Hardware Store Science Curriculum

Angular Velocity and Acceleration

ICP Maker-STEM Content Document

Lesson 1 – Describing Circular Motion: This module begins by discussing the characteristics of objects undergoing circular or rotational motion. This is followed up with an introduction to radians and their use as a measurement of rotational displacement. This is followed by methods of quantifying angular displacement. *(ICP.1.3, 1.4, 2.1, 2.2)*

Lesson 2 – Centripetal Force: Next, we will discuss forces that influence circular motion. We will apply basic geometry as a means of describing and quantifying circular motion. Then we will apply our understanding of balanced and unbalanced forces to circular motion. *(ICP. 3.3, 3.5, 3.6)*

Lesson 3 – Circular Motion Investigation: Armed with an understanding of how to quantify circular motion we take on the task of designing and building a model on which to investigate. Next, we will investigate the relationship between angular displacement, and forces acting on an object undergoing circular motion. We will collect and analyze data relating to circumference, revolutions, and time as they pertain to rotating objects. *(ICP. 1.3, 1.4, 2.1, 2.2, 3.3, 3.5, 3.6; IED-6.10.2, 6.10.4; POE-3.2, 5.5, 5.6)*

Lesson 4 – Advanced Circular Motion: Finally, after collecting experimental data on horizontal circular motion we discuss the motion of cars traveling around banked turns. We will focus our discussion on how road geometry influences the car's ability to safely navigate banked turns under varying road conditions. *(ICP.1.4, 2.1, 2.2; IED-0.1, 2.6; POE-5.3, 5.4)*

Module 11 Guiding Question

How are velocity and acceleration determined in circular motion?

Key Terms

Angular Velocity – rate an object rotates/revolves relative to another point. $\omega = \frac{\Delta\theta}{\Delta t}$

Angular Acceleration – the change in angular velocity of a rotating object. $\alpha = \frac{\Delta\omega}{\Delta t}$

Centripetal Acceleration – motion directed towards the axis of rotation, during circular motion. $a_R = \omega^2 r$

Centripetal Force –force directed toward the axis around which an object is moving. $F_R = m \times a_R$

Radian – describes the plane angle subtended by a circular arc as the length of the arc divided by the radius of the arc.

Riding Rollercoasters

Roller coaster rides are notorious for creating accelerations and *g-forces* which can give you the sense of weightlessness one minute and heaviness the next. The parts of the ride which are most responsible for the alternate sensations of weightlessness and heaviness are the clothoid loops. A clothoid loop has a constantly curving shape with sections which resemble the curve of a circle. As a result, roller coaster riders are continuously altering the direction of motion while moving through the loop. This change in direction is caused by the presence of unbalanced forces and results in the direction of acceleration continuously changing.

There are only two forces acting on roller coaster riders, they are the force of gravity and the normal force (the force of the seat pushing up on the rider). The force of gravity is always directed downwards, and the normal force is always directed perpendicular to the roller coaster seat. Since the orientation of the car on the track is continuously changing, the normal force is continuously changing its direction, causing rider to feel "weightless" at the top of the loop and "heavy" at the bottom.

Any moving object can be described using the kinematic concepts discussed from the Kinetic Energy & Projectile Motion module. The motion of a moving object can be explained using either Newton's Laws and vector principles or by means of the Work-Energy Theorem. The same concepts and principles used to describe and explain the motion of an object can be used to describe and explain the parabolic motion of a projectile. In this module, we will see that these same concepts and principles can also be used to describe and explain the motion of objects that either move in circles or moving through turns. Kinematic concepts and motion principles will be applied to the motion of objects in circles and can be extended to analyze the motion of such objects as roller coaster cars, a football player running passing routes, or a planet orbiting the sun.

Uniform circular motion can be described as the motion of an object in a circle at a constant speed. As an object moves in a circle, it is constantly changing its direction. At all instances, the object is moving tangent to a circle. Since the direction of the velocity vector is the same as the direction of the object's motion, the velocity vector is directed tangent to a circle as well.

By definition, an object moving in a circle is also accelerating. Accelerating objects are objects which are changing their velocity - either the magnitude of the velocity or the direction of the velocity. An object undergoing uniform circular motion is moving with a constant speed, while changing its direction and is therefore accelerating. The direction of the acceleration is towards the center of the traveled circular path.

The final motion characteristic for an object undergoing uniform circular motion is the net force. The net force acting upon an object undergoing circular motion must be directed towards the center of the circle if acceleration is towards the center of the circle. The inward net force is said to be the centripetal force. Without such an inward force, an object would continue in a straight line, never deviating from its direction. Yet, with the inward net force directed perpendicular to the velocity vector, the object is always changing its direction and undergoing an inward acceleration.

Lesson 1: Objectives

- **Define and calculate angular velocity and angular acceleration**

Content

- **Speed in Circular Motion**
- **Rotation Angle**
- **Angular Velocity**
- **Acceleration in Circular Motion**
- **Angular Acceleration**

Lesson 1: Describing Circular Motion

Up to this point our study of motion has been along a straight line and we discussed such concepts as displacement, velocity, and acceleration. Projectile motion dealt with motion in two dimensions when an object is projected into the air, while being subject to the gravitational force. In this lesson, we consider situations where an object moves in a circular motion.

Calculation of the average speed during circular motion

Circular motion at a constant speed is one of many forms of circular motion. An object moving in uniform motion covers the same linear distance in each unit of time. When moving in a circle, an object traverses a distance around the perimeter of the circle. If you were to move in a circle with a constant speed of 5 m/s, you would travel 5 meters along the perimeter of the circle during each second. The distance of one complete trip around the perimeter of a circle is equal to the circumference of the circle. With a uniform speed of 5 m/s, a car could make a complete cycle around a circle that had a circumference of 5 meters. Meaning that each cycle around the 5-m circumference circle would require 1 second. On the other hand, a circle with a circumference of 20 meters would take 4 seconds. This relationship between the circumference of a circle, the time to complete one cycle around the circle and the speed of the object is just another way of applying the average speed equation from the Speed Velocity Acceleration module.

$$avg\ speed = \frac{distance}{time} = \frac{circumference}{time}$$

π is the symbol for the Greek lowercase letter pi.

Pi (π) is the ratio of the circumference of any circle to the diameter of that circle ($\pi = 3.14$)

The circumference of any circle can be determined from the radius of the circle.

$$circumference = 2\pi r$$

Combining these two equations we derive a new equation relating the speed of an object moving in uniform circular motion to the radius of the circle and the time to make one cycle around the circle (**period**).

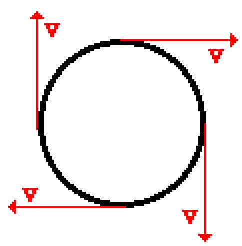
Period: *(noun)* the interval of time required to complete a cycle

$$avg\ speed = \frac{2\pi r}{t}$$

where r represents the radius of the circle and t represents the period. This suggests that for objects moving around circles of different radius in the same period, the object traveling the circle of larger radius must be traveling with the greater speed.

The Direction of the Velocity Vector

Objects moving in uniform circular motion will have a constant speed (magnitude). However, they will not have a constant velocity (magnitude and direction). The magnitude of the velocity vector is the instantaneous speed of the object. The direction of the velocity vector is directed in the same direction that the object moves. Since an object is moving in a circle, its direction is continuously changing. The diagram at the right shows the direction of the velocity vector at four different points for an object moving in a clockwise direction around a circle.



At one moment, the object is moving northward such that the velocity vector is directed to the north. One fourth of the way around the circle, the object would be moving eastward such that the velocity vector is directed to the east. As the object rounds the circle, the direction of the velocity vector is different than it was the instant before. While the magnitude of the velocity may be constant, the direction of the velocity is constantly changing. The best word that can be used to describe the direction of the velocity is **tangential**. The direction of the velocity at any moment is in the direction of a line drawn tangent to the circular path at the object's location. While the actual direction of the object is changing, it is always tangent to the observed circular path.

Tangential: *(adjective)* touching an arc or circle at only one point

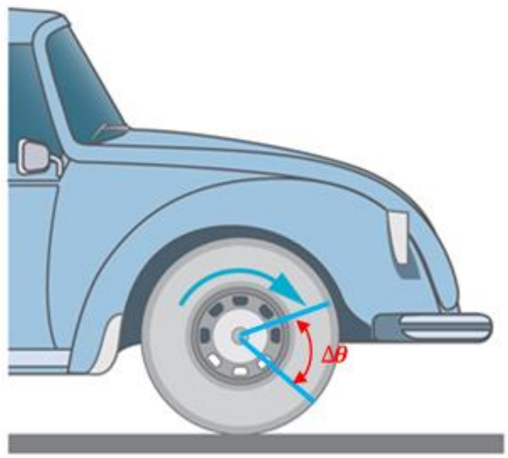
Because the speed is constant for circular motion, you may be tempted to think that there is no acceleration. You might reason that, if you are driving a car in a circle at a constant speed of 20 mi/hr, the speed is neither decreasing nor increasing; therefore, there must not be an acceleration. At the center of this common misconception is the wrong belief that acceleration has to do with speed and not with velocity. The truth is that an accelerating object is an object that is changing its velocity. And because velocity has both a magnitude and a direction, a change in either the magnitude or the direction constitutes a change in the velocity and is said to be accelerating.

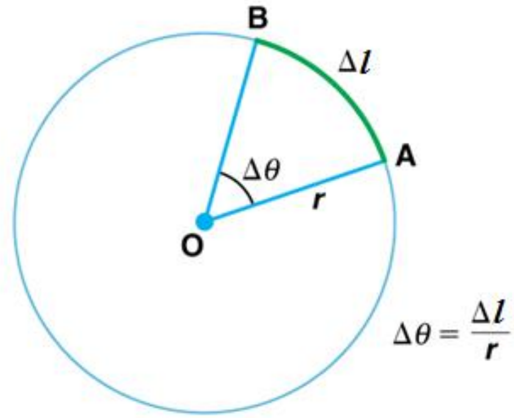
Rotation Angle

When objects rotate about some axis—for example, when a tire rotates on an axle – each point in the tire follows a circular arc. Consider a line from the center of the tire to its edge. Each point along the line moves through the same angle in the same amount of

Analogous: *(adjective)* similar or comparable to something else in some detail

time. The rotation angle is the amount of rotation and is analogous to a linear distance. We can define the rotation angle $\Delta\theta$ as the ratio of the arc length Δs to the radius r of curvature.





The points along the line from the center to the surface of the tire all move through the same angle $\Delta\theta$ in a time Δt. The arc length Δl is described on the circumference and is the distance traveled along a circular path, r is the radius of curvature of the circular path (the tire size in this case).

We know that for one complete revolution, the arc length is the circumference of a circle of radius r and that the circumference of a circle is $2\pi r$. We can therefore say that in one complete revolution the rotation angle is

$$revolution = \frac{2\pi r}{r} = 2\pi$$

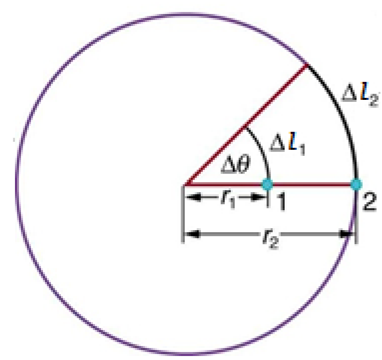
This result is the basis for defining radians (rad) as the units used to measure rotation angles $\Delta\theta$ defined so that

$$revolution = 2\pi \times rad$$

A comparison of some useful angles expressed in both degrees and radians is shown in the table to the left.

Degree Measures	Radian Measure
30^o	$\frac{\pi}{6}$
60^o	$\frac{\pi}{3}$
90^o	$\frac{\pi}{2}$
120^o	$\frac{2\pi}{3}$
135^o	$\frac{2\pi}{4}$
180^o	π

In the image to the right, points 1 and 2 rotate through the same angle ($\Delta\theta$), but point 2 moves through a greater arc length (Δl) because it is at greater distance from the center of rotation.



$$\boldsymbol{\Delta\theta = \frac{\Delta l_1}{r_1}}$$

$$\boldsymbol{\Delta\theta = \frac{\Delta l_2}{r_2}}$$

If $\Delta\theta = 2\pi$ rad, then the tire has made one complete revolution, and every point on the tire is back at its original position. Because there are 360^o in a circle or one revolution, the relationship between radians and degrees is thus

$$2\pi rad = 360^o$$

so that

$$rad = \frac{360^o}{2\pi} \approx 57.3^o$$

Angular Velocity

We now have the pieces for defining **angular velocity** (ω) as the rate of change of an angle.

ω is the symbol for the Greek lowercase letter omega.

$$\omega = \frac{\Delta\theta}{t}$$

where an angular rotation $\Delta\theta$ takes place in a time t. The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

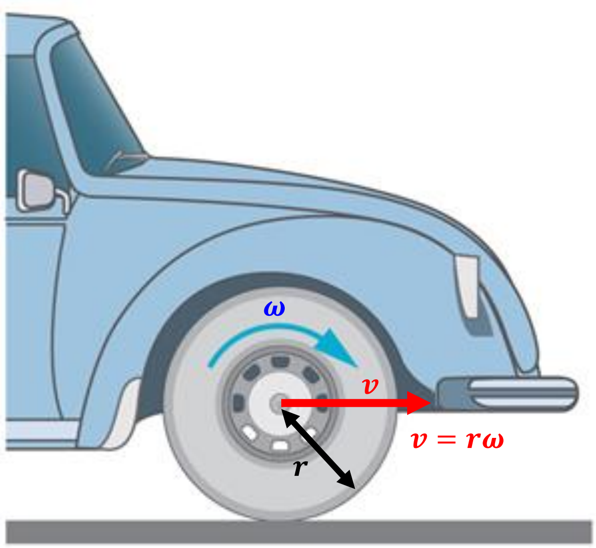
Angular velocity ω is analogous to linear velocity v. To get the precise relationship between angular and linear velocity, we again consider a point on the rotating tire. This point moves an arc length Δl in a time t, and so it has a linear velocity

$$v = \frac{\Delta l}{t}$$

From $\Delta\theta = \Delta l/r$ we see that $\Delta l = r\Delta\theta$. Substituting this into the expression for v gives

$$v = \frac{r\Delta\theta}{t} = r\omega$$

Analysis of this relationship provides two different insights. First, relationship in $v = r\omega$ states that the linear velocity v is proportional to the distance from the center of rotation, it is largest for a point on the surface of the tire. We can also call this linear speed v of a point on the surface of the tire the *tangential velocity*. The second relationship in $v = r\omega$ can be illustrated by considering the tire of a moving car. So that the faster the car moves, the faster the tire spins – large v means a large ω. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.



Sample Problem

What is the angular velocity of a 0.300 m radius car tire when the car travels at $15\, m/s$.

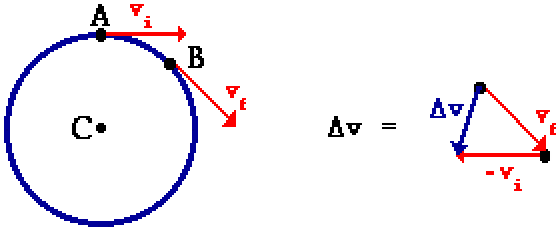
$$\omega = \frac{v}{r} = \frac{15.0\, m/s}{0.300m} = 50\, rad/s$$

Acceleration in circular motion

To understand this at a deeper level, we will have to combine the definition of acceleration with a review of some basic vector principles. During the Speed Velocity Acceleration module we learned that acceleration is a quantity defined as the rate an object changes its velocity. As such, it is calculated using the equation

$$avg\ accel = \frac{\Delta v}{t} = \frac{v_f - v_i}{t}$$

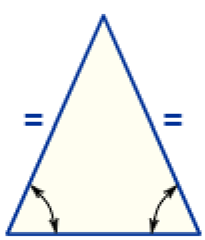
where v_i represents the initial velocity and v_f represents the final velocity after some unit of time t. The numerator of the equation is found by subtracting one vector (v_i) from a second vector (v_f). However, remember that the addition and subtraction of vectors from each other is done differently than the addition and subtraction of scalar quantities. Let's use the case of an object moving in a circle about point C as shown in the diagram below. In a time of t seconds, the object has moved from point A to point B. In this time, the velocity has changed from v_i to v_f. The process of subtracting v_i from v_f is shown in the vector diagram; this process yields the change in velocity.



Direction of the Acceleration Vector

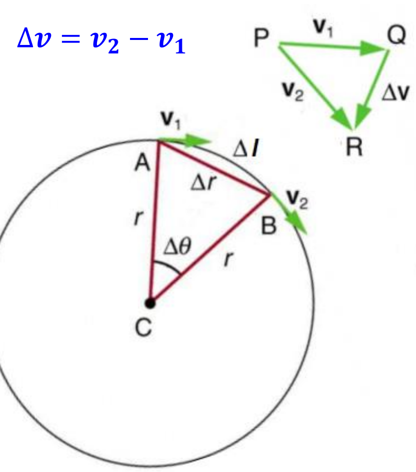
Notice that in the diagram above there is a velocity change for an object moving in a circle with a constant speed. A careful inspection of the velocity change vector in the above diagram shows that it points down and to the left. Along the arc connecting points A and B, the velocity change is directed towards point C, the center of the circle. The acceleration of the object is in the same direction as the velocity change, meaning the acceleration is directed towards point C as well. Objects moving in circles at a constant speed accelerate towards the center of the circle.

Isosceles triangles are triangles with two equal sides



The angles opposite the equal sides are also equal

We know the direction of centripetal acceleration, but what is its magnitude? The answer can be derived by knowing that the triangle formed by the velocity vectors and the one formed by the radii r and Δl are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain



$$\frac{\Delta v}{v} = \frac{\Delta l}{r}$$

Solving this expression for Δv results in

$$\Delta v = \frac{v \times \Delta l}{r}$$

Substituting this into our equation for acceleration results in

$$a = \frac{\Delta v}{\Delta t} \rightarrow \frac{v \times \Delta l}{r \times \Delta t}$$

Knowing that linear velocity at the edge of the circle is $v = \Delta l/\Delta t$, we arrive at a centripetal acceleration as

$$a_c = \frac{v^2}{r}$$

which is the acceleration of an object in a circle of radius r at a speed v.

We can also determine the centripetal acceleration for an object making a complete revolution around the perimeter of the circular path. Remember that the speed to make one cycle around the circle can be derived by $v = 2\pi r/t$. Substituting this into our equation for centripetal acceleration we obtain

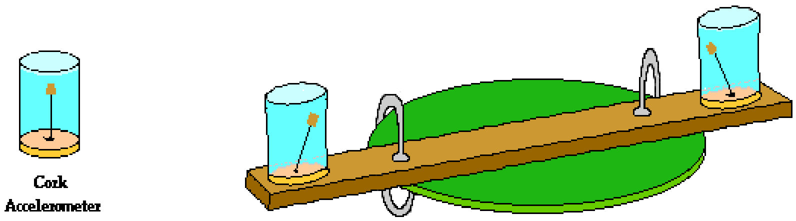
$$a_c = \frac{\left(\frac{2\pi r}{t}\right)^2}{r} = \frac{4 \times \pi^2 \times r}{t^2}$$

where r represents the radius of the circle and t represents the time it takes to complete one revolution.

These equations suggest that centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have experienced when riding in a car. But it may be a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h.

Visualizing Centripetal Acceleration

The acceleration of an object is often measured using a device known as an accelerometer. A simple accelerometer consists of an object immersed in a fluid such as water. Tie a piece of string around a cork and attach it to the lid of a jar filled with water. Invert the jar and attach it to the end of a section of a 1x4. A second accelerometer constructed in the same manner can be attached to the opposite end of the 1x4. Clamp the accelerometers to a rotating platform and spin in a circle, the direction of the acceleration can be clearly seen by the direction of lean of the corks.



Angular Acceleration

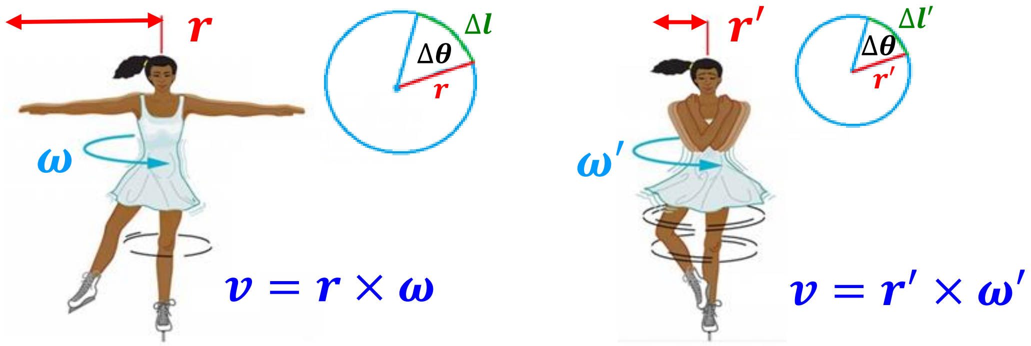
Recall that angular velocity ω was defined as the change of angle ($\Delta\theta$) per unit time (t):

$$\omega = \frac{\Delta\theta}{t}$$

where θ is the angle of rotation as seen the figure skater images below. The relationship between angular velocity ω and linear velocity v was also defined as

$$v = r\omega$$

where r is the radius of curvature, also seen in the image below. According to the sign convention, the counterclockwise direction is considered as the positive direction and clockwise direction as the negative direction.



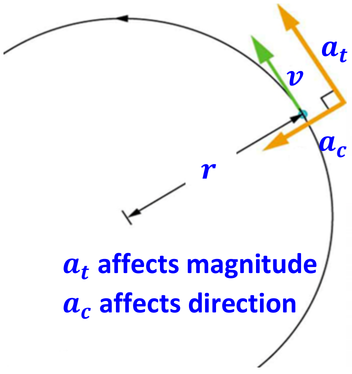
When the skater pulls in her arms, there is an increase in the angular velocity which can be termed an *angular acceleration*. Angular acceleration α is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

where $\Delta\omega$ is the change in angular velocity and Δt is the time required to make the change. The units of angular acceleration are (rad/s)/s, or rad/s^2. If ω increases, then α is positive. If ω decreases, then α is negative.

This relationship is analogous to what happens with linear motion. This connection between circular motion and linear motion needs to be explored more deeply. It is useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is tangent to the circle at the point of interest, as seen in the diagram to the right. Thus, linear acceleration is called tangential acceleration a_t.

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. Centripetal acceleration refers to changes in the direction of the velocity but not the magnitude. These two accelerations, a_t and a_c are perpendicular and independent of one another.



Now we can find the exact relationship between linear acceleration a_t and angular acceleration α. Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined to be

$$a_t = \frac{\Delta v}{\Delta t}$$

For circular motion, $v = r\omega$, so that

$$a_t = \frac{\Delta(r\omega)}{\Delta t}$$

The radius r is constant for circular motion, and so $\Delta(r\omega) = r(\Delta\omega)$. Thus,

$$a_t = \frac{r(\Delta\omega)}{\Delta t}$$

By definition, $\alpha = \Delta\omega/\Delta t$. Thus,

$$a_t = r\alpha$$

The relationship between linear and angular acceleration will be explored during the Torque and Gearing Module

This equation indicates that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius affects acceleration such that the smaller the radius – the smaller the linear acceleration for a given angular acceleration α.

Lesson 2: Centripetal Force

As we discussed in the preceding lesson, an object moving in a circle is experiencing an acceleration directed towards the center of the circle. According to Newton's second law of motion, an object which experiences an acceleration must also be experiencing a net force. In previous modules we learned the net force acts in the same direction as the acceleration. An object moving in a circle must have an inward force acting upon it in order to cause the inward acceleration. For object's moving in circular motion, we call this inward force the centripetal force (F_c) because it causes the object to seek the center.

To understand the importance of a centripetal force, it is important to have an understanding of Newton's first law of motion - **the law of inertia**. The law of inertia states that objects in motion tend to stay in motion with the same speed and the same direction unless acted upon by an unbalanced force.

According to this law the natural tendency of all moving objects to continue in motion in the same direction that they are moving, unless some form of unbalanced force acts upon the object to alter its motion from its straight-line path. This is because moving objects tend to naturally travel in straight lines and the presence of an unbalanced force is required for objects to move in circles.

Lesson 2: Objectives

- **Describe centripetal force and how it influences circular motion, using angular velocity and angular acceleration**

Content

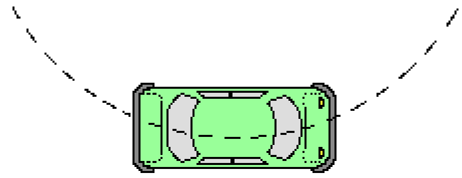
- **Inertia, Force and Acceleration in Passenger Car**
- **Centripetal Force and Direction Change**
- **Centripetal Force vs Centrifugal Force**

Inertia, Force and Acceleration for an Automobile Passenger

We experience the phenomenon of inertia when we travel in an automobile. For example, when a traffic light turns green and the car accelerates from rest. Relative to the seat your body has the sensation of moving backwards. Your body being at rest tends to stay at rest while the car moves forward. The seat pushing on your back gives you the feeling of moving backwards, resulting from your body resisting the car's acceleration.

The same principle applies when approaching a stoplight. The brakes are applied, and the car comes to a stop. However, your body tends to continue in motion while the car is stopping. Your body pressing against the seat belt gives you the sensation of moving forward. You are once more left with the false feeling of being pushed in a direction which is opposite the car's acceleration.

Motion of a car making left-hand turn



The passenger's inertia causes the "sensation of an outwards acceleration"

Suppose that the car makes a sharp turn to the left at constant speed. During the turn, the friction force acting upon the turned wheels of the car cause an unbalanced force resulting in centripetal acceleration directed towards the inside of the circular path. Your body however tends to stay in motion. While the car is accelerating inward, you continue in a straight line. If you are sitting on the passenger side of the car, you eventually bump against the outside door of the car as it turns inward. You are once more left with the false feeling of being pushed in a direction that is opposite your acceleration.

Centripetal Force and Direction Change

During centripetal acceleration there is some physical force pushing or pulling the object towards the center of the circular path. The force acting on the object causes it to deviate from its straight-line path, accelerate inwards and move along a circular path. This centripetal force alters the direction of the object without altering its speed. Remember

Refer to the Energy Storage Module for more information on Work and Energy

from the Energy Storage module that work is a force acting upon an object to cause a **displacement**. The amount of work done upon an object is found using the equation

$$W = F \times d \times \cos\theta$$

where the θ represents the angle between the force and the displacement. Centripetal force always acts perpendicular to the straight-line motion of the object being displaced.

$$W = F \times d \times \cos\theta = F \times d \times \cos 90^o = 0$$

Thus, the work done by the centripetal force in the case of uniform circular motion is 0 Joules. When no work is done on an object by external forces, the total mechanical energy of the object remains constant. If an object is moving in a horizontal circle at constant speed, the centripetal force cannot alter the total energy of the object. For an unbalanced force to change the speed of the object, there would have to be a force acting to increase/decrease the speed of the object.

Centrifugal, not to be confused with **centripetal**, means away from the center or outward, and its use combined with the common sensation of an outward lean when experiencing circular motion, often creates/reinforces a common misconception that objects in circular motion are experiencing an outward force.

There are three mathematical quantities we need to analyze in relation to the motion of objects in circles. These three quantities are speed, acceleration and force. In the previous lesson we learned that the speed of an object moving in a circle is given by

$$v = \frac{distance}{time} = \frac{2 \times \pi \times r}{t}$$

where r represents the radius of the circular path and t represents the time required to complete one revolution

The centripetal acceleration of an object moving in a circle can be determined by either of the following equations.

$$a_c = \frac{v^2}{r}$$

or

$$a_c = \frac{4 \times \pi^2 \times r}{t^2}$$

The net force acting on an object undergoing uniform circular motion is a product of the object's mass m and the centripetal acceleration a_c of the object. This can be expressed in three different equations

$$F = m \times a_c$$

$$F = m \times \frac{v^2}{r}$$

or

$$F = m \times \frac{4 \times \pi^2 \times r}{t^2}$$

This set of circular motion equations can be used in two ways to guide our thinking about how an alteration in one quantity would affect a second quantity. The process of solving a circular motion problem is much like any other problem we have worked up to this point and involves a careful reading of the problem, the identification of the known and required information in variable form, the selection of the relevant equation(s), substitution of known values into the equation, and finally algebraic manipulation of the equation to determine the answer.

Lesson 3: Objectives

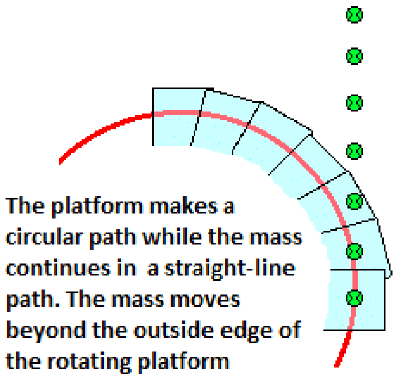
- **Investigate circular motion and relate angular velocity and angular acceleration to the collected data**

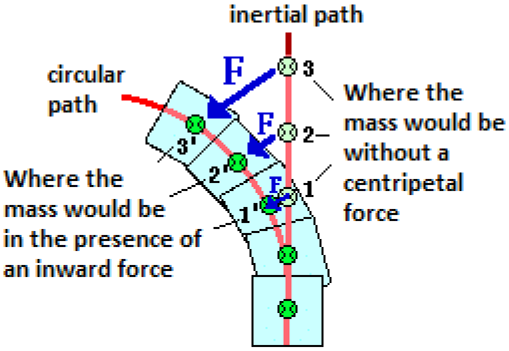
Content

- **Rotational Motion and Inertia**
- **Circular Motion Investigation**
- **Data Analysis**

Lesson 3: Circular Motion Experiment

A common physics scenario involves using a suspended mass attached to a rotating platform. When the platform rotates in a circular motion the suspended mass wishes to travel in a straight line. Without the mass being attached to the platform it would continue in this straight-line motion. Relative to the circular motion of the rotating platform, the mass moves away from the center of the circle. This observation can be explained by the tendency of an object in motion to continue in motion in the same direction (inertia).





An object moving in a circular motion is at all times moving tangent to the circle; the velocity is directed tangentially. The last lesson pointed out that to make the circular motion, there must be a net or unbalanced force directed towards the center of the circle. This path is an inward force - a centripetal force. We also identified three mathematical quantities for explaining the motion of objects in circles. The first of these was speed, defined in this case as

$$v = \frac{2\pi r}{t}$$

The next was acceleration, determined by either of the following equations.

$$a_c = \frac{v^2}{r}$$

or

$$a_c = \frac{4 \times \pi^2 \times r}{t^2}$$

The third was the force acting on the object in the same direction as the acceleration, obtained by one of the following three equations:

$$F = m \times a_c$$

$$F = m \times \frac{v^2}{r}$$

and

$$F = m \times \frac{4 \times \pi^2 \times r}{t^2}$$

All of these equations express the mathematical relationship between the variables present in the observed motion. For instance, the equation for Newton's second law identifies how centripetal acceleration is related to the net force and the mass of an object.

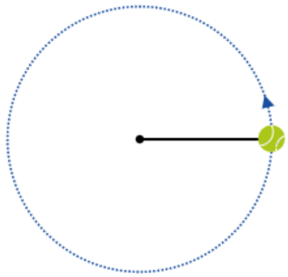
$$a_c = \frac{F}{m}$$

The relationship expressed by the equation is that the centripetal acceleration of an object is directly proportional to the net force acting upon it. In other words, as centripetal acceleration increases, the force increases.

Intuition: *(noun)* The ability to understand something immediately, without the need for conscious reasoning

Rotational inertia is a property of any object that can be rotated. It tells us how difficult it is to change the rotational velocity of the object around a given axis. When a mass moves further from the axis of rotation it becomes increasingly more difficult to change the rotational velocity of the system. Intuition tells us this is because the mass is now carrying more momentum with it around the circle (due to the higher speed).

Rotational inertia is given the symbol I. For a single body such as the tennis ball of mass m, rotating at radius r from the axis of rotation the inertia is



$$I = mr^2$$

Rotational inertia is also commonly known as the *second moment of mass*; the 'second' here refers to the fact that it depends on the length of the moment arm (r) squared.

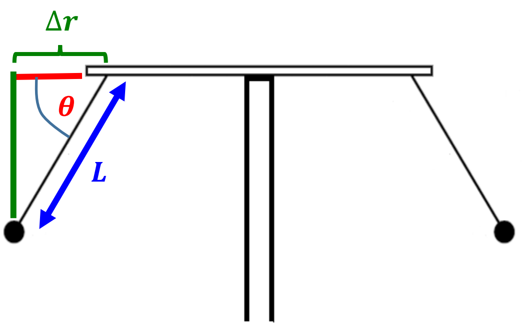
A moment arm is the length between a joint axis and the line of force acting on that point. This will be discussed in depth during the Torque and Gearing Module

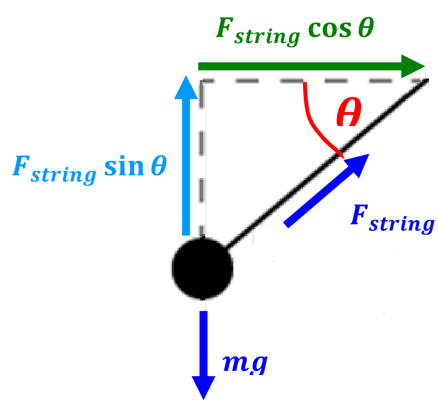
The setup for this investigation will be such that when the platform stationary the force acting on the string is

$$F_{string} = mg$$

where F_{string} is the tension in the string and m is a mass of the suspended object.

Because of the downward force of gravity on the mass, when it moves in a horizontal circle the string will extend at an angle θ to the horizontal support, as shown in the figure below. In this configuration, L is the length of the string, measured from the bottom of the platform to the center of the mass. The radius Δr of the change in circular path of the ball is given by $\Delta r = L\cos\theta$.





The forces on the ball are gravity and the tension in the string. The tension in the string is directed along the string while gravity is directed downward. The free-body diagram for a rotating mass is given in the diagram above. Since the mass moves in a horizontal circle, its linear acceleration is horizontal. This makes it convenient to use coordinates that are horizontal and vertical, and in the force diagram F_{string} has been resolved into its horizontal and vertical components. The gravitational force pulling the mass downward is balanced by the vertical component of F_{string}.

$$F_y = mg - F_{string} \sin\theta = 0$$

The horizontal component of F_{string} can be determined as

$$F_x = F_{string} \cos\theta = ma_c = m\frac{4 \times \pi^2 \times r}{t^2}$$

where $\Delta r = L\cos\theta$. Therefore,

$$F_{string} = m\frac{4 \times \pi^2 \times L}{t^2}$$

In this equation, F_{string} is the centripetal force causing the circular motion of the mass m suspended from the platform and L is the length of string from the center of the mass to bottom of the platform. Note that the angle θ that the string makes with the horizontal does not appear in calculation. This assumes that the acceleration of the mass moving at constant speed v in a circular path of radius r has the magnitude v^2/r, that is radially inward toward the center of the circular path.

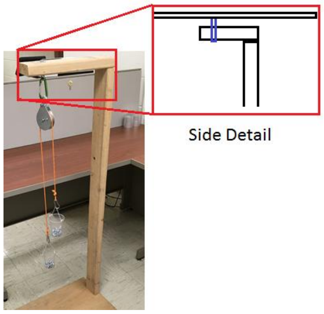
Newton's second law reveals that the acceleration of an object is inversely proportional to mass of the object. In other words, the bigger the mass value is, the smaller that the acceleration value will be. As mass increases, the acceleration decreases. This also shows that the net force required for an object to move in a circle is directly proportional to the square of the speed of the object. For a constant mass and radius, the F is proportional to the v^2.

The magnitude by which the force is altered is the square of the magnitude by which the speed is altered. Subsequently, if the speed of the object is doubled, the force required for that object's circular motion is quadrupled. And if the speed of the object is halved (decreased by a factor of 2), the force required is decreased by a factor of 4.

Circular Motion Investigation

An investigation into angular velocity, centripetal acceleration, and forces acting on an object undergoing circular motion can be accomplished by building a frame to support a rotating platform. Identical masses can be suspended from both ends of the platform using nylon string. Placing identical masses at opposing positions from the central axis of the platform will ensure the platform remains balanced during the experimental process.

This investigation can be constructed by modifying the support stand utilized in the Pulley, Energy Storage, and Circular Motion investigation apparatus



A simple test apparatus includes a wood base made from 1/2-inch plywood with a vertical support and horizontal extension made from 2x4s. The boards can be assembled using construction screws (or some equivalent screw). If we make the height of the vertical support around 48-inches, and the base a minimum of a 12” by 12” square, we end up with sufficient stability and space for making detailed observations of the pulley in action.

A 1/4” x 3” bolt and nut can be inserted through the horizontal extension in a manner that the excess bolt material extends above the surface of the horizontal extension. A 3-foot piece of 1” x 3” MDF board can be used as a platform and balanced on top of the 3” bolt. This can be done by drilling a 1/2-inch deep 5/16-inch hole in the center of the MDF board and placed on top of the 3” bolt.

A setup like this will allow us to investigate how changing the mass, speed of rotation and distance from the center effect observable forces acting on circular motion. Using the circumference of motion, and the number of revolutions per given time period, we can determine the angular velocity, centripetal acceleration, and forces acting on the suspended masses. The use of 3/4-inch Zinc hex nuts provides accurate results due to the non-excessive masses of each hex nut. Other items could work as weights however, the masses must be able to be suspended from the rotating platform in a manner that ensures they do not come detached and fly across the room.

Initial Circular Motion Experiment

The overall objective of this experiment is to determine the angular velocity, centripetal acceleration, and centripetal force associated with an object undergoing circular motion.

Hypothesis: increases to rotational speed will result in increases to centripetal force in a 1:1 ratio.

Remember, it is good to make an initial hypothesis based upon what you already know. Perhaps based on our knowledge from previous investigations we could hypothesize that as the rate of rotational speed increases the associated centripetal force will increase in a one-to-one ratio. We will use this hypothesis to guide the analysis of our experimental results.

As always, our results need to be recorded in a lab notebook or instructor provided lab packet. Make sure you record your data as well as a description of the experiment(s) conducted. Also, record and any special comments, for example the platform is tilted to one side. Often these extra comments become important when thinking about our data (the platform tilting to one side might result in it rubbing on something, causing it to slow down excessively and result in calculation errors when using averages). An actual set of lab results for the initial experiment are shown here.

Revolution End Time	Revolution Start Time	Time Per Revolution
7.13s	**6.13s**	**1.0s**
8.16s	**7.13s**	**1.03s**
9.22s	**8.16s**	**1.06s**
10.34s	**9.22s**	**1.12s**
11.50s	**10.34s**	**1.16s**
Avg. time for 1 revolution		**1.07s**

Rotational Data of platform with no additional mass

No added mass	Radius (r)	Avg time per revolution (t)	Linear Velocity ($v = \frac{2\pi r}{t}$)	Angular Velocity ($\omega = \frac{v}{r}$)	Centripetal Acceleration ($a_c = \frac{v^2}{r}$)
Pos #1	$0.3556m$	$1.07s$	$2.09\,m/s$	$5.88\,rad/s$	$12.28\,m/s^2$
Pos #2	$0.3048m$	$1.07s$	$1.79\,m/s$	$5.87\,rad/s$	$10.51\,m/s^2$
Pos #3	$0.254m$	$1.07s$	$1.49\,m/s$	$5.87\,rad/s$	$8.74\,m/s^2$
Pos #4	$0.2032m$	$1.07s$	$1.19\,m/s$	$5.86\,rad/s$	$6.97\,m/s^2$
Pos #5	$0.1524m$	$1.07s$	$0.89\,m/s$	$5.84\,rad/s$	$5.20\,m/s^2$
			Average Angular Velocity	$5.86\,rad/s$	

Rotational Data of platform with added mass

No added mass	Mass (m)	Radius (r)	Time per revolution (t)	Angular Velocity ($\omega = \frac{v}{r}$)	Centripetal Acceleration ($a_c = \frac{v^2}{r}$)	Centripetal Force ($F_c = ma_c$)
Pos #1	$32g$	0.3556m	$2.0s$	$3.15\,rad/s$	$3.53\,m/s^2$	$0.113N$
Pos #2	$32g$	0.3048m	$1.9s$	$3.31\,rad/s$	$3.33\,m/s^2$	$0.107N$
Pos #3	$32g$	0.254m	$1.9s$	$3.31\,rad/s$	$2.78\,m/s^2$	$0.089N$
Pos #4	$32g$	0.2032m	$1.85s$	$3.37\,rad/s$	$2.34\,m/s^2$	$0.075N$
Pos #5	$32g$	0.1524m	$1.9s$	$3.31\,rad/s$	$1.67\,m/s^2$	$0.053N$

Let's take a moment and think about the provided data. We already know that when a fulcrum is located in the center of a balanced beam the relationship of mass and distance to fulcrum on one side must equal the product of mass and distance on the other side. The equation $m_1 l_1 = m_2 l_2$ tells us the beam will balance during this investigation. Ensuring the masses are suspended at equal distances from the axis of rotation will ensure that frictional forces are at a minimum and can be ignored in analysis of forces during the investigation.

For more information on balancing a beam with a stationary fulcrum refer to the Lever and Mechanical Equilibrium Module

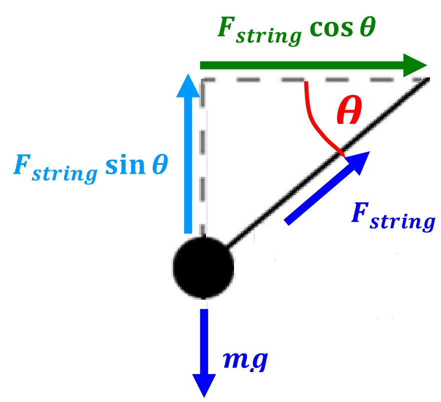
Use of the stopwatch camera app will allow for determining the time required for the platform to make one complete revolution. The times displayed above were collected to determine if the times for consecutive rotations can be averaged and used to obtain consistent angular velocity values for the proposed mass locations. Notice that the results were fairly consistent and resulted in an average angular velocity value of $5.86\,rad/s$, which allows us to have confidence in the proposed experimental setup. A more accurate measure of this value could be obtained by collecting more than five rotational speeds.

StopwatchCamera app available at app stores makes it possible to slow down the motion and collect accurate data

Next, look at the collected data for the suspended masses. This data set was collected with fairly consistent rotational speeds (ranging from $2s$ to $1.85s$ per revolution) with an average of $1.91s$, bolstering confidence in the collected data with additional mass. It now becomes a matter of completing calculations and comparing the data to ensure they are consistent with observable results.

The average angular velocity is $3.29\, rad/s$ with a high of $3.37\, rad/s$ and a low of $3.15\, rad/s$. Once again, we could say that more data collected would increase our accuracy, however we are still able to have confidence in the obtained results. We observe that centripetal acceleration decreased as the suspended mass got closer to the axis of rotation. This is consistent with the observed expected centripetal acceleration calculated with the platform when no mass was added.

Centripetal force values are a product of the mass and centripetal acceleration and followed the same decreasing pattern as the acceleration values. These values show that there should be more deflection of the suspended mass when suspended further from the axis of rotation than when the mass is suspended closer to the axis of rotation. This is consistent with the explanation that the hex nut is “pulled’ by the tension force.



The only forces acting on the hex nut are gravity and the tension in the string as seen in the free-body diagram to the right. The hex nut travels a circular path in a horizontal plane as a result of a horizontal acceleration directed to the center of the circular path. We have seen that the use of a coordinate system can resolve the force into x and y components. The gravitational force pulling the hex nut down is balanced by the vertical component of F_{string}.

$$F_y = mg - F_{string} \sin\theta = 0$$

The horizontal component of F_{string} can be determined as

$$F_x = F_{string} \cos\theta = ma_c = m\frac{4 \times \pi^2 \times r}{t^2}$$

where $F_{string} \cos\theta = F_c$. Therefore,

$$F_c = ma_c$$

In this equation, F_c is the centripetal force causing the circular motion of the hex nut of mass m suspended from the platform and a_c is the centripetal acceleration towards rotational axis of the platform. We can assume that the acceleration of the hex nut moving at constant speed v in a circular path of radius r has the centripetal acceleration v^2/r, that is radially inward toward the center of the circular path.

Refined Circular Motion Experiments

The first thing to question is whether something could be wrong with the experiment – experimental artifacts are a major source of errors in science. What could have gone wrong? Some ideas:

1. The system is out of align where the platform is not perfectly balanced. Verify alignment and redo the experiment to see if results change.

2. There is a problem with the angular velocity. Redo the experiment with a larger sample of rotations from which to obtain an average value for the suspended mass at each location.
3. The platform was not rotated fast enough to obtain an accurate centripetal acceleration and force magnitude. Increase/decrease the speed at which the platform is rotated and see if change in speed has an unexpected effect on the centripetal acceleration and force. Note: be aware of the mathematical relationships inherent in the formulas used to obtain an acceleration and/or force value?

If after these additional experiments, the initial observation still holds we can have confidence in our investigation results.

Inquiry Experiments

After you have completed the analysis in Lessons 3 you may want to improve understanding of what is happening in order to ask better questions about circular motion. Some interesting questions to explore include:

1. How does the relative distance from the central axis affect the motion of an object during circular motion? Is it directly or inversely proportional?
2. Can you manipulate the center of gravity (CG) to predict at what point the block will flip over? Will it ever flip, if the platform is shaped like a bowl?
3. How does the size affect the motion of the block during circular motion? What would happen if an object that contained water was used instead of a block?
4. How do you think this activity relates to the handling of a car? Would a car act the same as the block? Can you design an experiment that would allow you to test this hypothesis?
5. What other materials that can be added to the block in this experiment? What about the stationary surface? Find a variety of materials to pair up. Which surfaces can you find with the lowest friction coefficients? Which have the highest?
6. What would happen if your tether were attached to a hinge instead of the block itself? Would the block flip?

Lesson 4: Advanced Circular Motion

In order for circular motion to be uniform the motion must be at a constant speed. If we define the radius of the circular path as r and the time it takes to complete one trip around the circular path as t, then the speed can be obtained by dividing the circumference of the circular path by the time period it takes to complete a single trip around the circular path.

$$v = \frac{2\pi r}{t}$$

A similar equation relates the magnitude of the acceleration to this derived speed:

$$a = \frac{v^2}{r}$$

This is known as the centripetal acceleration and is the special form of acceleration when we're dealing with objects experiencing uniform circular motion.

Warning: Often when talking about circular motion, the term centripetal force is used, and this can be misleading. The phrase centripetal acceleration rant than centripetal force

Lesson 4: Objectives

- **Describe how circular motion on an incline compares to circular motion on a flat surface.**

Content

- **Uniform Circular Motion**
- **Go Karts on an Unbanked Curve**
- **Go Karts on a Banked Curve**

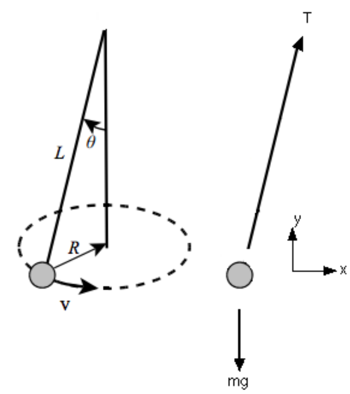
will be utilized whenever possible. Centripetal force can be a misleading term because, unlike other forces like tension, gravitational force, normal force, and the force of friction, centripetal force should not appear on a free-body diagram. You do NOT put a centripetal force on a free-body diagram for the same reason that ma does not appear on a free body diagram; $F = ma$ is the net force, and in the case of circular motion the net force happens to have the special form

$$F = \frac{mv^2}{r}$$

The centripetal force is not something that mysteriously appears whenever an object is traveling in a circle; it is simply the special form of the net force.

Uniform Circular Motion

We know that whenever an object experiences a change in its motion there will be a net force acting on it. During uniform circular motion the net force acting on the object will be pointing towards the center of the circular path, and because it points in to the center of the circle, at right angles to the velocity, the force will change the direction of the velocity but not the magnitude.



Sample Problem 1 – Take for example an object tied to a rope in a horizontal circle. (Note that the object travels in a horizontal circle, but the rope itself is not horizontal).

The free-body diagram shows just two forces, the tension and gravity. We can set up a coordinate system that is horizontal and vertical to the object, because the centripetal acceleration will be horizontal, and there is no vertical acceleration.

This divides the tension, T, into horizontal and vertical components. Now we can solve for the tension by adding all the forces in the y direction as follows

$$F_y = T\sin\theta + mg = 0$$

This can be rearranged and solved for the angle

$$\theta = \sin^{-1}\frac{mg}{T} = \sin^{-1}\frac{(3.7kg)(9.8\,m/s^2)}{100N} = 21.3^o$$

The x direction forces are equal to the mass times the centripetal acceleration:

$$F_x = T\cos\theta = ma_x = \frac{mv^2}{r}$$

If the tension in the rope is 100 N, the object's mass is 3.7 kg, and the rope is 1.4 m long, what is the angle of the rope with respect to the horizontal, and what is the speed of the object? We know the mass, tension, length of rope and the angle. We are only left with determining the radius r of the circular path the ball travels. In this case r is not simply the length of the rope. It is the horizontal component of the 1.4 m (let's call this L, for length), and there is a cosine factor coming into the r as well ($r = L\cos\theta$). Therefore, …

$$T\cos\theta = \frac{mv^2}{r} = \frac{mv^2}{L\cos\theta}$$

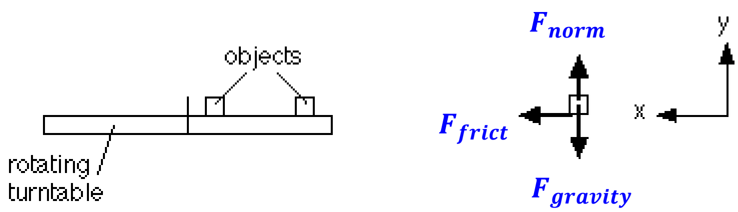
Rearranging this to solve for the speed gives:

$$v = \sqrt{\frac{(T\cos\theta)(L\cos\theta)}{m}} = \sqrt{\frac{(100N \times \cos 21.3)(1.4m \times \cos 21.3)}{3.7kg}} = 5.73\,m/s$$

which gives a speed of $v = 5.73\,m/s$.

Sample Problem 2 – In the case of identical objects on a turntable, different distances from the center, the free-body diagram has three forces, the force of gravity, the normal force, and a frictional force. The friction here is static friction, because even though the objects are rotating, they are not moving relative to the turntable. Recall that if there is no relative motion, you have static friction. The frictional force also points towards the center; the frictional force acts to oppose any relative motion, and the object has a tendency to go in a straight line which, relative to the turntable, would carry it away from the center. So, a static frictional force points in towards the center.

Refer to the Sliding Friction Module for a reference on friction forces



Summing forces in the y-direction tells us that the normal force is balanced out by the gravitational force ($F_y = F_{norm} + F_{gravity} = 0$). The only force in the x-direction is the frictional force.

$$F_x = F_{frict} = ma_x = \frac{mv^2}{r}$$

The maximum possible value of the static force of friction is

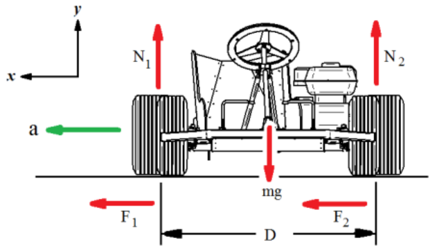
$$F_{frict} = \mu_s F_{norm}$$

As the velocity increases, the frictional force has to increase to provide the necessary force required to keep the object spinning in a circle. If we continue to increase the rotation rate of the turntable, thereby increasing the speed of an object sitting on it, at some point the frictional force won't be large enough to keep the object traveling in a circle, and the object will move towards the outside of the turntable and fall off.

Advanced Topic: Go-Kart On An Unbanked Curve

Suppose a go kart is going around an unbanked (level) curve. The diagram to the right shows a free-body diagram of the forces acting on the go-kart.

The downward force labeled "mg" is the weight of the kart. The upward force, labeled "N" is the normal force the road exerts on the car perpendicular to the surface of the road. We know that the kart is not accelerating vertically, these two forces must cancel, so that the vertical net force on the go-kart is zero.



However, there must be a horizontal net force on the go-kart. In the diagram, the horizontal force labeled "F" represents the horizontal friction force that the road exerts on the go-kart's tires. This friction force points toward the center of the turn and is responsible for turning the go-kart. So, in an unbanked turn, the force responsible for turning the kart is the friction force between the tires and the road.

Another very important feature of the above diagram is, there is NO outward-pointing force. The friction force points toward the center of the turn (circle). From the free-body diagram above and Newton's Second Law we can write

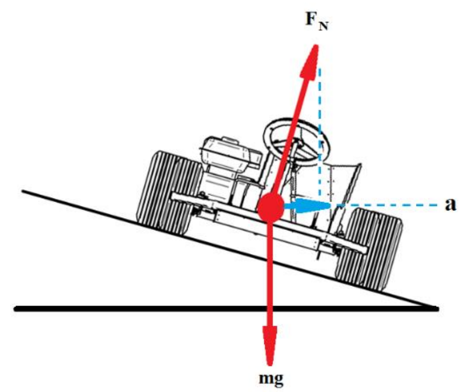
$N = mg$ (vertical forces cancel)

$F_{net} = F_c = F_{frict}$ (horizontal net force)

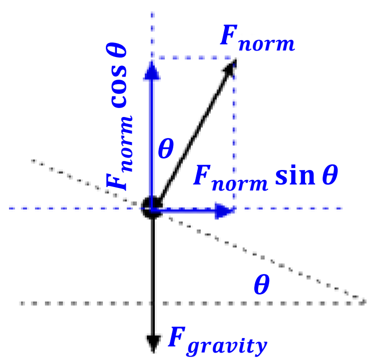
Remember that $F_c = mv^2/r$ and the friction force, $F_{frict} = \mu F_{norm}$, and you can handle these types of scenarios.

Advanced Topic: Banked Curve – No Friction

If a go-kart is on a level frictionless surface (unbanked turn), the forces acting on the go-kart are its weight, mg, and the normal force, F_{norm}, due to the road. Both forces act in the vertical direction. If there is no friction, there is no horizontal force that can supply the centripetal force required to make the go-kart move in a circular path, there is no way that the go-kart can turn.



On the other hand, if the go-kart is on an unlevel surface (banked turn) the normal force (which is always perpendicular to the road's surface) now has a horizontal and vertical component. The horizontal component can act as the centripetal force on the go-kart. Given just the right speed, a go-kart could safely negotiate a banked curve using the horizontal component of the normal force.



A free-body diagram for the go-kart on the banked turn is shown at left. The normal force, F_{norm}, has been resolved into horizontal and vertical components (the blue vectors).

In the vertical direction there is no acceleration, and:

$$F_{norm}\cos\theta = mg$$

so:

$$F_{norm} = \frac{mg}{\cos\theta}$$

In the horizontal direction:

$$F_{norm} = F_c = F_{norm}\sin\theta = \left(\frac{mg}{\cos\theta}\right)\sin\theta = mg\tan\theta$$

Since $F_{norm} = F_c$:

$$mg\tan\theta = \frac{mv^2}{r}$$

Solving for v gives:

$v = \sqrt{rg\tan\theta}$

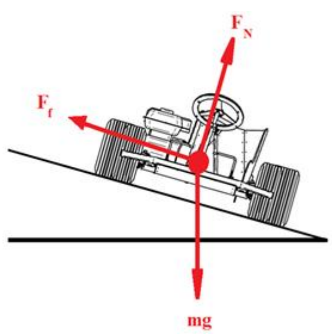
A go-kart moving at velocity v will successfully round the curve.

Note: Your initial thought might have been to resolve the weight vector parallel and perpendicular to the road - after all, that is what was done for all of those lovely inclined plane problems, remember? The difference is that we expected the object to accelerate parallel to the incline, so it made sense to have the vectors pointing parallel and perpendicular to the incline. Here, though, the acceleration is horizontal - toward the center of the car's circular path - so it makes sense to resolve the vectors horizontally and vertically.

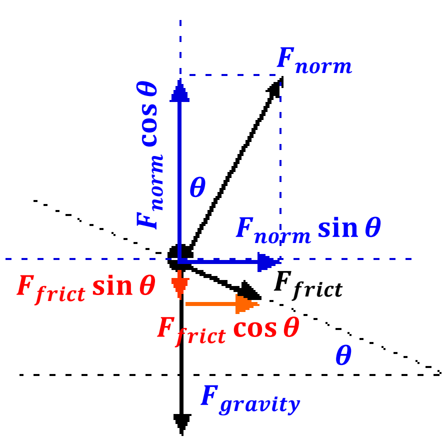
Advanced Topic: Banked Curve – With Friction

Suppose we consider a go-kart going around a banked turn where the centripetal force *needed* to turn the go-kart (mv^2/r) depends on the speed of the go-kart (since the mass of the go-kart and the radius of the turn are fixed) - more speed requires more centripetal force, less speed requires less centripetal force. The centripetal force *available* to turn the go-kart (the horizontal component of the normal force $F = mg\tan\theta$ if you followed the mathematics previously) is fixed (since the mass of the go-kart and the bank angle are fixed). It should make sense that we found one particular speed at which the centripetal force needed to turn the go-kart equals the centripetal force supplied by the road. This is the "ideal" speed, v_{ideal}, at which the go-kart will negotiate the turn - even if it is covered with perfectly smooth ice. Any other speed will require a friction force between the go-kart's tires and the surface to keep the go-kart from sliding up or down the embankment.

$\boldsymbol{v < v_{ideal}}$ **:** If the speed of the go-kart, v, is less than the ideal (no friction) speed for the turn, v_{ideal}. In this case, the horizontal component of the normal force will be greater than the required centripetal force and the go-kart will "want to" slide down the incline toward the center of the turn. If there is a friction force present between the go-kart's tires and the road it will oppose this relative motion and pull the go-kart up the incline.



A free-body diagram for the kart is shown below. Both the normal force, F_{nrom} (blue components) and the friction force, F_{frict} (red components) have been resolved into horizontal and vertical components. Notice that there are now 3 vectors in the vertical direction (there were 2 vectors in the no-friction case), and:



$$F_{norm}\cos\theta = mg + F_{frict}\sin\theta$$

Using the approximation $F_{frict} = \mu F_{norm}$, where μ is the coefficient of friction, gives:

$$F_{norm}\cos\theta - \mu F_{norm}\sin\theta = mg$$

Which can be rearranged and solved for F_{norm} as

$$F_{norm} = \frac{mg}{(\cos\theta - \mu\sin\theta)}$$

If the coefficient of friction was zero, this would reduce to the same normal force as we derived for no-friction scenario earlier, which is reassuring. If the coefficient of friction is not zero, the normal force will be larger.

In the horizontal direction:

$$F_{norm} = F_{norm}\sin\theta + F_{frict}\cos\theta = F_{norm}\sin\theta + \mu F_{norm}\cos\theta$$

Which can be rearranged and solved for F_{norm} as

$$F_{norm} = \frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} \times mg$$

Here, the term $F_{frict}\cos\theta$ is friction's contribution to the centripetal force. Since the net force provides the centripetal force to turn the kart:

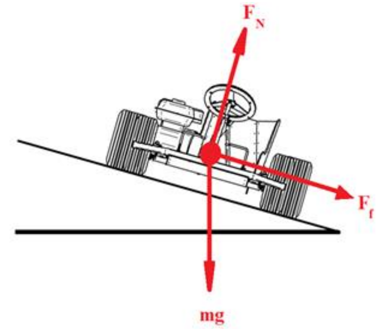
$$F_{norm} = F_c$$

Therefore

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} \times mg = \frac{mv^2}{r}$$

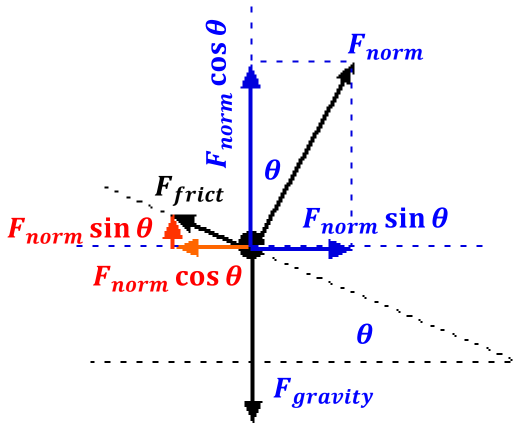
This can be solved for the velocity v as

$$v = \sqrt{\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right) \times rg} = \sqrt{\left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right) \times rg}$$



$\boldsymbol{v > v_{ideal}}$ **:** If the speed of the go-kart, v, is greater than the ideal speed for the turn, v_{ideal}, the horizontal component of the normal force will be less than the required centripetal force, and the go-kart will "want to" slide up the incline, away from the center of the turn. The friction force will oppose this motion and will act to pull the go-kart down the incline, in the general direction of the center of the turn.

A free-body diagram for the kart is shown at left. Both the normal force, F_{norm} (blue components) and the friction force, F_{frict} (red components) have been resolved into horizontal and vertical components. Notice that the friction force acts up the incline, to keep the kart from sliding toward the center of the turn. This derivation is very similar to the previous case.



In the vertical direction:

$$F_{norm}\cos\theta + F_{frict}\sin\theta = mg$$

$$F_{norm}\cos\theta + \mu F_{norm}\sin\theta = mg$$

Which can be rearranged and solved for F_{norm} as

$$F_{norm} = \frac{mg}{(\cos\theta + \mu\sin\theta)}$$

In the horizontal direction:

$$F_{norm} = F_{norm}\sin\theta - F_{frict}\cos\theta = F_{norm}\sin\theta - \mu F_{norm}\cos\theta$$

Which can be rearranged and solved for F_{norm} as

$$F_{norm} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} \times mg$$

Once again, the term $F_{frict}\cos\theta$ is friction's contribution to the centripetal force. Therefore

$$\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} \times mg = \frac{mv^2}{r}$$

This can be solved for the velocity v as

$$v = \sqrt{\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right) \times rg} = \sqrt{\left(\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right) \times rg}$$

Module 11 Activities and Resources

Lesson 1: Describing Circular Motion PowerPoint Slide Deck
Angular Velocity & Acceleration Student Activity Sheet

Lesson 2: Circular Motion and Force PowerPoint Slide Deck
Centripetal Acceleration & Force Student Activity Sheet

Lesson 3: Circular Motion Investigation Data Sheet

Lesson 4: Advanced Circular Motion PowerPoint Slide Deck
Advanced Circular Motion Student Activity Sheet
Circular Motion Review PowerPoint Slide Deck
Circular Motion Practice Problems Review Sheet

The following is an incomplete list of referenced resources

http://www.physicsclassroom.com
https://courses.lumenlearning.com/physics/chapter/10-1-angular-acceleration/
https://openstax.org/books/college-physics
https://courses.lumenlearning.com/physics/chapter/6-3-centripetal-force/

The Purpose behind all resources within the hardware store science curriculum is to research the effective integration of STEM subjects into a physical science classroom. All material is organized from outside sources and solely intended to provide the researchers a framework for the development of original content based on experimental findings.

