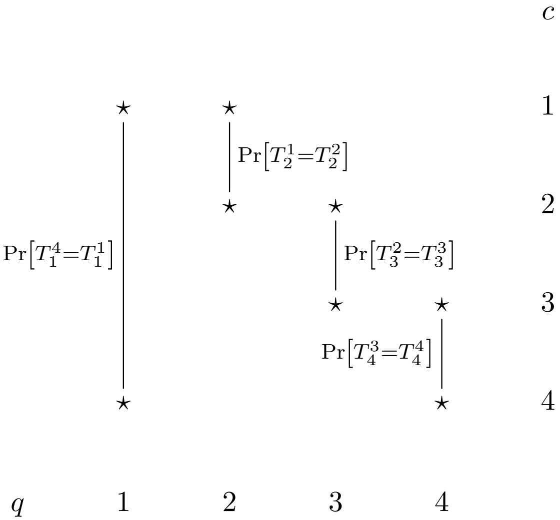

Contextuality-by-Default: Measures of contextuality

$$\Delta_{\max} = \sum \max_{\text{couplings } (T_q^c, T_{q'}^{c'}) \text{ of } R_q^c, R_{q'}^{c'}} \Pr\left[T_q^c = T_{q'}^{c'}\right]$$

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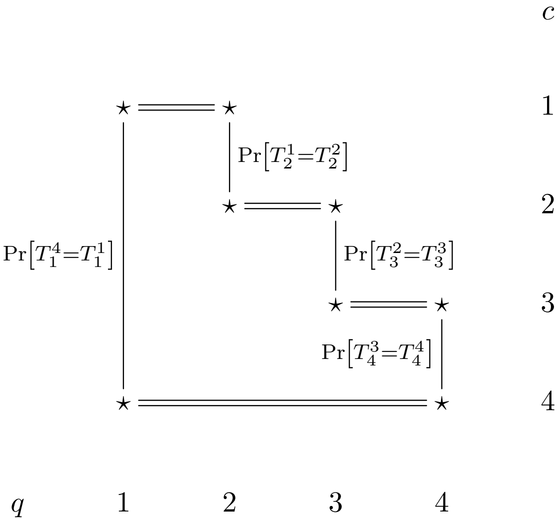

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$$\text{CNTX}_1 = \Delta_{\max} - \Delta_{\text{achievable}}$$

Contextuality-by-Default: Measures of contextuality

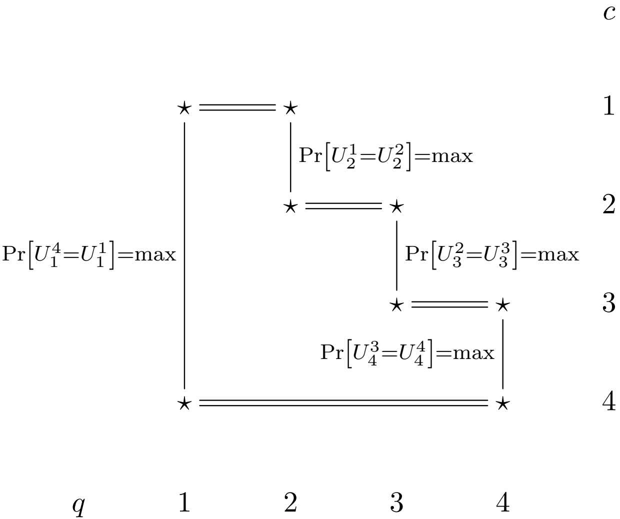
$$\mathsf{CNTX}_1 = \Delta_{\max} - \Delta_{\mathsf{achievable}}$$

Theorem (*Kujala-Dzhafarov, 2016*)

In a cyclic system of dichotomous random variables, CNTX_1 *equals*

$$\frac{1}{2}\max_{(\iota_1,\ldots,\iota_k)\in\{-1,1\}^n:\prod_{i=1}^{n}\iota_i=-1}\sum_{i=1}^{n}\iota_i\left\langle R_i^i R_{i\oplus 1}^i\right\rangle - n + 2 - \sum_{i=1}^{n}\left|\left\langle R_i^i\right\rangle - \left\langle R_i^{i\ominus 1}\right\rangle\right|.$$

Contextuality-by-Default: Measures of contextuality



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$$\mathsf{CNTX}_2 = \frac{\text{minimal total variation of a quasi-distribution}}{\text{over all quasi-couplings of the system}} - 1$$

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- Relationship between CNTX_2 and CNTX_2 is not well-understood:
  - they are conjectured to be proportional for all cyclic systems: $\text{CNTX}_2 = \text{CNTX}_1/(\text{rank-1})$.
    - (conjectured by Cervantes and Kujala based on numerical calculations)

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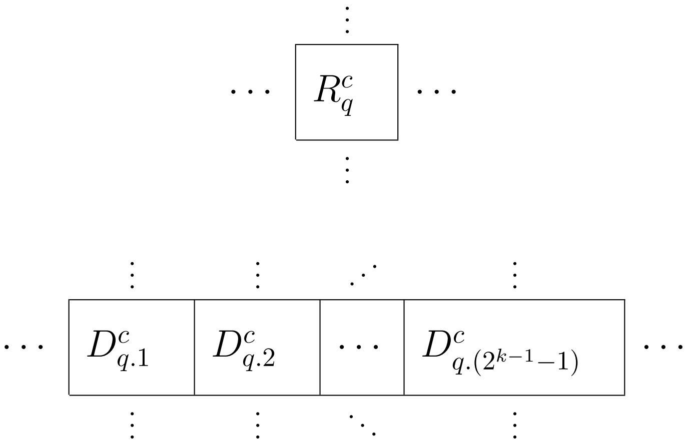

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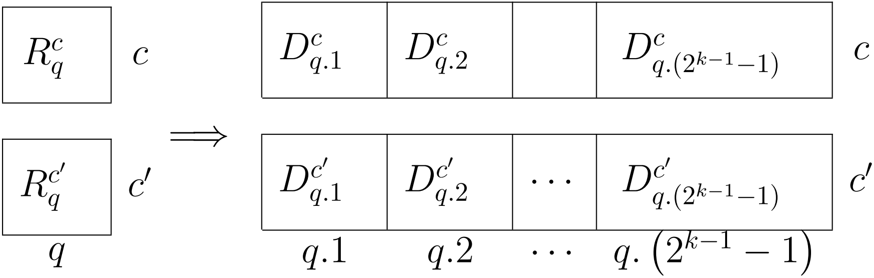

Contextuality-by-Default: Canonical systems

Theorem (*Dzhafarov-Cervantes-Kujala, 2017*)

A canonical system is noncontextual only if for every pair of (original) content-sharing R_q^c *and* $R_q^{c'}$ *one of them nominally dominates the other.*

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Definition

R_q^c *nominally dominates* $R_q^{c'}$ if $\Pr\left[R_q^c = i\right] < \Pr\left[R_q^{c'} = i\right]$ for no more than one value of $i = 1, \ldots, k$.

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Example

$\updownarrow$	1	2	3	4	5
R_q^c	0.1	0.2	0.2	0.5	0
$R_q^{c'}$	0.1	0.2	0.2	0.5	0

,

$\updownarrow$	1	2	3	4	5
R_q^c	0.1	0.2	0.2	0.5	0
$R_q^{c'}$	0.2	0.1	0.2	0.5	0

,

$\downarrow$	1	2	3	4	5
R_q^c	0.1	0.2	0.2	0.5	0
$R_q^{c'}$	0.5	0	0.1	0.4	0

,

$\times$	1	2	3	4	5
R_q^c	0.1	0.2	0.2	0.5	0
$R_q^{c'}$	0.3	0.3	0.1	0.2	0.1