The sheaf-theoretic description of contextuality
Part II: contextuality and valuation algebras

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Introduction

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- This leads to the idea of developing a **contextual semantics**, an all-comprehensive theory which captures the essence of all such contextual phenomena.
- All the different instances of contextuality share a common trait: they concern pieces of **information**, which agree locally, but disagree globally.

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3. Projection: $\Phi \times \mathscr{P}(V) \to \Phi :: (\phi, S) \mapsto \phi^{\downarrow S}$, for all $S \subseteq d(\phi)$,

such that axioms (A1)–(A6) are satisfied:

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(A6) **Domain**: Given $\phi \in \Phi$,

$$\phi^{\downarrow d(\phi)} = \phi$$

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Intuitively, a valuation $\phi \in \Phi$ represents information about the possible values of a finite set of variables $d(\phi) = \{x_1, \ldots x_n\} \subseteq V$, which constitutes the domain of ϕ. For any finite set of variables $S \subseteq V$, we denote by

$$\Phi_S := \{\phi \in \Phi \mid d(\phi) = S\}$$

the set of valuations with domain S. Thus, we have

$$\Phi = \bigcup_{S \subseteq V} \Phi_S.$$

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(A7) *Commutative monoid*: For each $S \subseteq V$, there exists a *neutral element* $e_S \in \Phi_S$ such that

$$\phi \otimes e_S = e_S \otimes \phi = \phi$$

for all $\phi \in \Phi_S$. Such neutral elements must satisfy the following identity:

$$e_S \otimes e_T = e_{S \cup T}$$

for all subsets $S, T \subseteq V$.

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(A8) *Nullity*: For each $S \subseteq V$ there exists a *null element* $z_S \in \Phi_S$ such that

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Moreover, for all $S, T \subseteq V$ such that $S \subseteq T$, we have, for each $\phi \in \Phi_T$,

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If Φ satisfies axioms (A7)–(A9) it is called an **information algebra**.

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- We will denote by $\mathbf{x}_{\downarrow T}$ the cartesian projection of a tuple $\mathbf{x} \in \Omega_S$ to Ω_T, where $T \subseteq S$.

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- The algebra has neutral elements and null elements, but it is idempotent only if $R = \mathbb{B}$.

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- Consider the following data table:

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- A **database instance** is a family of relations.

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A **tuple system** over $\mathscr{P}(V)$, where V is a set of variables, is a set T equipped with two operations $d : T \to \mathscr{P}(V)$ and $\downarrow : T \times \mathscr{P}(V) \to T$ satisfying the following axioms:

(T1) If $Q \subseteq d(\mathbf{t})$, then $d(\mathbf{t}_{\downarrow Q}) = Q$.

(T2) If $Q \subseteq U \subseteq d(\mathbf{t})$, then $\left(\mathbf{t}_{\downarrow U}\right)_{\downarrow Q} = \mathbf{t}_{\downarrow Q}$.

(T3) If $d(\mathbf{t}) = Q$, then $\mathbf{t}_{\downarrow Q} = \mathbf{t}$.

(T4) For $d(\mathbf{t}) = Q$, $d(\mathbf{u}) = U$ such that $\mathbf{t}_{\downarrow Q \cap U} = \mathbf{u}_{\downarrow Q \cap U}$, there exists $\mathbf{g} \in T$ such that $d(\mathbf{g}) = Q \cup U$, $\mathbf{g}_{\downarrow Q} = t$ and $\mathbf{g}_{\downarrow U} = \mathbf{u}$.

(T5) For $d(\mathbf{t}) = Q$ and $Q \subseteq U$, there exists $\mathbf{g} \in T$ such that $d(\mathbf{g}) = U$ and $\mathbf{g}_{\downarrow Q} = \mathbf{t}$.

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$$S_1 \otimes S_2 := S_1 \bowtie S_2 = \{\mathbf{t} \in T_{Q \cup U} \mid \mathbf{t}_{\downarrow S} \in S_1 \ \wedge \ \mathbf{t}_{\downarrow U} \in S_2\},$$

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- The truth valuation gives information about **all** the variables appearing in K, while each ϕ_i only concerns a set of the variables $d(\phi_i) \subseteq D$.

Definition

We say that $\phi_1, \dots, \phi_n$ **agree** (or **agree globally**) if there exists a (global) **truth valuation** $\gamma \in \Phi_D$ such that, for all $1 \leq i \leq n$,

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- Clearly, agreement implies local agreement.

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$$\Phi : \mathscr{P}(V)^{op} \longrightarrow \mathbf{Set}$$

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- In general, this description does not capture composition.

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- Recall that an **empirical model** on a scenario $\langle X, \mathscr{M}, (O_m)_{m\in X}\rangle$ is a **compatible family** $e = \{e_C\}_{C\in\mathscr{M}}$ for the presheaf $\mathscr{D}_R\mathscr{E}$.

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- Therefore, contextuality simply arises as an instance of **a locally agreeing knowledgebase that disagrees globally**.
- This is a very general concept, which has meaningful realisations in many fields captured by the valuation algebraic framework.

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- A first example in relational databases, taken from real sources. Breast cancer guidelines from 3 different medical associations:
  - *Screening with mammography annually, clinical breast exam annually or biannually*
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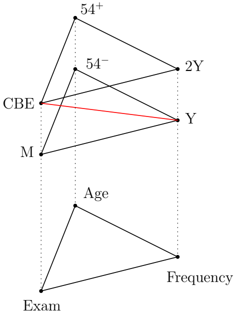

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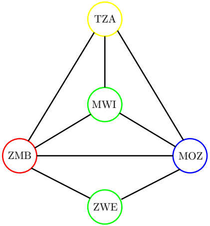

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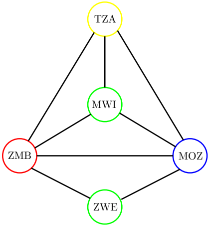

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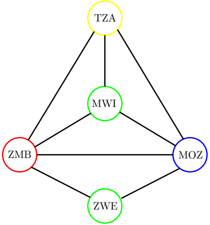

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- This can be seen as an instance of local agreement (LA) vs global disagreement (GD) both for the algebra of CSP-information sets, and the algebra of CSP-formulae.

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- Also in this case, this is an instance of LA vs GD both for the algebra of propositional information sets, and the algebra of propositional formulae.

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- The equations are locally consistent (i.e. every pair of equations admit solutions for their common variables), yet if we sum them all we obtain $0 = 1$, which means that there is no global solution, i.e. the knowledgebase $\{e_{1,2,3,4}\}$ disagrees globally.

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- These equations are exactly those used by Mermin to prove strong contextuality of the GHZ model.

A (new!) dictionary

Valuation algebras	**Empirical models**
variables	measurements
frame Ω_x	outcome set O_x
knowledgebase domains	measurement scenario
domain of valuation	context
tuple	event
local agreement	no-signalling
locally-agreeing knowledgebase	empirical model
projection	restriction (marginalisation)
combination	glueing
truth valuation	global section
disagreement	contextuality

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- This theorem can be generalised to the level of valuation algebras:

Theorem

Every locally agreeing knowledgebase $\{\phi_1, \ldots, \phi_n\}$ on a set of domains $\mathscr{D} := \{d_1, \ldots, d_n\}$ agrees globally iff the simplicial complex described by $\mathscr{D}$ is acyclic.

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- More generally speaking, we would like to apply the wide range of methods and algorithms of the valuation algebraic framework to study contextuality.

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- There is a vast class of **algorithms** designed to solve inference problems efficiently.
- Can we turn the problem of detecting disagreement in a inference problem?

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- Given a valuation algebra Φ on a set of variables V, and a valuation $\phi \in \Phi_S$ for some $S \subseteq V$, one could ask whether it is possible to **quantify** the amount of information carried by ϕ and compare it to other valuations of Φ_S.

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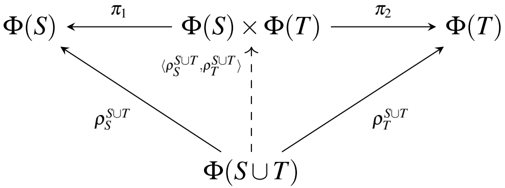

Proposition

Let Φ be an ordered algebra in the list above.
The composition law $\otimes : \Phi(S) \times \Phi(T) \to \Phi(S \cup T)$ is uniquely characterised as the right adjoint of $\langle \rho_S^{S\cup T}, \rho_T^{S\cup T} \rangle$. In other words, it is the unique map such that

$$\mathsf{id}_{\Phi(S\cup T)} \leq \otimes \circ \langle \rho_S^{S\cup T}, \rho_T^{S\cup T} \rangle, \qquad \langle \rho_S^{S\cup T}, \rho_T^{S\cup T} \rangle \circ \otimes \leq \mathsf{id}_{\Phi(S)\times\Phi(T)},$$

where $\leq$ is the pointwise order inherited by the partial order $\preceq$.

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Let Φ be a lossy valuation algebra on a set of variables V. Let $K = \{\phi_1, \dots, \phi_n\} \subseteq \Phi$ be a knowledgebase. Let

$$\gamma = \bigotimes_{i=1}^{n} \phi_i. \tag{1}$$

Then $\phi_1, \dots, \phi_n$ agree globally if and only if $\gamma^{\downarrow d(\phi_i)} = \phi_i$. In this case, γ is the most informative of all the possible truth valuations.

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- In other words, a truth valuation can only be obtained by combining all of the valuations in a knowledgebase.
- Consequently, in order to determine whether a knowledgebase $\{\phi_1, \dots \phi_n\}$ disagrees globally, all we have to do is to solve the inference problem

$$(\phi_1 \otimes \cdots \otimes \phi_n)^{\downarrow d(\phi_i)}$$

for all $1 \leq i \leq n$.

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- **Strong contextuality** is an instance of strong disagreement, where the information algebra in question is the one of **boolean distributions**.

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- One can use these efficient algorithm to detect contextuality in measurement scenarios.
- In particular, faster algorithms for non-locality detection can be implemented in specific scenarios.