The sheaf-theoretic description of contextuality
Part II: contextuality and valuation algebras

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Introduction

The high level of generality of the sheaf theoretic description of contextuality led to unexpected connections with fields unrelated to quantum mechanics: contextual behaviour has been observed in

- Relational databases (Abramsky 2013)
- Constraint satisfaction problems (Abramsky, Gottlob, Kolaitis 2013)
- Logic (Abramsky, Barbosa, Kishida, Lal, Mansfield 2015)

This leads to the idea of developing a contextual semantics, an all-comprehensive theory which captures the essence of all such contextual phenomena.

All the different instances of contextuality share a common trait: they concern pieces of information, which agree locally, but disagree globally.
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Valuation algebras

Valuation algebras are a general framework to model concepts such as information and knowledge.

Definition

Let $V$ be a set of variables. A valuation algebra over $V$ is a set $\Phi$ equipped with three operations:

1. Labelling: $\Phi \rightarrow \mathcal{P}(V) :: \phi \mapsto d(\phi)$

2. Combination: $\Phi \times \Phi \rightarrow \Phi :: (\phi, \psi) \mapsto \phi \otimes \psi$

3. Projection: $\Phi \times \mathcal{P}(V) \rightarrow \Phi :: (\phi, S) \mapsto \phi_{\downarrow S}$, for all $S \subseteq d(\phi)$, such that axioms (A1)–(A6) are satisfied:

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(A1) Commutative Semigroup: 
\((\Phi, \otimes)\) is associative and commutative.

(A2) Labelling: For all \(\phi, \psi \in \Phi\), 
\[d(\phi \otimes \psi) = d(\phi) \cup d(\psi)\]

(A3) Projection: Given \(\phi \in \Phi\) and \(S \subseteq d(\phi)\), 
\[d(\phi \downarrow S) = S\]

(A4) Transitivity: Given \(\phi \in \Phi\) and \(S \subseteq T \subseteq d(\phi)\), 
\[d(\phi \downarrow T) \downarrow S = d(\phi \downarrow S)\]

(A5) Combination: For \(\phi, \psi \in \Phi\), with \(d(\phi) = S, d(\psi) = T\) and \(U \subseteq V\) such that \(S \subseteq U \subseteq S \cup T\), 
\[d(\phi \otimes \psi) \downarrow U = d(\phi \otimes \psi) \downarrow U \cap T\]

(A6) Domain: Given \(\phi \in \Phi\), 
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The elements of $\Phi$ are called valuations. A set of valuations is called a knowledgebase. A set of variables $D \subseteq V$ is called a domain. The domain of a valuation $\varphi \in \Phi$ is the set $d(\varphi) = \{x_1, \ldots, x_n\} \subseteq V$, which constitutes the domain of $\varphi$. For any finite set of variables $S \subseteq V$, we denote by $\Phi_S = \{\varphi \in \Phi | d(\varphi) = S\}$ the set of valuations with domain $S$. Thus, we have $\Phi = \bigcup S \subseteq V \Phi_S$. 

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Intuitively, a valuation $\phi \in \Phi$ represents information about the possible values of a finite set of variables $d(\phi) = \{x_1, \ldots, x_n\} \subseteq V$, which constitutes the domain of $\phi$. 
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The elements of $\Phi$ are called \textbf{valuations}. A set of valuations is called a \textbf{knowledgebase}. A set of variables $D \subseteq V$ is called a \textbf{domain}. The \textbf{domain of a valuation} $\phi$ is the set $d(\phi)$.

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the set of valuations with domain $S$. Thus, we have

$$\Phi = \bigcup_{S \subseteq V} \Phi_S.$$
Information algebras

It is often desirable to add additional postulates, which collectively give rise to the notion of information algebra.

**Definition**

Let $\Phi$ be a valuation algebra on $V$. We say that $\Phi$ has neutral elements if it satisfies

$$(A7) \text{ Commutative monoid: For each } S \subseteq V, \text{ there exists a neutral element } e_S \in \Phi_S \text{ such that } \phi \otimes e_S = e_S \otimes \phi = \phi \text{ for all } \phi \in \Phi_S.$$ 

Such neutral elements must satisfy the following identity:

$$e_S \otimes e_T = e_{S \cup T}$$

for all subsets $S, T \subseteq V$. 

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(A8) Nullity: For each $S \subseteq V$ there exists a null element $z_S \in \Phi_S$ such that $\phi \otimes z_S = z_S \otimes \phi = z_S$.

Moreover, for all $S, T \subseteq V$ such that $S \subseteq T$, we have, for each $\phi \in \Phi_T$,

$\phi \downarrow S \iff \phi = z_T$.

We say that $\Phi$ is idempotent if it satisfies

(A9) Idempotency: For all $\phi \in \Phi$ and $S \subseteq d(\phi)$, it holds that $\phi \otimes \phi \downarrow S = \phi$.

If $\Phi$ satisfies axioms (A7)–(A9) it is called an information algebra.
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Frames and tuples

For each variable \( x \in V \), we denote by \( \Omega_x \) its frame, i.e. the set of possible values for \( x \).

A tuple with finite domain \( S \subseteq V \) is an element \( x \) of \( \Omega_S = \prod_{x \in S} \Omega_x \).

We will denote by \( x \downarrow T \) the cartesian projection of a tuple \( x \in \Omega_S \) to \( \Omega_T \), where \( T \subseteq S \).
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  - **Labelling**: Given $\phi : \Omega_S \rightarrow R$, define $d(\phi) := S$. 

The algebra has neutral elements and null elements, but it is idempotent only if $R = B$. 

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- The algebra has neutral elements and null elements, but it is idempotent only if \( R = \mathbb{B} \).
Examples

Relational databases

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<tbody>
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Each column is labelled by an attribute. Each entry of the table is a tuple specifying a value for each of the attributes. The full table is simply a set of tuples, i.e. a relation. The set of attributes of a relation \( R \) is called its schema, denoted \( \text{schema}(R) \). A database instance is a family of relations.
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Relational databases

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A database instance is a family of relations.
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Information algebra of relational databases

Define a valuation algebra $\Phi$ such that...

Let $S \subseteq V$.

The neutral element is $e_S$.

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The algebra of relational databases can be generalised by elevating the concept of tuple to a higher level:

**Definition**

A tuple system over $P(V)$, where $V$ is a set of variables, is a set $T$ equipped with two operations $d: T \to P(V)$ and $\downarrow: T \times P(V) \to T$ satisfying the following axioms:

1. (T1) If $Q \subseteq d(t)$, then $d(t \downarrow Q) = Q$.
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3. (T3) If $d(t) = Q$, then $t \downarrow Q = t$.
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General information sets
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Examples
General information sets

- Cartesian tuples (cartesian projection) $\Rightarrow$ relational databases
- Probability distributions (marginalisation) $\Rightarrow$ probability distribution information sets
- Propositional truth valuations $v: L \rightarrow \{0, 1\}$ (function restriction) $\Rightarrow$ propositional information sets

More generally, given any logical 'context' $\langle L, M, | = \rangle$, one can define both an algebra of information sets, and an algebra of formulae, e.g. $\rightarrow$-Predicate logic $\rightarrow$-Linear equations $\rightarrow$-Constraint satisfaction problems $\rightarrow$...
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Disagreement

Disagreement between sources is a fundamental aspect of information. Despite this, there is no general definition of disagreement in the valuation algebraic approach, which focuses more on the problem of extracting information (more on that later).

We propose a natural formulation:

Consider a valuation algebra $\Phi$ on a set of variables $V$, let $K = \{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ be a knowledgebase, with $D = \bigcup_{i=1}^{n} d(\phi_i)$.

To say that the information sources agree is equivalent to say that there is a truth which is agreed upon by all the sources. The truth valuation gives information about all the variables appearing in $K$, while each $\phi_i$ only concerns a set of the variables $d(\phi_i) \subseteq D$.

Definition: We say that $\phi_1, \ldots, \phi_n$ agree (or agree globally) if there exists a (global) truth valuation $\gamma \in \Phi^D$ such that, for all $1 \leq i \leq n$, $\gamma \downarrow d(\phi_i) = \phi_i$. 

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$$D := \bigcup_{i=1}^{n} d(\phi_i).$$

To say that the information sources agree is equivalent to say that there is a truth which is agreed upon by all the sources.
Disagreement

- **Disagreement** between sources is a fundamental aspect of information.
- Despite this, there is no general definition of disagreement in the valuation algebraic approach, which focuses more on the problem of *extracting information* (more on that later).
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  To say that the information sources agree is equivalent to say that there is a *truth* which is agreed upon by all the sources.
- The truth valuation gives information about all the variables appearing in $K$, while each $\phi_i$ only concerns a set of the variables $d(\phi_i) \subseteq D$.

**Definition**

We say that $\phi_1, \ldots, \phi_n$ agree (or agree globally) if there exists a (global) truth valuation $\gamma \in \Phi_D$ such that, for all $1 \leq i \leq n$,

\[
\gamma_{\upharpoonright d(\phi_i)} = \phi_i.
\]
Local disagreement

A necessary condition for global agreement is that each pair of information sources agree at their common variables. This property is captured by the notion of local agreement:

Definition

Let $K = \{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ be a knowledgebase. We say that $K$ agrees locally if $\downarrow_d(\phi_i) \cap \downarrow_d(\phi_j) \subseteq \downarrow_d(\phi_i) \cup \downarrow_d(\phi_j)$ for all $1 \leq i, j \leq n$.

Clearly, agreement implies local agreement.
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- Clearly, agreement implies local agreement.
Valuations algebras and sheaf theory

Remarkably, many properties of valuations algebras can be described by sheaf theory. Let $\Phi$ be a valuation algebra on a set of variables $V$. Consider the powerset $\mathcal{P}(V)$ as a discrete topological space. We can describe $\Phi$ as a presheaf:

$$
\Phi : \mathcal{P}(V)^{\text{op}} \to \text{Set}
$$

by letting $\Phi(S) = \Phi_S$ for all $S \subseteq V$ and $\Phi(S \subseteq T) : \Phi_T \to \Phi_S$ :: $\phi \mapsto \phi \downarrow T$.

Functoriality is guaranteed by axioms (A4) and (A6), indeed, for all $S \subseteq V$ and for all $\phi \in \Phi_S$, $\rho_S \rho_S(\phi) = \phi \downarrow S = \phi \downarrow \text{d}(\phi)$ (A6) = $\phi$, and, by (A4), for all $S \subseteq T \subseteq U \subseteq V$ and $\phi \in \Phi_U$, $\rho_T \rho_U(\phi) = (\phi \downarrow T) \downarrow S$ (A4) = $\phi \downarrow S = \rho_U \rho_S(\phi)$.

In general, this description does not capture composition.
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Disagreement and sheaf theory

We can rephrase the definitions of local and global disagreement in sheaf theoretic terms:

Definition

Let \( U \subseteq P(V) \), with \( D = \bigcup U \).

A set of local sections \( \{ s_U \in \Phi(U) \} \) \( U \in U \) of \( \Phi \) agrees locally if it is compatible.

It agrees globally if there exists a global section \( \gamma \in \Phi(D) \) such that \( \gamma|_U = s_U \) for all \( U \in U \).
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  for all $U \in \mathcal{U}$.
Recall that an empirical model on a scenario $\langle X, M, (O_m)_{m \in X} \rangle$ is a compatible family $e = \{ e_C \}_{C \in M}$ for the presheaf $D_{R(E)}$. This can be seen as a locally agreeing knowledgebase of the valuation algebra of $R$-distributions. We say that $e$ is non-contextual if there exists a global $R$-distribution $g \in D_{R(E)}(X)$ such that $g|_C = e_C$, for all $C \in M$. Therefore, contextuality simply arises as an instance of a locally agreeing knowledgebase that disagrees globally. This is a very general concept, which has meaningful realisations in many fields captured by the valuation algebraic framework.
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Local agreement vs global disagreement: Examples

A first example in relational databases, taken from real sources. Breast cancer guidelines from 3 different medical associations:

- Screening with mammography annually, clinical breast exam annually or biannually
- Women aged 50 to 54 years should get mammograms. Women aged 55 years and older should switch to clinical breast exams
- Women aged 50 to 54 years should undergo an exam every year. Women aged 55 years and older should be examined every 2 years

<table>
<thead>
<tr>
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<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>CBE</td>
</tr>
<tr>
<td>Y</td>
<td>2Y</td>
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Samson Abramsky & Giovanni Carù (Oxford CS)
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For an example in CSPs, consider a graph colorability problem:

Consider the problem of coloring a political map with 3 colors. We focus on the geographical region surrounding Malawi:

Zambia
Mozambique
Tanzania
Malawi
Zimbabwe

One can easily show that it is impossible to color the map using only 3 colors. This can be seen as an instance of local agreement (LA) vs global disagreement (GD) both for the algebra of CSP-information sets, and the algebra of CSP-formulae.
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For an example from logic, consider the liar's cycle $S_1$:

$S_2$: $S_3$ is true,

$S_3$: $S_4$ is true,

$\ldots$

$S_{n-1}$: $S_n$ is true,

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Local agreement vs global disagreement: Examples

Finally, an example concerning linear equations. Consider the following system of equations in $\mathbb{Z}_2$:

\begin{align*}
e_1 & : (x_1 \oplus x_2 \oplus x_3) = 1 \\
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\end{align*}

The equations are locally consistent (i.e. every pair of equations admit solutions for their common variables), yet if we sum them all we obtain $0 = 1$, which means that there is no global solution, i.e. the knowledgebase $\{e_1, e_2, e_3, e_4\}$ disagrees globally.

These equations are exactly those used by Mermin to prove strong contextuality of the GHZ model.
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## A (new!) dictionary

<table>
<thead>
<tr>
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<th>Empirical models</th>
</tr>
</thead>
<tbody>
<tr>
<td>variables</td>
<td>measurements</td>
</tr>
<tr>
<td>frame $\Omega_x$</td>
<td>outcome set $O_x$</td>
</tr>
<tr>
<td>knowledgebase domains</td>
<td>measurement scenario</td>
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<td>domain of valuation</td>
<td>context</td>
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<tr>
<td>tuple</td>
<td>event</td>
</tr>
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<td>local agreement</td>
<td>no-signalling</td>
</tr>
<tr>
<td>locally-agreeing knowledgebase</td>
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</tr>
<tr>
<td>projection</td>
<td>restriction (marginalisation)</td>
</tr>
<tr>
<td>combination</td>
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</tr>
<tr>
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Samson Abramsky & Giovanni Carù (Oxford CS) | Contextuality and valuation algebras | Winer Memorial Lectures 2018
Translating results...

The valuation algebraic definition of disagreement allows us to translate definitions, methods, results and algorithms from one field to the other.

Example

A key result in contextuality is the characterisation of all scenarios that do not admit contextual behavior. The following result was proven by Barbosa, via an adaptation of Vorob'ev's theorem:

\[ \text{Every empirical model on a scenario } \langle X, M, O \rangle \text{ is non-contextual iff the simplicial complex described by } M \text{ is acyclic.} \]

This theorem can be generalised to the level of valuation algebras:

**Theorem**

Every locally agreeing knowledgebase \( \{ \phi_1, \ldots, \phi_n \} \) on a set of domains \( D = \{ d_1, \ldots, d_n \} \) agrees globally iff the simplicial complex described by \( D \) is acyclic.
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*Every locally agreeing knowledgebase $\{\phi_1, \ldots, \phi_n\}$ on a set of domains $D := \{d_1, \ldots, d_n\}$ agrees globally iff the simplicial complex described by $D$ is acyclic.*
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- This theorem then specialises to results for each specific valuation algebra:
  - **Probability distributions**: Vorob’ev’s theorem.
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  - **CSPs – graph colorability**: Every tree is \( k \)-colorable, for any \( k \geq 2 \).
Theorem

*Every locally agreeing knowledgebase \( \{\phi_1, \ldots, \phi_n\} \) on a set of domains \( \mathcal{D} := \{d_1, \ldots, d_n\} \) agrees globally iff the simplicial complex described by \( \mathcal{D} \) is acyclic.*

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Translating results...

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- More generally speaking, we would like to apply the wide range of methods and algorithms of the valuation algebraic framework to study contextuality.
Inference problems

The classic problem of extracting relevant knowledge about a given query out of a certain set of information sources can be effectively modelled in the valuation algebraic framework:

Definition
Given a valuation algebra $\Phi$, a knowledgebase $\{\phi_1, \ldots, \phi_n\} \subseteq \Phi$, and a set of domains $x = \{x_1, \ldots, x_k\}$, with $x_i \subseteq d(\phi_1 \otimes \cdots \otimes \phi_n)$, we call an inference problem the task of computing $(\phi_1 \otimes \cdots \otimes \phi_n) \downarrow x_i$.

The valuation $\phi = (\phi_1 \otimes \cdots \otimes \phi_n)$ is called joint valuation or objective function, while each domain $x_i$ is called a query.

There is a vast class of algorithms designed to solve inference problems efficiently.

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Given a valuation algebra $\Phi$ on a set of variables $V$, and a valuation $\phi \in \Phi$ for some $S \subseteq V$, one could ask whether it is possible to quantify the amount of information carried by $\phi$ and compare it to other valuations of $\Phi_S$.

This idea is captured by the notion of ordered valuation algebras

Definition

Let $\Phi$ be a valuation algebra on $V$. We say that $\Phi$ is ordered if there exists a partial order $\preceq$ on $\Phi$ such that:

(A10) Partial order: For all $\phi, \psi \in \Phi$, $\phi \preceq \psi$ implies $d(\phi) = d(\psi)$.

Moreover, for every $S \subseteq V$ and $\Psi \subseteq \Phi_S$, the infimum $\inf(\Psi)$ exists.

(A11) Null element: For all $S \subseteq V$, we have $\inf(\Phi_S) = z_S$.

(A12) Monotonicity of combination: For all $\phi_1, \phi_2, \psi_1, \psi_2 \in \Phi$ such that $\phi_1 \preceq \phi_2$ and $\psi_1 \preceq \psi_2$, we have $\phi_1 \otimes \psi_1 \preceq \phi_2 \otimes \psi_2$.

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Most valuations algebras have an order structure: relational databases, propositional information, algebra of sentences, any algebra related to general notions of language and models, etc. (not probability distributions!)

All of these algebras have a key common property. Their composition operation is described by the same categorical construction:

\[ \Phi(S) \otimes \Phi(T) = \Phi(S \cup T) \]

**Proposition**

Let \( \Phi \) be an ordered algebra in the list above. The composition law \( \otimes: \Phi(S) \times \Phi(T) \to \Phi(S \cup T) \) is uniquely characterised as the right adjoint of \( \langle \rho_{S \cup T}, \rho_{S \cup T} \rangle \). In other words, it is the unique map such that

\[ \text{id}_{\Phi(S \cup T)} \leq \otimes \circ \langle \rho_{S \cup T}, \rho_{S \cup T} \rangle, \langle \rho_{S \cup T}, \rho_{S \cup T} \rangle \circ \otimes \leq \text{id}_{\Phi(S) \times \Phi(T)}, \]

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Disagreement as an inference problem

We call algebras with this structure lossy. They have the following key property:

**Proposition**

Let $\Phi$ be a lossy valuation algebra on a set of variables $V$. Let $K = \{\phi_1, \ldots, \phi_n\} \subseteq \Phi$ be a knowledgebase. Let $\gamma = \bigotimes_{i=1}^{n} \phi_i$.

Then $\phi_1, \ldots, \phi_n$ agree globally if and only if $\gamma \downarrow_d (\phi_i) = \phi_i$. In this case, $\gamma$ is the most informative of all the possible truth valuations.

In other words, a truth valuation can only be obtained by combining all of the valuations in a knowledgebase. Consequently, in order to determine whether a knowledgebase $\{\phi_1, \ldots, \phi_n\}$ disagrees globally, all we have to do is to solve the inference problem $(\phi_1 \otimes \cdots \otimes \phi_n) \downarrow_d (\phi_i)$ for all $1 \leq i \leq n$. 
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Strong disagreement

Let $\Phi$ be a lossy information algebra. By the previous proposition, we know that $\gamma = \bigotimes_{i=1}^{n} \phi_i$ is the most informative of all the possible candidate truth valuations. Even when the knowledgebase disagrees, $\gamma$ represents the portion of the information on which the sources do agree. In extreme cases, it can happen that the information sources disagree completely. In this case, the truth valuation is the least informative valuation, i.e. the null element of the information algebra.

Definition
We say that a knowledgebase $\{\phi_1, \ldots, \phi_n\}$, with $D = \bigcup_{i=1}^{n} d(\phi_i)$, disagrees strongly if $\gamma = z_D$.

Strong contextuality is an instance of strong disagreement, where the information algebra in question is the one of boolean distributions.

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The valuation algebraic framework provides a wide range of algorithms to solve inference problems. In particular, the paradigm of local computation has proved particularly useful to efficiently solve single query inference problems:

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