"CENTROID" AND "CENTER OF MASS" BY INTEGRATION

Learning Objectives

1). To determine the *volume, mass, centroid* and *center of mass* using integral calculus.

2). To do an *engineering estimate* of the volume, mass, centroid and center of mass of a body.

Definitions

Centroid: **Geometric** *center* of a line, area or volume.

Center of Mass: **Gravitational** *center* of a line, area or volume.

The *centroid* and *center of mass* coincide when the **density** is **uniform** throughout the part.

Centroid by Integration

a). *Line*:

$$L = \int dL \qquad L\,\bar{x} = \int x_c \, dL \qquad L\,\bar{y} = \int y_c \, dL$$

b). *Area*:

$$A = \int dA \qquad A\,\bar{x} = \int x_c \, dA \qquad A\,\bar{y} = \int y_c \, dA$$

c). *Volume*:

$$V = \int dV \qquad V\,\bar{x} = \int x_c \, dV$$

$$V\,\bar{y} = \int y_c \, dV \qquad V\,\bar{z} = \int z_c \, dV$$

where: $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the centroid of the line, area or volume.

$(x_c)_i$, $(y_c)_i$, $(z_c)_i$ represent the centroid of the differential element under consideration.

Center of Mass by Integration

$$m = \int dm = \int \rho \, dV$$

$$m \, x_G = \int x_c \, dm = \int x_c \, (\rho \, dV)$$

$$m \, \bar{y} = \int y_c \, dm = \int y_c \, (\rho \, dV)$$

$$m \, \bar{z} = \int z_c \, dm = \int z_c \, (\rho \, dV)$$

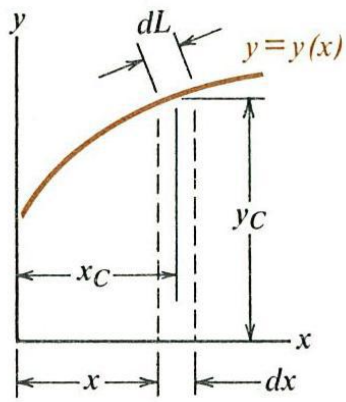
Note:

- For a homogeneous body ρ = constant, thus

$$m = \int \rho \, dV = \rho \int dV = \rho V$$

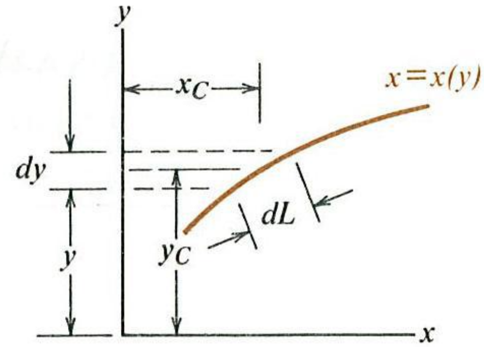
- Tabulated values of the *centroid* and *center of mass* of several standard shapes can be found on the back inside cover of the textbook.

Arch Length



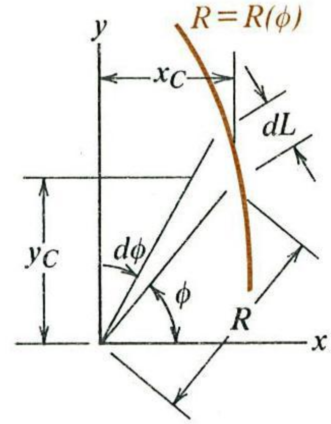
$dL = [1 + (\frac{dy}{dx})^2]^{1/2} dx$

$x_C = x,\ y_C = y(x)$



$dL = [1 + (\frac{dx}{dy})^2]^{1/2} dy$

$x_C = x(y),\ y_C = y$

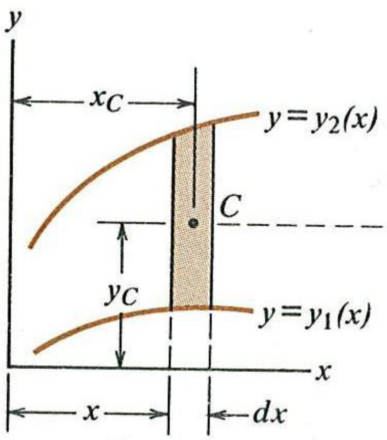


$dL = [(\frac{dR}{d\phi})^2 + R^2]^{1/2} d\phi$

$x_C = R(\phi) \cos \phi$

$y_C = R(\phi) \sin \phi$

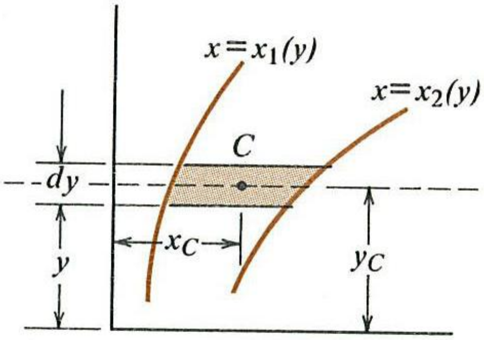
Planar Area



$d\mathscr{A} = [y_2(x) - y_1(x)]dx$

$x_C = x$

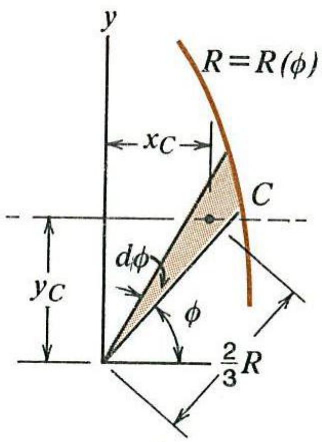
$y_C = \frac{1}{2}[y_2(x) + y_1(x)]$



$d\mathscr{A} = [x_2(y) - x_1(y)]dy$

$x_C = \frac{1}{2}[x_2(y) + x_1(y)]$

$y_C = y$

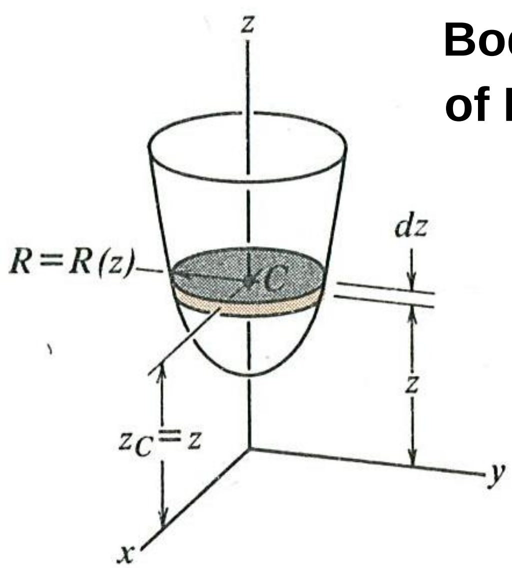


$d\mathscr{A} = \frac{1}{2}R^2 d\phi$

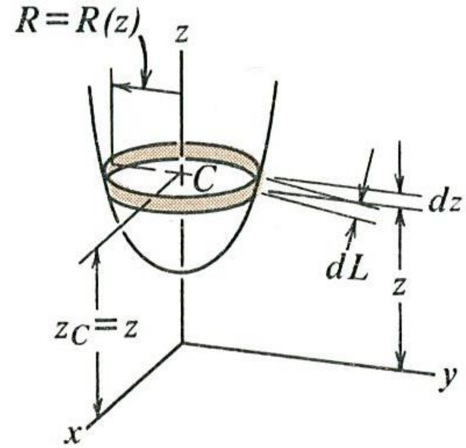
$x_C = \frac{2}{3}R(\phi) \cos \phi$

$y_C = \frac{2}{3}R(\phi) \sin \phi$

Body or Shell of Revolution



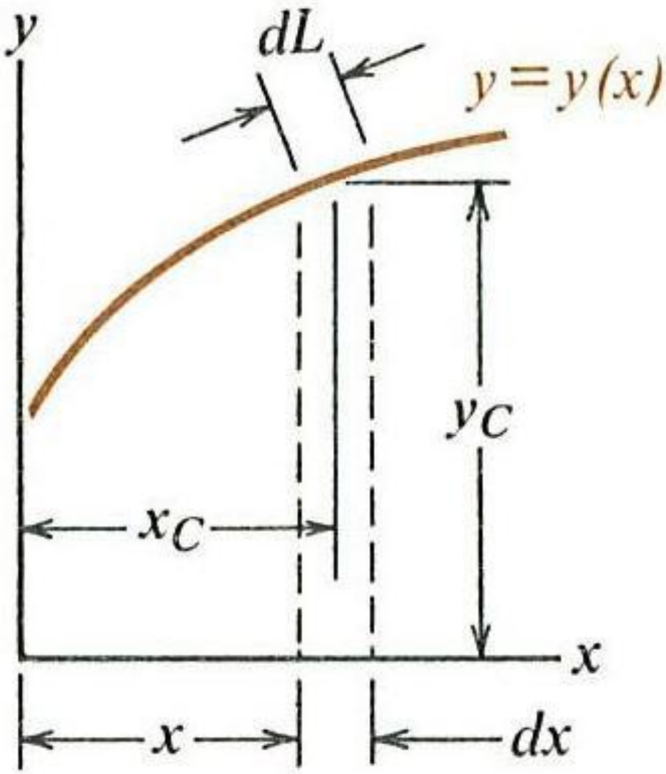
$d\mathscr{V} = \pi [R(z)]^2 dz$

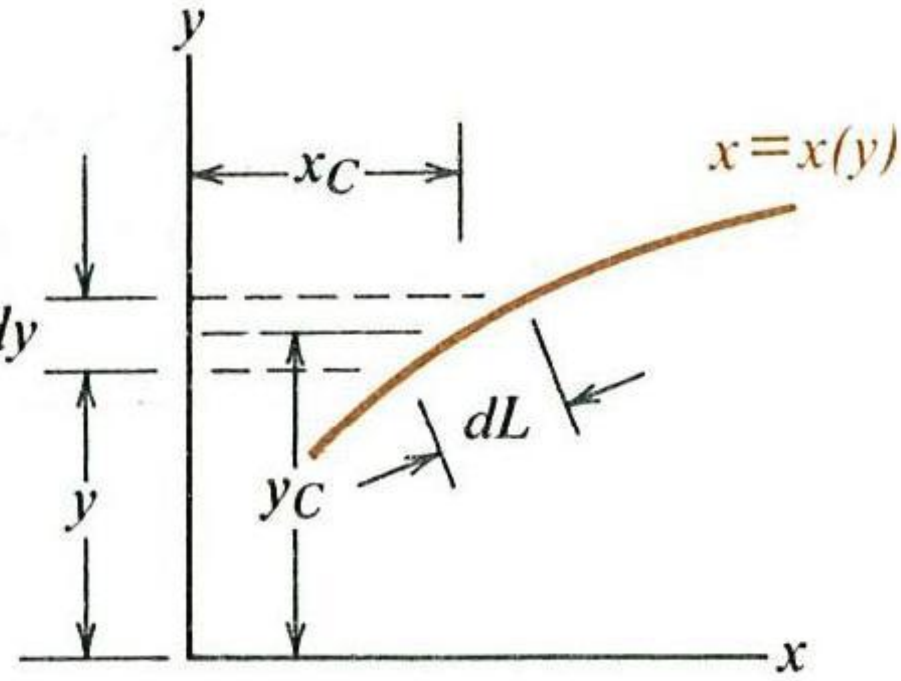


$d\mathscr{A} = 2\pi R(z) dL$

$= 2\pi R(z)[1 + (\frac{dR}{dz})^2]^{1/2} dz$

Arc Length





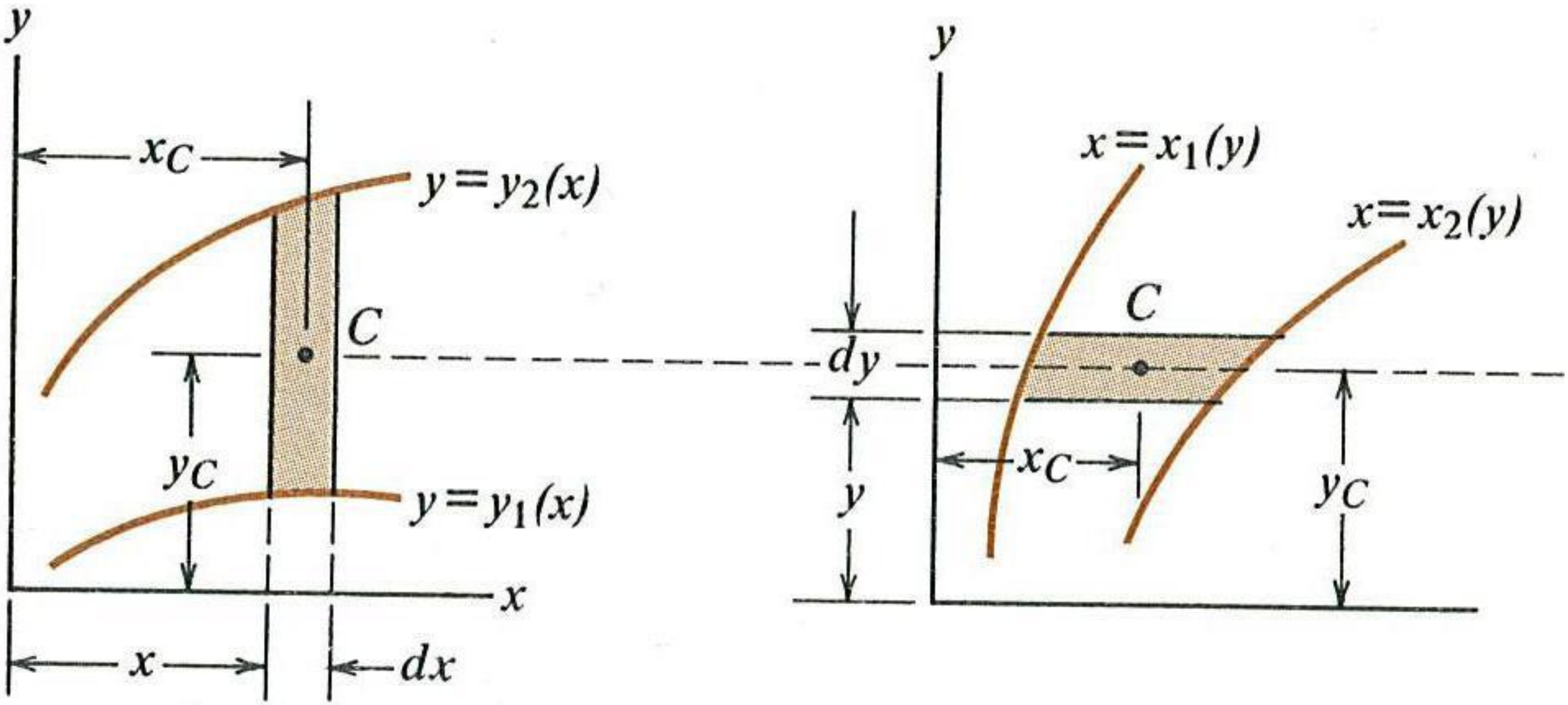
$$dL = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

$$x_C = x, \; y_C = y(x)$$

$$dL = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2} dy$$

$$x_C = x(y), \; y_C = y$$

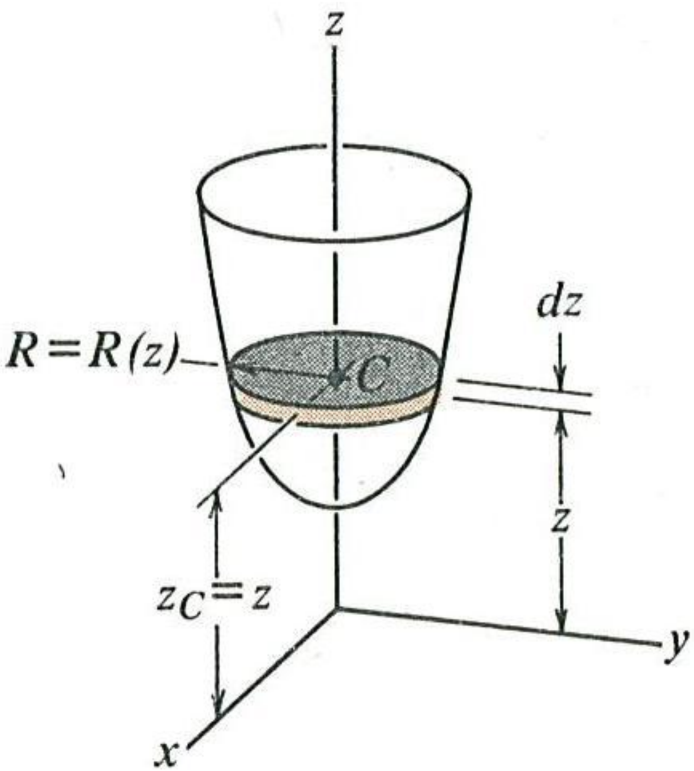
Planar Area



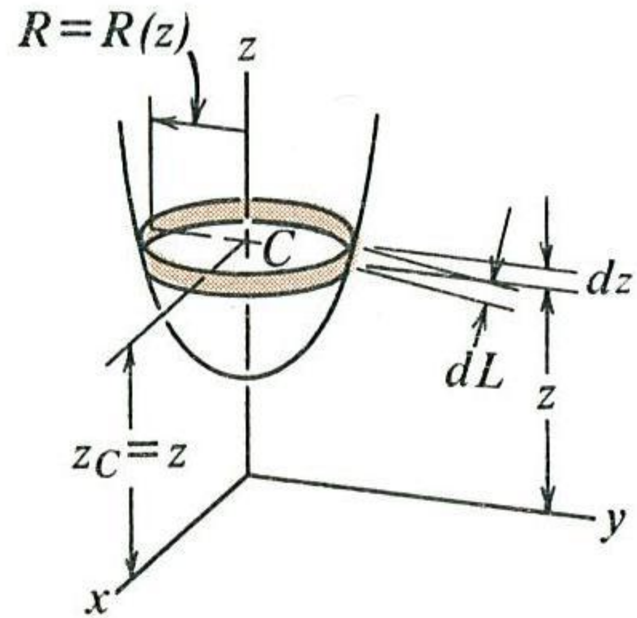
$$d\mathscr{A} = [y_2(x) - y_1(x)]\,dx$$
$$x_C = x$$
$$y_C = \tfrac{1}{2}[y_2(x) + y_1(x)]$$

$$d\mathscr{A} = [x_2(y) - x_1(y)]\,dy$$
$$x_C = \tfrac{1}{2}[x_2(y) + x_1(y)]$$
$$y_C = y$$

Body or Shell of Revolution



$$d\mathscr{V} = \pi [R(z)]^2 dz$$



$$d\mathscr{A} = 2\pi R(z) dL$$
$$= 2\pi R(z) [1 + (\frac{dR}{dz})^2]^{1/2} dz$$

Centroids and Center of Mass By Integration

Example 1

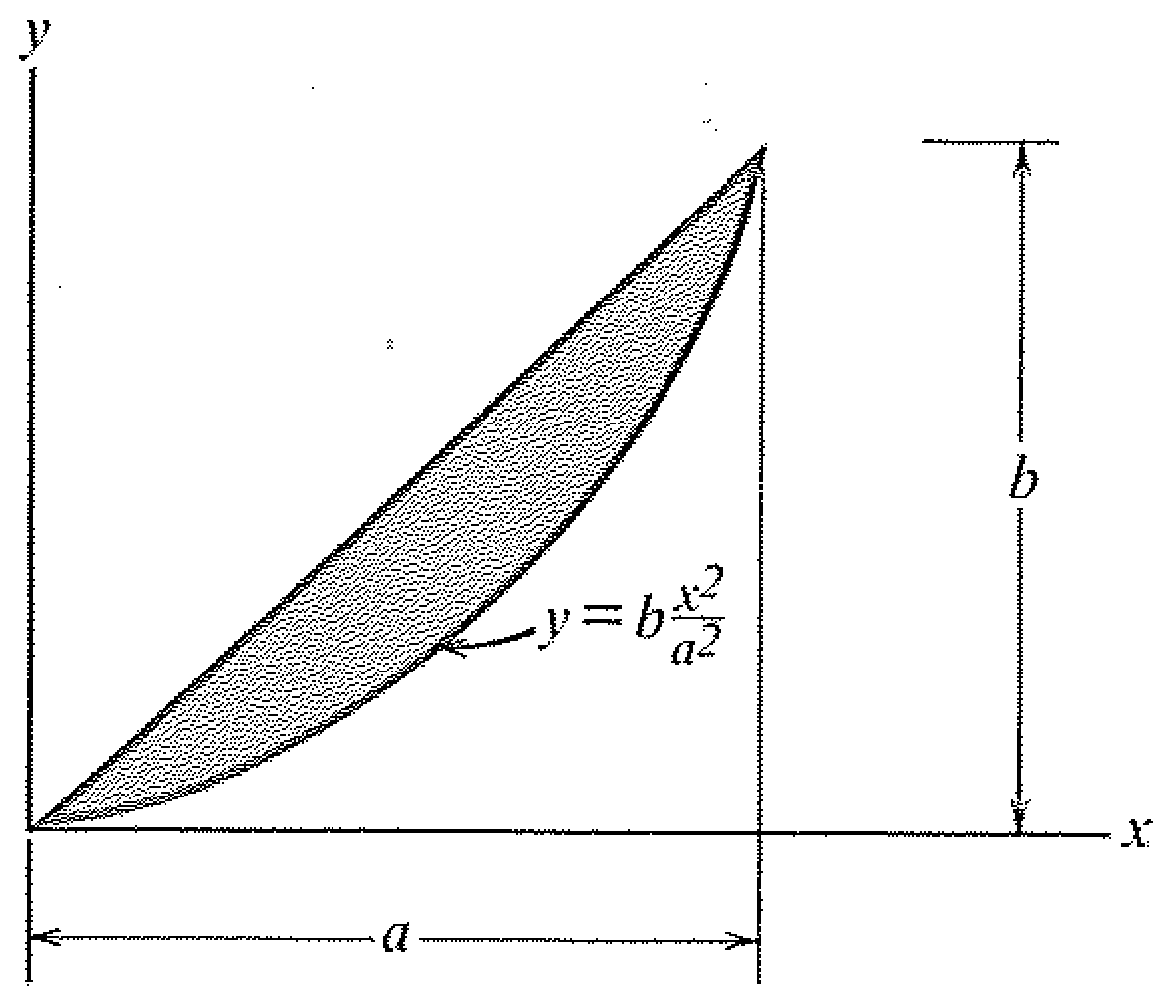
Given: It is desired to determine the area and centroids of the shaded shape.

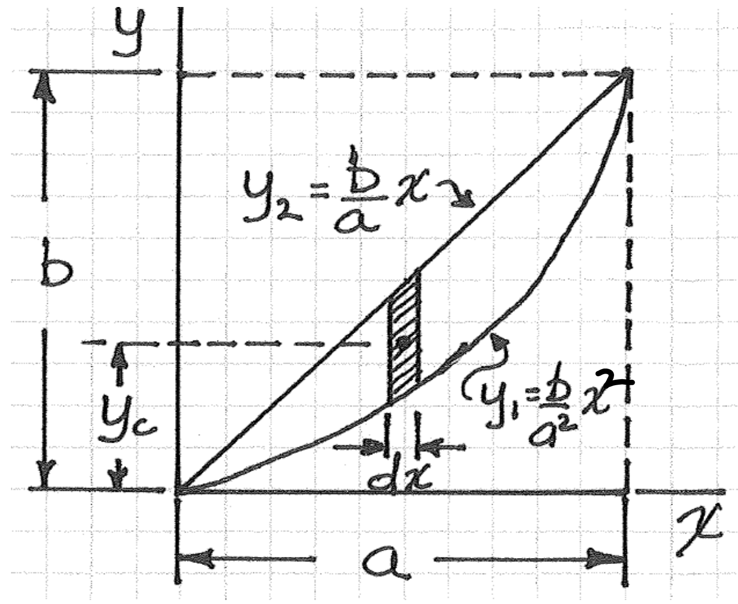
Find: For the shaded shape provided,

a) Estimate the area and the x and y centroids.

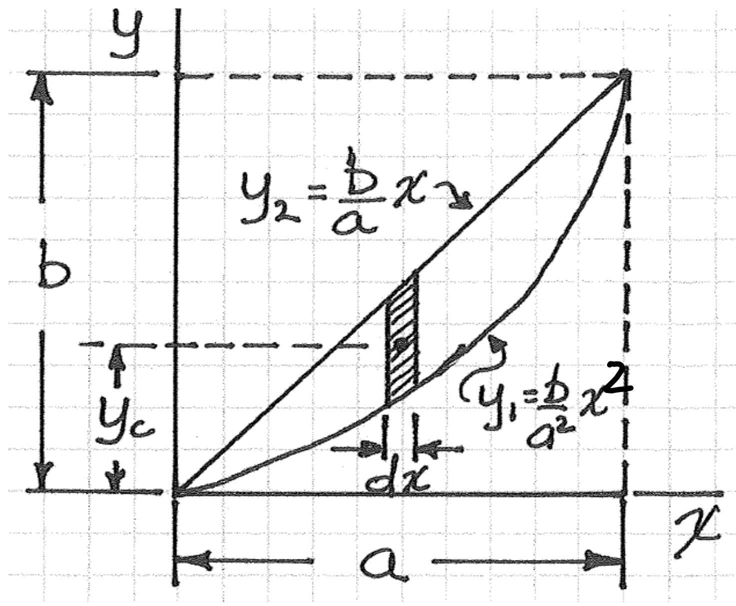
b) Calculate the area of the shape.

c) Calculate the x and y centroids of the shape.

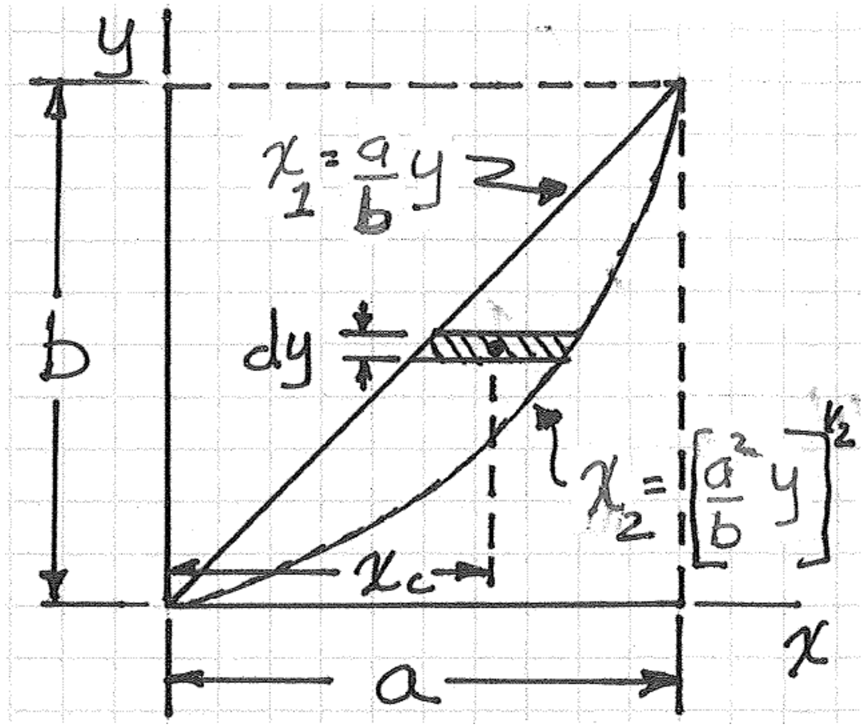


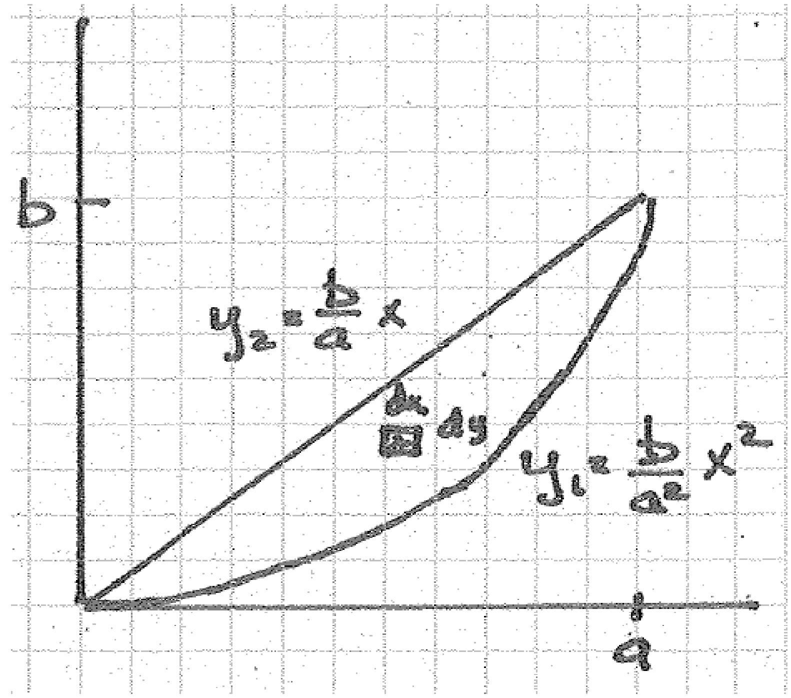


y
$y_2 = \frac{b}{a}x$
b
y_c
$y_1 = \frac{b}{a^2}x^2$
dx
x
a



y
$y_2 = \frac{b}{a}x$
b
y_c
$y_1 = \frac{b}{a^2}x^2$
dx
x
a


y
x
b
a
dy
x1 = a/b y
x2 = [a²/b y]^1/2
xc



b
$y_2 = \frac{b}{a}x$
dx
dy
$y_1 = \frac{b}{a^2}x^2$
a

Centroids and Center of Mass By Integration

Example 4

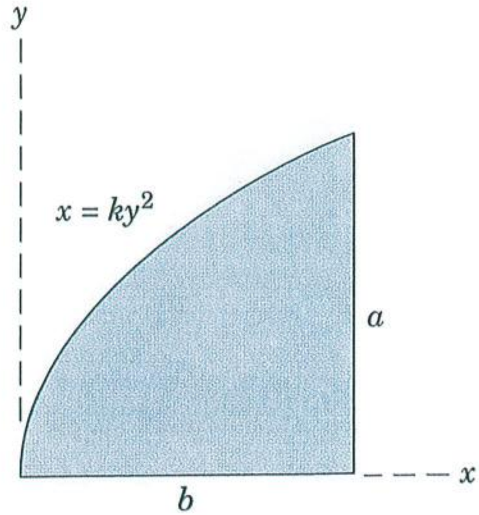
Given: The shaded area is bound by two curves.

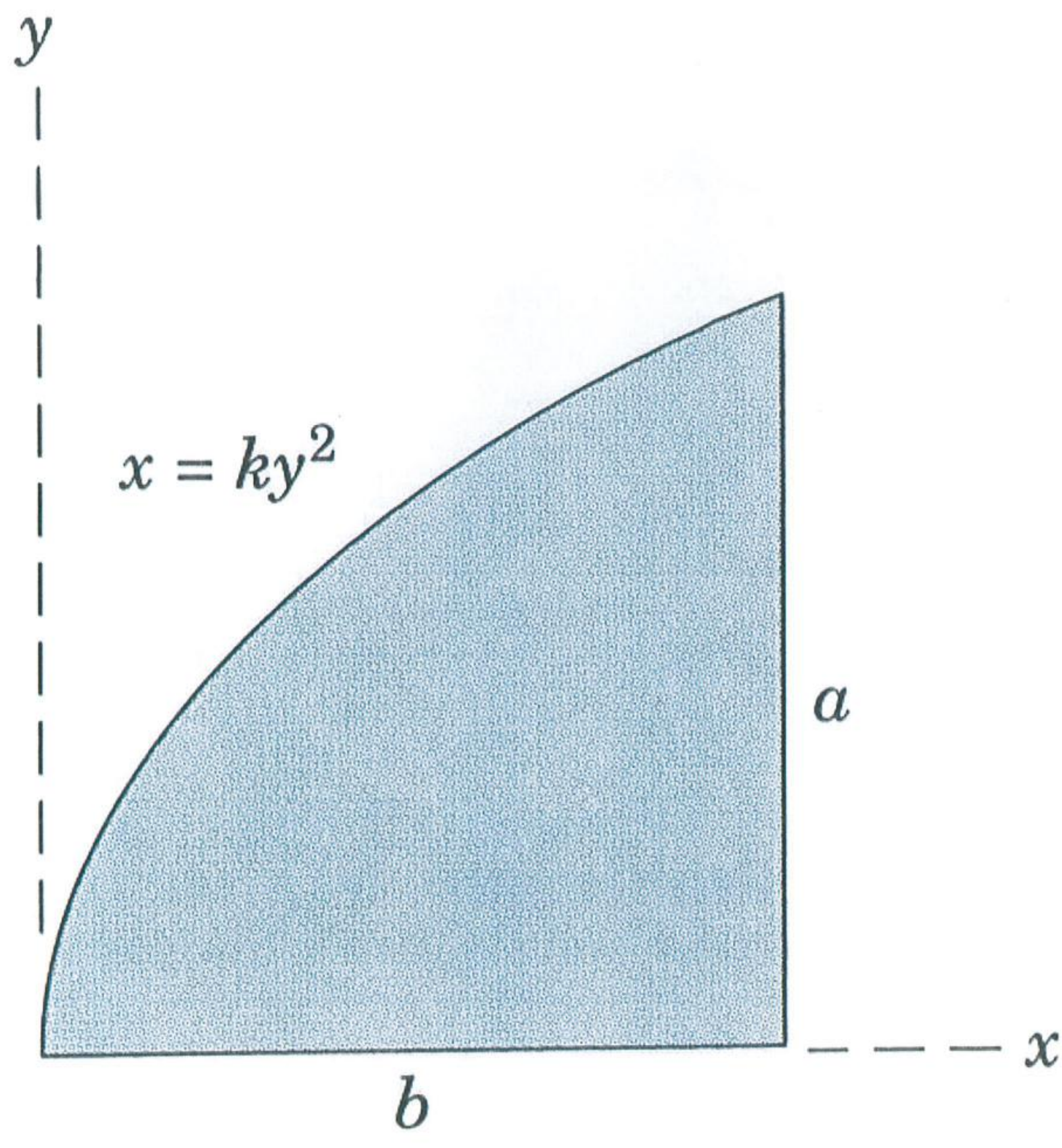
Find:

a) Estimate and then calculate the shaded area.

b) Estimate and then calculate the x-centroid of the shaded area.

c) Estimate and then calculate the y-centroid of the shaded area.





y
$x = ky^2$
a
x
b

Centers of Mass & Centroids: By Integration
Group Quiz 1

Group #: ____________ Group Members: 1) ______________________________
(Present Only)
Date: ______________ Period: _________ 2) ______________________________

3) ______________________________

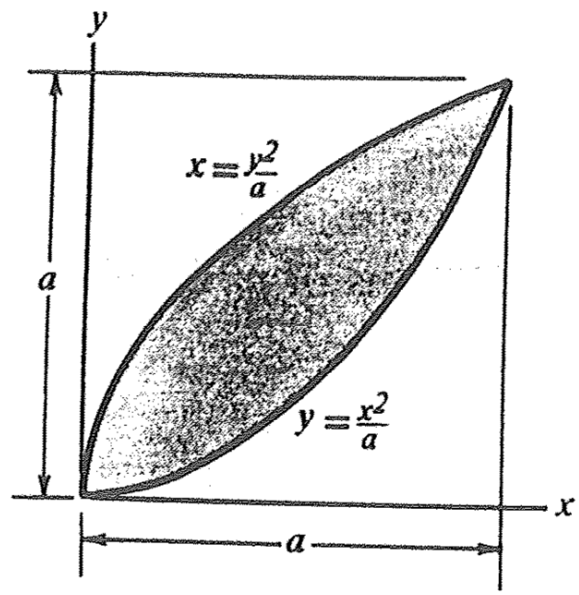
4) ______________________________

Given: A shaded area is bounded by two lines given by $x = y^2/a$ and $y = x^2/a$.

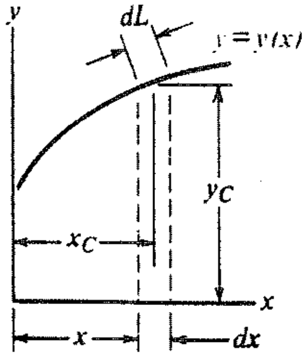
Find:

a) Do an engineering estimate of the shaded area and the centroid of the shaded area $(\bar{x}, \bar{y})$.

b) Determine the location of the centroid $(\bar{x}, \bar{y})$ by the method of integration.

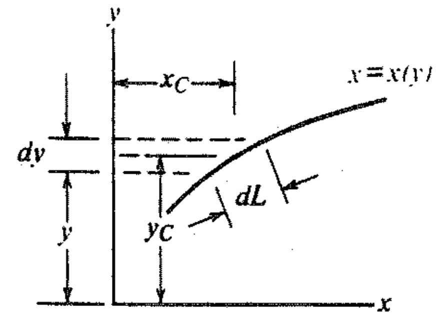


Solution:



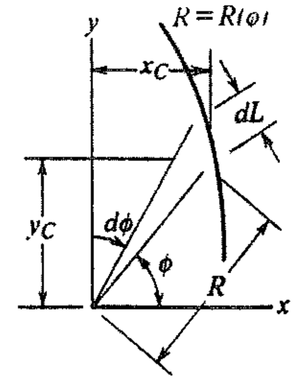
$dL = [1 + (\frac{dy}{dx})^2]^{1/2} dx$

$x_C = x,\ y_C = y(x)$



$dL = [1 + (\frac{dx}{dy})^2]^{1/2} dy$

$x_C = x(y),\ y_C = y$



$dL = [(\frac{dR}{d\phi})^2 + R^2]^{1/2} d\phi$

$x_C = R(\phi) \cos \phi$

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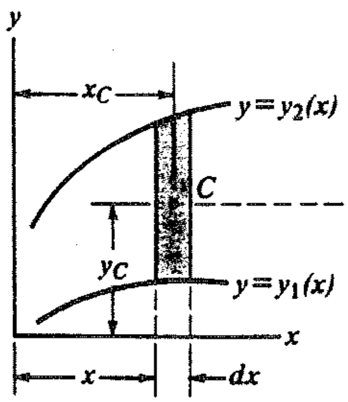
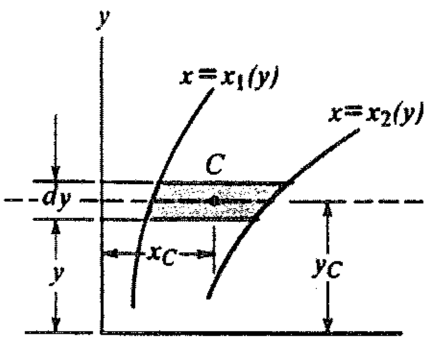
FIGURE 8



$d\mathscr{A} = [y_2(x) - y_1(x)]dx$

$x_C = x$

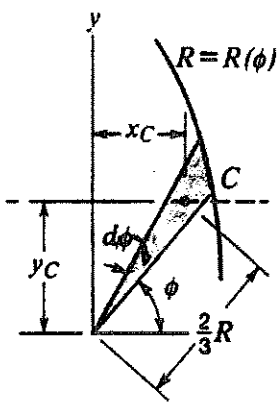
$y_C = \frac{1}{2}[y_2(x) + y_1(x)]$



$d\mathscr{A} = [x_2(y) - x_1(y)]dy$

$x_C = \frac{1}{2}[x_2(y) + x_1(y)]$

$y_C = y$

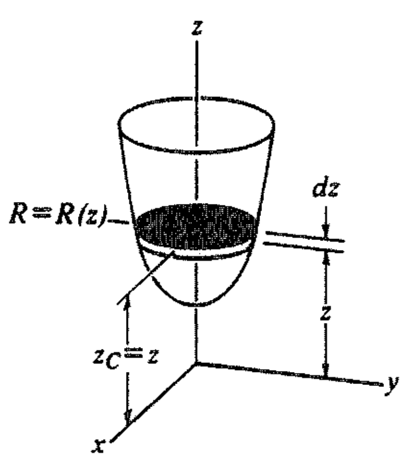


$d\mathscr{A} = \frac{1}{2} R^2 d\phi$

$x_C = \frac{2}{3} R(\phi) \cos \phi$

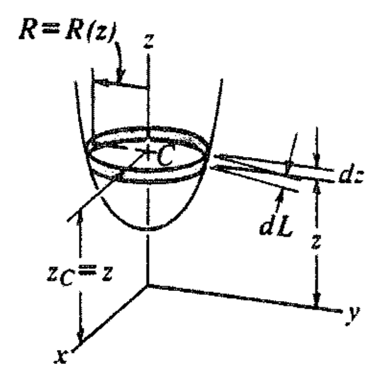
$y_C = \frac{2}{3} R(\phi) \sin \phi$

FIGURE 7



$d\mathscr{V} = \pi [R(z)]^2 dz$

FIGURE 9a



$d\mathscr{A} = 2\pi R(z) dL$

$= 2\pi R(z)[1 + (\frac{dR}{dz})^2]^{1/2} dz$

FIGURE 9b