Introduction to the sheaf-theoretic approach to contextuality

Samson Abramsky

Department of Computer Science, University of Oxford
Why sheaves?

Sounds intimidating – it isn’t!

Connects to beautiful and powerful mathematical ideas

One of now several approaches which develop a

general theory of

contextuality, rather than a collection of examples:

▶ Spekkens,
▶ Contextuality by Default (Dzhakfarov and Kujala),
▶ graph-theoretic (Cabello, Severini, Winter),
▶ hypergraphs (Acin, Fritz, Leverrier, Sainz).

See recent exposition of some of this by Marcelo Terra Cunha and Barbara

Comparison with other approaches, e.g. the CSW graph-theoretic approach:

both have useful features, the “sheaf” approach exposes some additional

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What is Contextuality?

What, then, is the essence of contextuality? In broad terms, we propose to describe it as follows:

Contextuality arises where we have a family of data which is

**locally consistent, but globally inconsistent**
Contextuality Analogy: Local Consistency
Contextuality Analogy: Local Consistency
Contextuality Analogy: Global Inconsistency
Empirical Data
Empirical Data

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<tr>
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\[o_A \in \{0, 1\}\]

\[m_A \in \{a, a'\}\]

\[o_B \in \{0, 1\}\]

\[m_B \in \{b, b'\}\]
Formalizing Contextuality: Measurement scenarios

These are types in logical/CS terms. Types of experimental set-up. A scenario is \((X, M, O)\), where \(X\) is a set of variables or measurement labels. \(M\) is a family of subsets of \(X\) – the contexts, or compatible subsets. \(O\) is a set of outcomes or values for the variables. Can be refined to \(O_x, x \in X\).

Two variants of \(M\), which is a hypergraph: either the maximal contexts (no inclusions), or closure under subsets. In the latter case, we have an abstract simplicial complex.
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Two variants of **\(\mathcal{M}\)**, which is a hypergraph: either the maximal contexts (no inclusions), or closure under subsets.

In the latter case, we have an **abstract simplicial complex**.
Example

In this table, the set of variables is $X = \{ a, a', b, b' \}$. The measurement contexts are:

- $\{ a_1, b_1 \}$
- $\{ a_2, b_1 \}$
- $\{ a_1, b_2 \}$
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The outcomes are $O = \{ 0, 1 \}$. 
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\[
\begin{array}{cccccccc}
U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 & U_9 \\
A & A & H & H & B & I & P & P & Q \\
B & E & I & K & E & K & Q & R & R \\
C & F & C & G & M & N & D & F & M \\
D & G & J & L & N & O & J & L & O \\
\end{array}
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The original K-S construction used 117 variables!
Basic events are local sections

A basic event is to measure all the variables in a context $\mathcal{C} \in M$, and observe the outcomes. This is represented by a function $s: \mathcal{C} \to \mathcal{O}$, i.e. $s \in \mathcal{O}^\mathcal{C}$, or more generally $s \in \prod_{x \in \mathcal{C}} \mathcal{O}_x$. Example: if $\mathcal{C} = \{a, b\}$, $\mathcal{O} = \{0, 1\}$, such an outcome might be $s = \{a \mapsto 0, b \mapsto 1\}$. This is a local section, since it is defined only on $\mathcal{C}$, not on the whole of $X$!

Basic operation of restriction: if $\mathcal{C} \subseteq \mathcal{C}'$, $s \in \mathcal{O}_{\mathcal{C}'}$, then $s|_{\mathcal{C}} \in \mathcal{O}_{\mathcal{C}}$. E.g. $s|_{\{a\}} = \{a \mapsto 0\}$. 

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Thus we have a family of probability distributions over **different**, but **coherently related**, sample spaces.

(The coherent relationship is functoriality!)
Restriction and Compatibility

We would like to express the condition that an empirical model is compatible, i.e. "locally consistent." We want to do this by saying that the distributions "agree on overlaps." For all $C, C' \in M$:

$$e_{C|C \cap C'} = e_{C'|C \cap C'}.$$

Cf. the usual notion of compatibility of a family of functions defined on subsets.

Marginalization of distributions: if $C \subseteq C'$, $d \in \text{Prob}(O_{C'})$, $d|_C(s) := \sum_{t \in O_{C'}|C} d(t)$. Compatibility is a general form of the important physical principle of No-Signalling; this general form is also known as No Disturbance.
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Contextuality defined

An empirical model \( \mathcal{C} \in \text{M} \) on a measurement scenario \((X, \mathcal{M}, \mathcal{O})\) is non-contextual if there is a distribution \(d \in \text{Prob} (\mathcal{O} | X)\) such that, for all \(\mathcal{C} \in \text{M}\):

\[
    d | \mathcal{C} = e_{\mathcal{C}}.
\]

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered. We call such a \(d\) a global section.

If no such global section exists, the empirical model is contextual.

Thus contextuality arises where we have a family of data which is locally consistent but globally inconsistent.

The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.
Contextuality defined

An empirical model \( \{ e_C \}_{C \in \mathcal{M}} \) on a measurement scenario \((X, \mathcal{M}, O)\) is **non-contextual** if there is a distribution \( d \in \text{Prob}(O^X) \) such that, for all \( C \in \mathcal{M} \):

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Samson Abramsky (Department of Computer Science) Introduction to the sheaf-theoretic approach to context
Contextuality defined

An empirical model \( \{ e_C \}_{C \in \mathcal{M}} \) on a measurement scenario \((X, \mathcal{M}, O)\) is **non-contextual** if there is a distribution \( d \in \text{Prob}(O^X) \) such that, for all \( C \in \mathcal{M} \):

\[
d|_C = e_C.
\]

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

We call such a \( d \) a **global section**.

If no such global section exists, the empirical model is **contextual**.
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The import of Bell’s theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.
Logical Contextuality

- Ignore precise probabilities
- Events are possible or not
- E.g. the Hardy model:

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### Strong Contextuality

The PR Box

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<tr>
<th>A</th>
<th>B</th>
<th>(0, 0)</th>
<th>(1, 0)</th>
<th>(0, 1)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$b_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$b_2$</td>
<td>1</td>
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</tr>
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The PR Box
**Bundle Pictures**

**Strong Contextuality**
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![Diagram of PR box](image-url)
Visualizing Contextuality

The Hardy table and the PR box as bundles
Comparison with the graph-theoretic CSW approach

Deriving an orthogonality graph $G$ from an empirical model $e$

$$V = \{ (C, s) \mid C \in M, s \in \mathcal{O}_C \}$$

$$(C, s) \dashv (C', s') \iff \exists x \in C \cap C'. s(x) \neq s'(x).$$

(de Silva 2016): $e$ is strongly contextual iff the independence number of $G_e$ is less than $|M|$. There is more structure in an empirical model $e$ than in $G_e$. 

Samson Abramsky (Department of Computer Science, University of Oxford)
Comparison with the graph-theoretic CSW approach

Deriving an orthogonality graph $G_e = (V, E)$ from an empirical model $e$

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Liar cycles

A Liar cycle of length $N$ is a sequence of statements $S_1$: $S_2$ is true, $S_2$: $S_3$ is true, \ldots, $S_{N-1}$: $S_N$ is true, $S_N$: $S_1$ is false.

For $N = 1$, this is the classic Liar sentence $S$: $S$ is false.

Following Cook, Walicki et al. we can model the situation by boolean equations:

\[ x_1 = x_2, \ldots, x_{n-1} = x_n, x_n = \neg x_1. \]

The “paradoxical” nature of the original statements is now captured by the inconsistency of these equations.
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Contextuality, Logic and Paradoxes

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The “paradoxical” nature of the original statements is now captured by the inconsistency of these equations.
Contextuality in the Liar; Liar cycles in the PR Box

We can regard each of these equations as fibered over the set of variables which occur in it:

\{ x_1, x_2 \}:
\[ x_1 = x_2 \]

\{ x_2, x_3 \}:
\[ x_2 = x_3 \]

... 

\{ x_{n-1}, x_n \}:
\[ x_{n-1} = x_n \]

\{ x_n, x_1 \}:
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Any subset of up to \( n-1 \) of these equations is consistent; while the whole set is inconsistent.

Up to rearrangement, the Liar cycle of length 4 corresponds exactly to the PR box.

The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.
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The usual reasoning to derive a contradiction from the Liar cycle corresponds precisely to the attempt to find a univocal path in the bundle diagram.
Paths to contradiction

Suppose that we try to set $a_2$ to 1. Following the path on the right leads to the following local propagation of values:

\[
\begin{align*}
    a_2 &= 1; \\
    b_1 &= 0; \\
    a_1 &= 0; \\
    b_2 &= 0; \\
    a_2 &= 1
\end{align*}
\]

We have discussed a specific case here, but the analysis can be generalised to a large class of examples.
Suppose that we try to set $a_2$ to 1. Following the path on the right leads to the following local propagation of values:

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Constraint Satisfaction

A (possibilistic) empirical model is a constraint satisfaction problem. Represent \( C \subseteq O \) as a formula.

Example: the PR Box

\[
\begin{array}{c|c}
\hline
& 0 & 1 \\
\hline
0 & ✓ & \times \\
1 & \times & ✓ \\
\hline
\end{array}
\]

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Local consistency is well-studied in (classical) CSP.
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<td>×</td>
<td>×</td>
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</tr>
<tr>
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Topological Characterization

Local consistency — global inconsistency

Contextuality is pervasive (e.g. physics, computation, logic, ...)

Goal: find the common mathematical structure in these diverse manifestations, and develop a widely applicable theory

Can be effectively visualised in topological terms

"Twisting" in bundle space gives rise to an obstruction to global consistency

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Why Cohomology?

A major theme of 20/21st century mathematics
Constructive witnesses for non-existence, instead of proofs by contradiction
Often computable
Increasingly coming into applications (e.g. persistent homology, TDA)
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Summary of Cohomological Characterization

We have a cover \( U = \{ C_1, \ldots, C_n \} \) of measurement contexts.

Given \( s = s_1 \in S \mathcal{e}(C_1) \), we define \( z = \delta_0(s_1, \ldots, s_n) \),

where \( s_1 | C_1 \cap C_i = s_i | C_1 \cap C_i \), \( i = 1, \ldots, n \).

This is a cocycle in the relative Čech cohomology with respect to \( C_1 \).

We define \( \gamma(s) = [z] \in \check{\mathcal{H}}_1(U, F\bar{C}_1) \)

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Here \( \gamma \) is in fact the connecting homomorphism of the long exact sequence.
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Basic Results

Proposition

The following are equivalent:

1. The cohomology obstruction vanishes:
   \[ \gamma(s_1) = 0. \]

2. There is a family \( \{ r_i \in F(C_i) \} \) with \( s_1 = r_1 \), and for all \( i, j \):
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Proposition

If the model \( e \) is possibilistically extendable, then the obstruction vanishes for every section in the support of the model. If \( e \) is not strongly contextual, then the obstruction vanishes for some section in the support. Thus non-vanishing of the obstruction provides a cohomological witness for contextuality.
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Notes on Cohomology

There are false positives because of negative coefficients in cochains.

We can effectively compute (mod 2) witnesses in many cases of interest:
GHZ, Kylachko, Peres-Mermin, large class of Kochen-Specker models, . . .

In Contextuality, Cohomology and Paradox (ABKLM 2015), we obtain very
general results in cases where the outcomes themselves have a module
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This yields cohomological characterisations

[All-vs.-Nothing](Mermin).
These account for most of the contextuality arguments in the
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**Theorem**

Let \( S \) be an empirical model on \( \langle X, M, R \rangle \).

A

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(\( S \))

\( \Rightarrow \)

S

C

A

f

S

C

S

C

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**Theorem**

Let $S$ be an empirical model on $\langle X, M, R \rangle$. Then:

$$\text{AvN}_R(S) \Rightarrow \text{SC(Aff } S) \Rightarrow \text{CSC}_R(S) \Rightarrow \text{CSC}_\mathbb{Z}(S) \Rightarrow \text{SC}(S).$$
Relational databases
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Samson Abramsky, ‘Relational databases and Bell’s theorem’, In In Search of Elegance in the Theory and Practice of Computation: Essays Dedicated to Peter Buneman, Springer 2013.
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<tr>
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<td>…</td>
<td>…</td>
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Consider again the Hardy model:

\[
\begin{align*}
(a_1, b_1) & : (0, 0) (0, 1) (1, 0) (1, 1) \\
(a_2, b_1) & : (0, 1) (1, 1) \\
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\end{align*}
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Change of perspective:

- \(a_1, a_2, b_1, b_2\) attributes
- \(0, 1\) data values
- joint outcomes of measurements tuples
From possibility models to databases

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<tr>
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  & (0, 0) & (0, 1) & (1, 0) & (1, 1) \\
(a_1, b_1) & 1 & 1 & 1 & 1 \\
(a_1, b_2) & 0 & 1 & 1 & 1 \\
(a_2, b_1) & 0 & 1 & 1 & 1 \\
(a_2, b_2) & 1 & 1 & 1 & 0 \\
\end{array}
\]

Change of perspective:

\[a_1, a_2, b_1, b_2\] attributes

\[0, 1\] data values

joint outcomes of measurements tuples
The Hardy model as a relational database

The four rows of the model turn into four relation tables:

<table>
<thead>
<tr>
<th></th>
<th>a₁</th>
<th>b₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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</table>

What is the DB property corresponding to the presence of non-locality/contextuality in the Hardy table?

There is no universal relation: no table whose projections onto \{aᵢ, bᵢ\}, \(i = 1, 2\), yield the above four tables.
The Hardy model as a relational database

The four rows of the model turn into four relation tables:

$$\begin{array}{cc}
  a_1 & b_1 \\
  0 & 0 \\
  0 & 1 \\
  1 & 0 \\
  1 & 1 \\
\end{array}$$

$$\begin{array}{cc}
  a_1 & b_2 \\
  0 & 1 \\
  1 & 0 \\
  1 & 1 \\
\end{array}$$

$$\begin{array}{cc}
  a_2 & b_1 \\
  0 & 1 \\
  1 & 0 \\
  0 & 1 \\
\end{array}$$

$$\begin{array}{cc}
  a_2 & b_2 \\
  0 & 0 \\
  1 & 0 \\
  0 & 1 \\
\end{array}$$

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whose projections onto \( \{a_i, b_i\}, \ i = 1, 2 \), yield the above four tables.
## A dictionary

<table>
<thead>
<tr>
<th>Relational databases</th>
<th>measurement scenarios</th>
</tr>
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<td>attribute</td>
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We can also consider probabilistic databases and other generalisations; cf. provenance semirings.

Samson Abramsky (Department of Computer Science, University of Oxford)
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Why do such similar structures arise in such apparently different settings? The phenomenon of contextuality is pervasive. Once we start looking for it, we can find it everywhere! Physics, computation, logic, natural language, ... biology, economics, ... The **Contextual semantics hypothesis**: we can find common mathematical structure in all these diverse manifestations, and develop a widely applicable theory. More than a hypothesis! Already extensive results in Quantum information and foundations: hierarchy of contextuality, logical characterisation of Bell inequalities, classification of multipartite entangled states, cohomological characterisation of contextuality, contextual fraction as a measure of contextuality, resource theory for contextuality, applications to quantum advantage, quantum homomorphisms and the quantum monad, developments towards quantum finite model theory ... And beyond: connections with databases, robust refinement of the constraint satisfaction paradigm, application of contextual semantics to natural language semantics, connections with team semantics in Dependence logics, ...
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References

Papers (available on arXiv):


The Penrose Tribar