Introduction to the sheaf-theoretic approach to contextuality

Samson Abramsky

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  - graph-theoretic (Cabello, Severini, Winter),
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- See recent exposition of some of this by Marcelo Terra Cunha and Barbara Amaral
- Comparison with other approaches, e.g. the CSW graph-theoretic approach: both have useful features, the "sheaf" approach exposes some additional mathematical structure, which plays a crucial role in gaining a wider perspective on contextuality

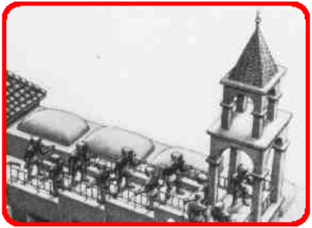
What is Contextuality?

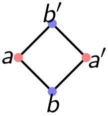
What, then, is the essence of contextuality? In broad terms, we propose to describe it as follows:

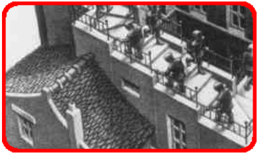
Contextuality arises where we have a family of data which is

locally consistent, but **globally inconsistent**

Contextuality Analogy: Local Consistency







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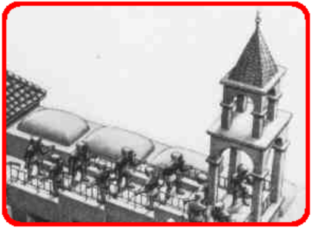
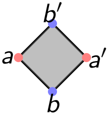
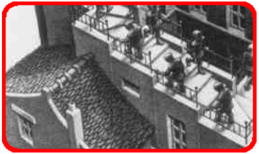
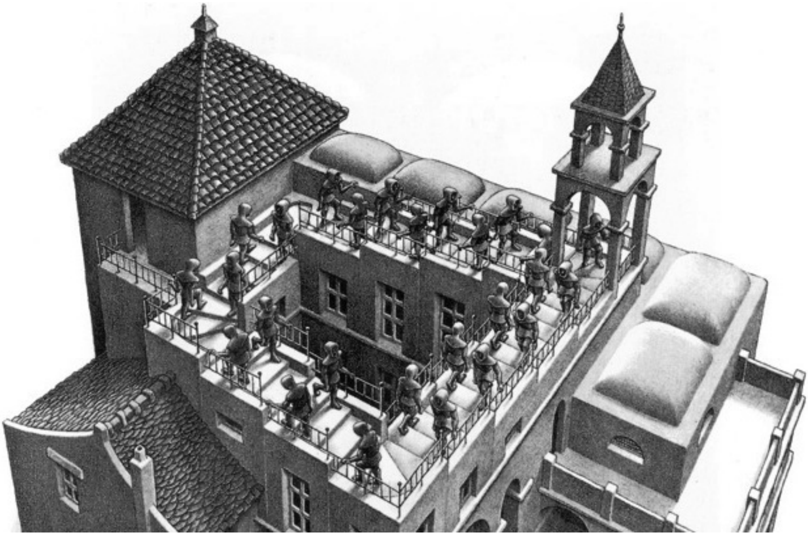






Contextuality Analogy: Global Inconsistency



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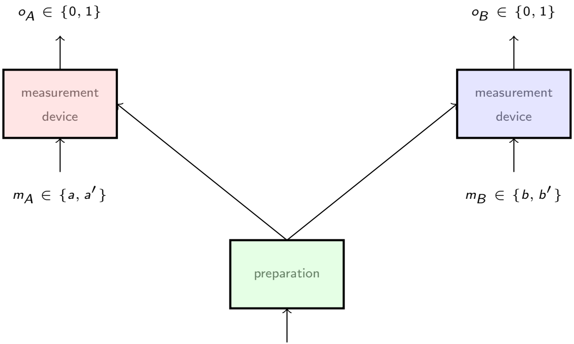

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In the latter case, we have an **abstract simplicial complex**.

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B	E	I	K	E	K	Q	R	R
C	F	C	G	M	N	D	F	M
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The original K-S construction used 117 variables!

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E.g. $s|_{\{a\}} = \{a \mapsto 0\}$.

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(The coherent relationship is functoriality!)

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Compatibility is a general form of the important physical principle of **No-Signalling**; this general form is also known as **No Disturbance**.

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The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

Bundle Pictures

Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

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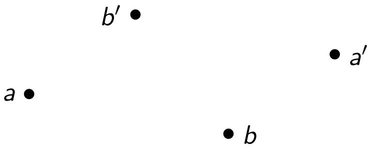

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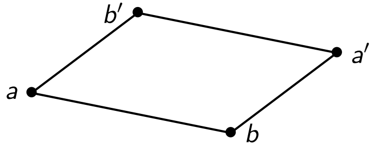

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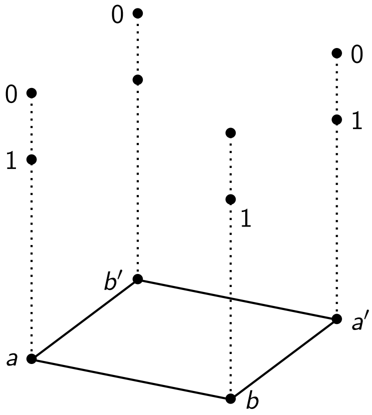

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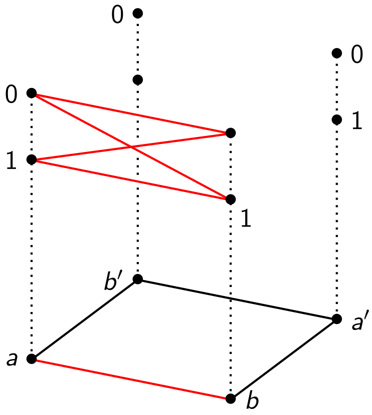

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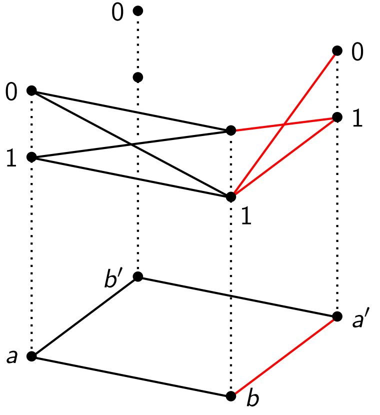

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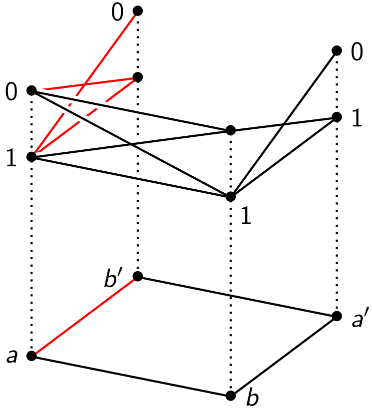

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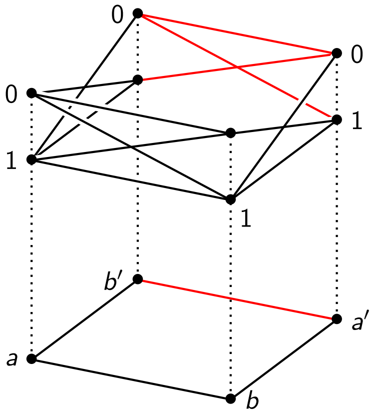

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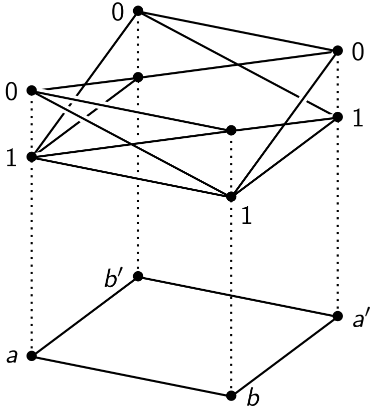

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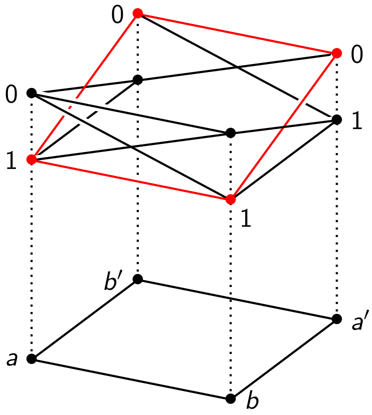

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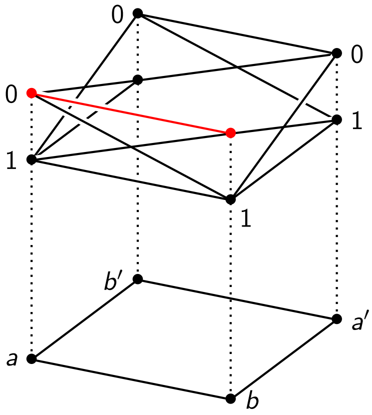

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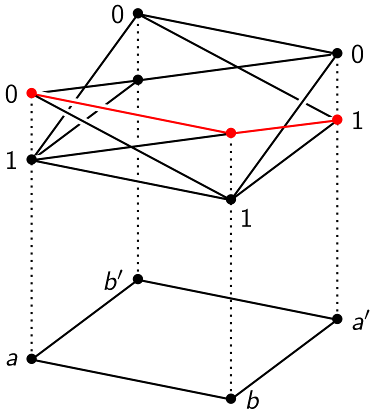

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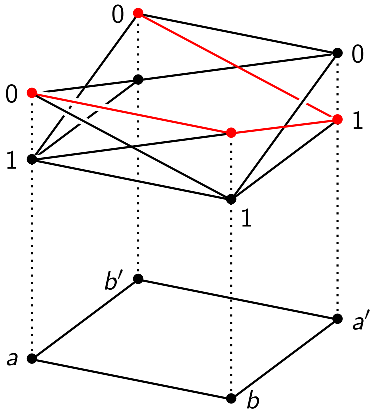

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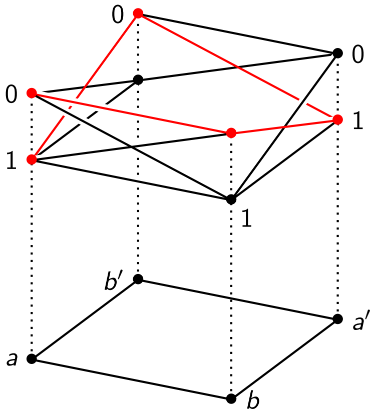

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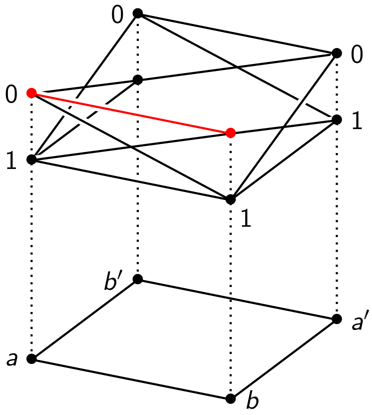

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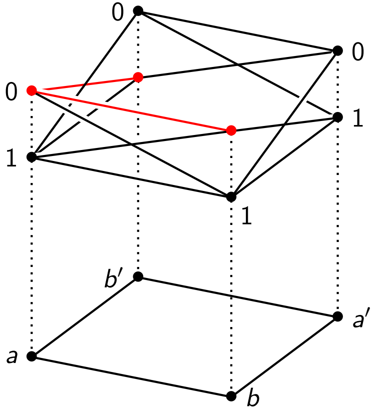

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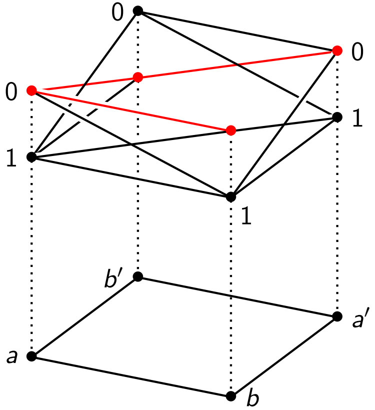

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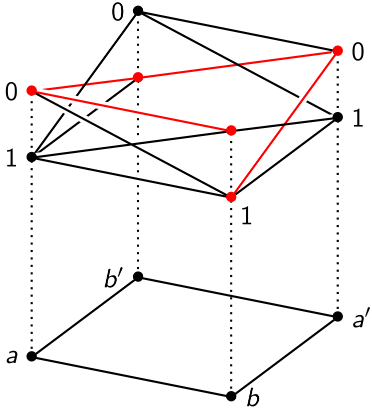

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Strong Contextuality

A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	1	0	0	1
a_1	b_2	1	0	0	1
a_2	b_1	1	0	0	1
a_2	b_2	0	1	1	0

The PR Box

Bundle Pictures

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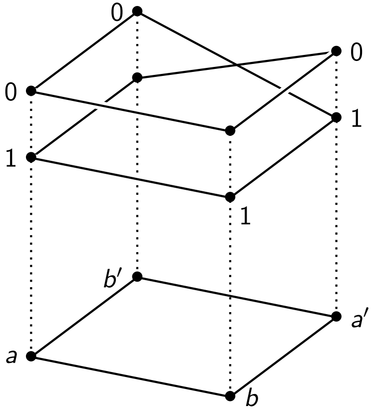
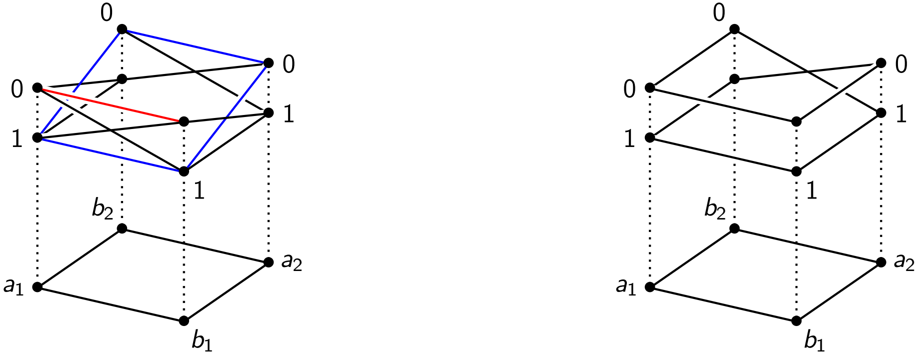


Visualizing Contextuality



The Hardy table and the PR box as bundles

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There is more structure in an empirical model e than in G_e.

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$$\begin{aligned} S_1 &: \; S_2 \text{ is true,} \\ S_2 &: \; S_3 \text{ is true,} \\ &\vdots \\ S_{N-1} &: \; S_N \text{ is true,} \\ S_N &: \; S_1 \text{ is false.} \end{aligned}$$

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The "paradoxical" nature of the original statements is now captured by the inconsistency of these equations.

Contextuality in the Liar; Liar cycles in the PR Box

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We can regard each of these equations as fibered over the set of variables which occur in it:

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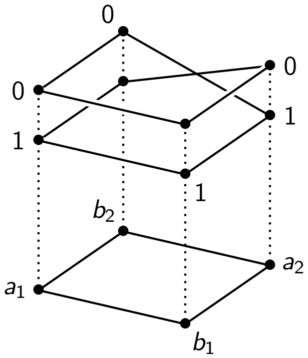
The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.

Paths to contradiction



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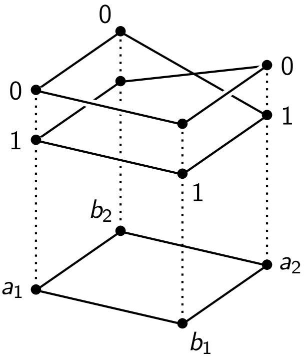

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$$a_2 = 1 \leadsto b_1 = 1 \leadsto a_1 = 1 \leadsto b_2 = 1 \leadsto a_2 = 0$$
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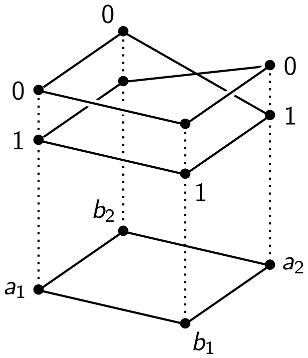

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We have discussed a specific case here, but the analysis can be generalised to a large class of examples.

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- Idea: use **cohomology** to characterize contextuality

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Here γ is in fact the **connecting homomorphism** of the long exact sequence.

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Proposition

The following are equivalent:

1. *The cohomology obstruction vanishes:* $\gamma(s_1) = 0$.
2. *There is a family* $\{r_i \in \mathcal{F}(C_i)\}$ *with* $s_1 = r_1$, *and for all* i, j*:*

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Thus non-vanishing of the obstruction provides a cohomological witness for contextuality.

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- In Contextuality, Cohomology and Paradox (ABKLM 2015), we obtain very general results in cases where the outcomes themselves have a module structure (over the same ring as the cohomology coefficients).
- This yields cohomological characterisations of **All-vs.-Nothing** proofs (Mermin). These account for most of the contextuality arguments in the quantum literature. In particular, we can find large classes of concrete examples in **stabiliser QM**.

Theorem

Let $\mathcal{S}$ be an empirical model on $\langle X, \mathcal{M}, R \rangle$. Then:

$$\mathsf{AvN}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathsf{Aff}\,\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_R(\mathcal{S}) \;\Rightarrow\; \mathsf{CSC}_{\mathbb{Z}}(\mathcal{S}) \;\Rightarrow\; \mathsf{SC}(\mathcal{S}) \;.$$

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Samson Abramsky, ‘Relational databases and Bell’s theorem’, In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.

Relational databases

This geometric picture and the associated methods can be applied to a wide range of situations in classical computer science.

In particular, as we shall now see, there is an isomorphism between the formal description we have given for the quantum notions of non-locality and contextuality, and basic definitions and concepts in relational database theory.

Samson Abramsky, 'Relational databases and Bell's theorem', In *In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman*, Springer 2013.

branch-name	**account-no**	**customer-name**	**balance**
Cambridge	10991-06284	Newton	£2,567.53
Hanover	10992-35671	Leibniz	€11,245.75
...	...	...	...

From possibility models to databases

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Consider again the Hardy model:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a_1, b_1)	1	1	1	1
(a_1, b_2)	0	1	1	1
(a_2, b_1)	0	1	1	1
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Change of perspective:

a_1, a_2, b_1, b_2	attributes
0, 1	data values
joint outcomes of measurements	tuples

The Hardy model as a relational database

The four rows of the model turn into four **relation tables**:

a_1	b_1
0	0
0	1
1	0
1	1

a_1	b_2
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What is the DB property corresponding to the presence of non-locality/contextuality in the Hardy table?

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There is no **universal relation**: no table

a_1	a_2	b_1	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$

whose projections onto $\{a_i, b_i\}$, $i = 1, 2$, yield the above four tables.

A dictionary

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Relational databases	measurement scenarios
attribute	measurement
set of attributes defining a relation table	compatible set of measurements
database schema	measurement cover
tuple	local section (joint outcome)
relation/set of tuples	boolean distribution on joint outcomes
universal relation instance	global section/hidden variable model
acyclicity	Vorob'ev condition

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We can also consider probabilistic databases and other generalisations;
cf. provenance semirings.

Contextual Semantics

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Why do such similar structures arise in such apparently different settings?

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Physics, computation, logic, natural language, . . . biology, economics, . . .

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- And beyond: connections with databases, robust refinement of the constraint satisfaction paradigm, application of contextual semantics to natural language semantics, connections with team semantics in Dependence logics, . . .

People

People












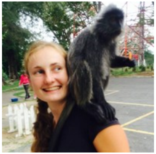
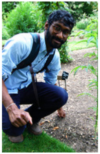

Adam Brandenburger, Lucien Hardy, Shane Mansfield, Rui Soares Barbosa, Ray Lal, Mehrnoosh Sadrzadeh, Phokion Kolaitis, Georg Gottlob, Carmen Constantin, Kohei Kishida. Giovanni Caru, Linde Wester, Nadish de Silva

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