ME 563 - Fall 2025

Homework No. 4

Due: Novemeber 17, 2025

ME 563 – Fall 25
Homework Problem 4.1

AI tools may be used to review background concepts (axial waves, dynamic boundary conditions) and to assist with plotting the two sides of the characteristic equation for visualization. They may not be used to derive the characteristic equation, solve it symbolically, or return the numerical roots η_n*. If AI is used as a study aid, please acknowledge it briefly in your submission (for example, "I consulted AI to review how* $\tan\eta$ *behaves near its asymptotes").*

A uniform elastic rod of length L, cross-sectional area A, modulus of elasticity E, and density ρ carries end masses M_1 at $x = 0$ and M_2 at $x = L$. Both end masses are mounted on rollers so that they are free to move axially in the x-direction. Let $u(x, t)$ denote the axial displacement of the rod, positive to the right.

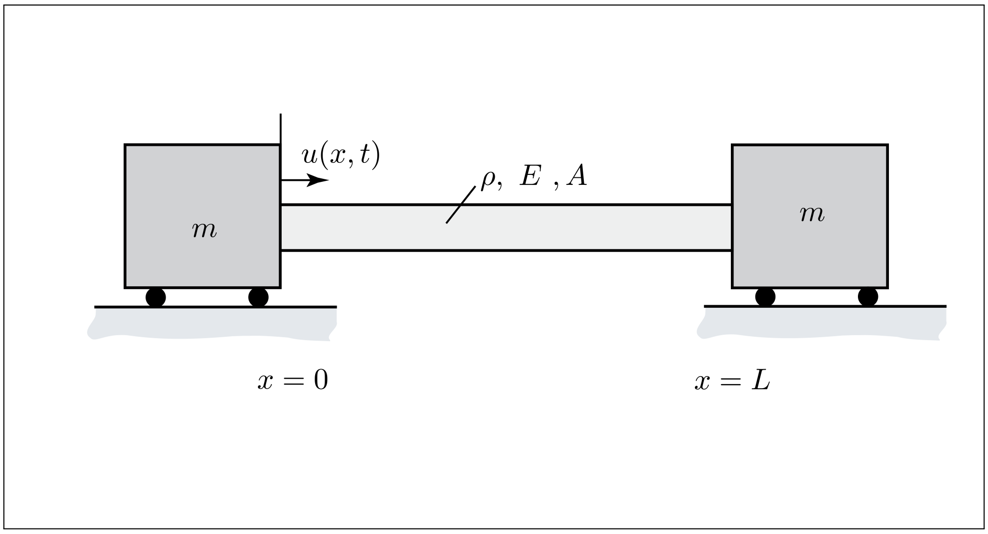

Figure 1: Uniform rod with two end masses on rollers. Positive displacement u is to the right.

(a) Governing PDE

Show that the axial vibration of the rod is governed by the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad c = \sqrt{\frac{E}{\rho}}, \qquad 0 < x < L.$$

(b) Boundary conditions (FBD approach)

Define the interface traction $N(t) = EA\,\frac{\partial u}{\partial x}$, which points in the normal direction from the cut on the beam. Show the following:

- At $x = 0$, equilibrium of mass M_1 gives

$$EA\,\frac{\partial u(0,t)}{\partial x} - M_1\,\frac{\partial^2 u(0,t)}{\partial t^2} = 0.$$

- At $x = L$, equilibrium of mass M_2 gives

$$EA\,\frac{\partial u(L,t)}{\partial x} + M_2\,\frac{\partial^2 u(L,t)}{\partial t^2} = 0.$$

(c) Characteristic equation

Assume harmonic motion $u(x, t) = U(x)T(t)$. The spatial solution is

$$U(x) = A\cos(\beta x) + B\sin(\beta x), \qquad \beta = \frac{\omega}{c}.$$

Show that applying the boundary conditions leads to the transcendental characteristic equation

$$\boxed{(\mu_1\mu_2\,\eta^2 - 1)\tan\eta = \eta(\mu_1 + \mu_2)}$$

where

$$\eta = \beta L = \frac{\omega L}{c}, \qquad \mu_1 = \frac{M_1}{m}, \qquad \mu_2 = \frac{M_2}{m}, \qquad m = \rho A L.$$

(d) Graphical solution

Plot

$$y_1(\eta) = (\mu_1\mu_2\eta^2 - 1)\tan\eta, \qquad y_2(\eta) = \eta(\mu_1 + \mu_2),$$

for $0 < \eta < 5\pi$ on the same axes. Estimate the first four roots $\eta_1, \eta_2, \eta_3, \eta_4$ by locating intersections. Pick values of μ_1 and μ_2. Do this by hand.

(e) Natural frequencies

Show that the natural frequencies are

$$\omega_n = \frac{c\,\eta_n}{L}, \qquad n = 1, 2, \ldots$$

(f) Mode shapes

From the boundary condition at $x = 0$, show that

$$B = -\mu_1\eta\,A.$$

Thus the unnormalized mode shape can be written

$$\boxed{U_n(x) = \cos\Big(\eta_n\,\tfrac{x}{L}\Big) - \mu_1\eta_n\sin\Big(\eta_n\,\tfrac{x}{L}\Big)}$$

for η_n satisfying the characteristic equation. Verify that $U_n(x)$ also satisfies the $x = L$ boundary condition. Note that the system also admits a rigid body mode is $U_1(x) = Ax + B$, where A and B can be determined from boundary conditions.

(g) Limiting cases and checks

Briefly justify how the characteristic equation reduces in the following cases:

1. $M_1, M_2 \to \infty$: both ends fixed $\Rightarrow$ $\eta = n\pi$.
2. $M_1 = 0$, $M_2 = M$: single tip mass at $x = L$ $\Rightarrow$ $-\tan\eta = \eta\frac{M}{m}$.
3. $M_1 = M_2 = 0$: free–free rod $\Rightarrow$ $\eta = (n)\pi$ and a rigid-body mode at $\eta = 0$.
4. $M_1 = M_2$: symmetric configuration — discuss even/odd families of modes.

(h) Numerical example

Take $M_1 = M_2 = m$ ($\mu_1 = \mu_2 = 1$). Using Matlab or Python, compute numerical approximations to the first three roots η_1, η_2, η_3 of the characteristic equation and report the corresponding frequencies $\omega_n = (c/L)\eta_n$.

ME 563 – Fall 25
Homework Problem 4.2

You may use AI tools to review beam theory concepts and to generate plots that aid root bracketing. Do not use AI to derive the characteristic equation or to compute the numerical roots λ_n. If AI is used as a study aid, please acknowledge it briefly in your submission (e.g., "I consulted AI to recall the cantilever basis functions $\cosh - \cos$ *and* $\sinh - \sin$*").*

A uniform Euler–Bernoulli beam of length L, flexural rigidity EI, and mass per unit length ρA is clamped at $x = 0$ and carries a concentrated tip mass M at $x = L$. Let $w(x,t)$ denote the transverse deflection (positive upward).

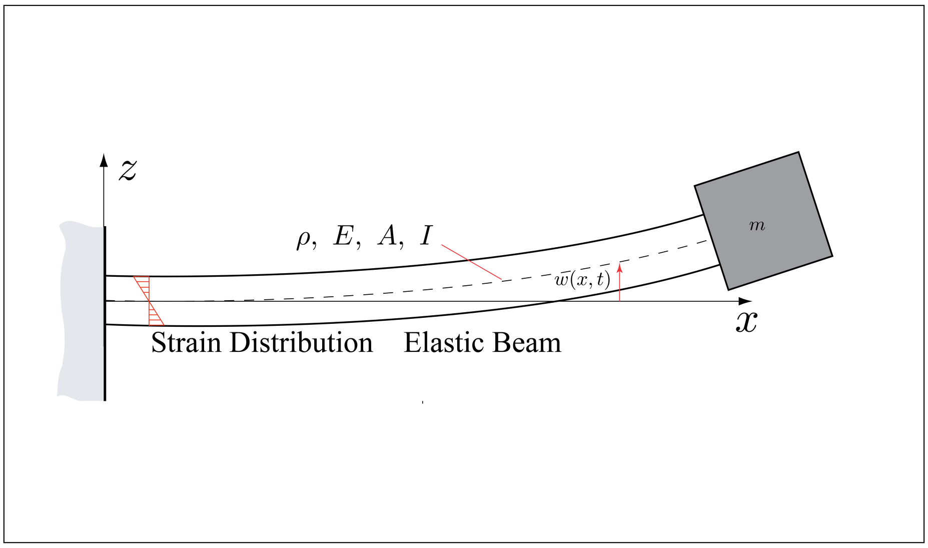

Figure 2: Cantilever beam with a tip mass at $x = L$. Sign convention: $w > 0$ upward; $M_b = EI\,w''$, $V = EI\,w'''$.

Modeling assumptions. Uniform, prismatic beam; Euler–Bernoulli kinematics; small deflections; no tip rotary inertia and no damping.

(a) Governing PDE

Show that the transverse motion satisfies

$$\frac{\partial^2 w_(x,t)}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 w(x,t)}{\partial x^4} = 0, \qquad 0 < x < L.$$

(b) Boundary conditions

State and justify each boundary condition using a short free-body diagram (FBD).

1. **Clamped root at $x = 0$:** $w(0,t) = 0, \quad w_x(0,t) = 0.$
2. **Free of applied moment at $x = L$:** $EI\,w_{xx}(L,t) = 0.$
3. **Shear balances tip inertia at $x = L$:** $EI\,w_{xxx}(L,t) = M\,w_{tt}(L,t).$

(c) Separation of variables and spatial basis

Assume $w(x,t) = W(x)\,T(t)$ and define

$$\beta^4 = \frac{\rho A\,\omega^2}{EI}, \qquad \lambda = \beta L.$$

Show that $W'''' - \beta^4 W = 0$ and that the clamped conditions at $x = 0$ reduce the general solution to

$$W(x) = C_1\big(\cosh\beta x - \cos\beta x\big) + C_2\big(\sinh\beta x - \sin\beta x\big).$$

(d) Characteristic equation (nondimensional form)

Apply the two $x = L$ boundary conditions to eliminate C_1, C_2 and derive the transcendental equation

$$\boxed{1 + \cosh\lambda\,\cos\lambda \; + \; \mu\,\lambda\big(\sinh\lambda\,\cos\lambda - \cosh\lambda\,\sin\lambda\big) = 0,}$$

where the *tip mass ratio* is $\mu = \dfrac{M}{\rho A L}$.

Sanity checks. (i) $\mu = 0 \Rightarrow 1 + \cosh\lambda\cos\lambda = 0$ (classic cantilever). (ii) $\mu \to \infty$ shifts the roots upward (heavier tip $\Rightarrow$ higher λ for the same ω).

(e) Graphical root bracketing

For a chosen μ, plot on the same axes over $0 < \lambda < 15$:

$$g_1(\lambda) = 1 + \cosh\lambda\cos\lambda, \qquad g_2(\lambda) = -\,\mu\,\lambda\big(\sinh\lambda\cos\lambda - \cosh\lambda\sin\lambda\big).$$

Use intersections of g_1 and g_2 to bracket the first three roots $\lambda_1, \lambda_2, \lambda_3$.

(f) Natural frequencies

Show that for each root λ_n of the characteristic equation,

$$\boxed{\omega_n = \left(\frac{\lambda_n}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}}.$$

(g) Mode shapes and normalization

Using the $x = L$ conditions, express C_2/C_1 at each λ_n and write an (unnormalized) mode shape as

$$W_n(x) = \Big(\cosh\beta_n x - \cos\beta_n x\Big) - \frac{\cosh\lambda_n - \cos\lambda_n}{\sinh\lambda_n + \sin\lambda_n}\Big(\sinh\beta_n x - \sin\beta_n x\Big), \quad \beta_n = \frac{\lambda_n}{L}.$$

Normalization: (i) set $W_n(L) = 1$; (ii) set the *modal mass*

$$m_n = \rho A \int_0^L W_n^2\,dx \; + \; M\,W_n(L)^2$$

to unity.

(h) Numerical Verification

Take $\mu = 1$. Use your plot from part (e) or a numerical root finder to estimate λ_1, λ_2 (to three significant figures) and compute the corresponding

$$\omega_n = \left(\frac{\lambda_n}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}, \qquad n = 1, 2.$$

Provide $W_n(L)$-normalized mode shapes $W_n(x)/W_n(L)$.

ME 563 – Fall 25
Homework Problem 4.3

An ideal flexible string under tension produces harmonic overtones $f_n = nf_1$. Real piano strings, however, possess finite bending stiffness that causes the overtones to be *anharmonic*, i.e., $f_n \neq nf_1$. The transverse motion of a stiff string (neglecting damping) is governed by

$$EI\,\frac{\partial^4 y(x,t)}{\partial x^4} + T\,\frac{\partial^2 y(x,t)}{\partial x^2} = \rho A\,\frac{\partial^2 y(x,t)}{\partial t^2}, \qquad 0 < x < L,\; t > 0, \tag{1}$$

where E is Young's modulus, I is the area moment of inertia, T is the string tension, ρA is the linear mass, and $y(x,t)$ is the transverse displacement.

(a) Non-dimensional form.

Introduce the dimensionless coordinate $\xi = x/L$ and derive the nondimensional PDE. Let the nondimensional coordinate be

$$\zeta = \frac{x}{L}.$$

By the chain rule we have

$$\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\,\frac{\partial}{\partial \zeta} = \frac{1}{L}\,\frac{\partial}{\partial \zeta}.$$

Hence, for any field $y(x,t) = y(\zeta,t)$,

$$\frac{\partial y}{\partial x} = \frac{1}{L}\,\frac{\partial y}{\partial \zeta} = \frac{1}{L} y_\zeta.$$

You can repeat the procedure for higher derivatives. Once you have the nondimensional form, show that the behavior depends on a single stiffness parameter

$$B = \frac{\pi^2 EI}{TL^2}. \tag{2}$$

Discuss physically what large or small values of B imply. Hint: Use the modeshapes of the fixed-fixed string, i.e., $U(\zeta) = S_n \sin(n\pi\zeta)$.

(b) Frequency equation and approximate modes.

Use the original dimensional PDE with assume pinned–pinned ends. Starting from the general solution $y = C_1 \sin(kx) + C_2 \cos(kx) + C_3 \sinh(kx) + C_4 \cosh(kx)$, apply boundary conditions to obtain the approximate natural frequencies

$$f_n \approx nf_1\sqrt{1 + Bn^2}, \qquad f_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho A}}. \tag{3}$$

(c) Numerical comparison.

For a steel piano string with

$$E = 2.0 \times 10^{11}\ \text{Pa}, \qquad \rho = 7800\ \text{kg/m}^3,$$
$$d = 1.2\ \text{mm}, \qquad L = 0.80\ \text{m}, \qquad T = 600\ \text{N},$$

compute B, the first six natural frequencies f_n, and the percentage deviation of $f_n/(nf_1)$ from unity. Present the results in a table.

(d) Discussion.

1. How does bending stiffness shift the higher overtones?
2. Why are long, thin, high-tension strings (treble) nearly harmonic, while short, thick, low-tension bass strings exhibit noticeable inharmonicity?

3. Which boundary condition (pinned or clamped) more closely models a real piano string? Justify briefly.

Wrapped bass string. For a wound string with steel core $d_c = 1.0$ mm and outer diameter $D = 4.0$ mm, use Fletcher's empirical relation for effective linear density

$$\rho A \approx 5.43 D^2 - 0.62 d_c^2 \quad \text{[g/cm]}$$

and assume bending stiffness dominated by the steel core. Estimate B and comment on the trend in inharmonicity compared with the solid string from part 3.

Reference: Harvey Fletcher, *"Normal Vibration Frequencies of a Stiff Piano String," Journal of the Acoustical Society of America*, Vol. 36, No. 1, 1964.