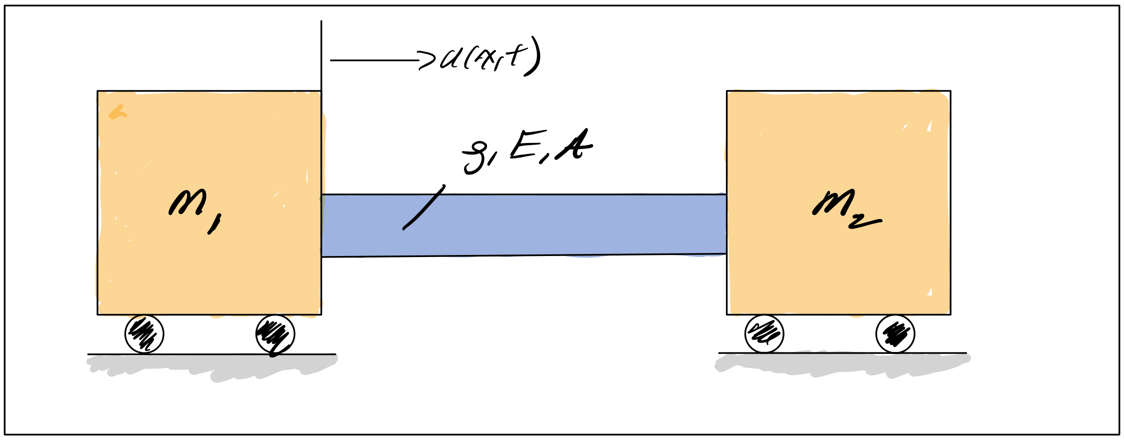
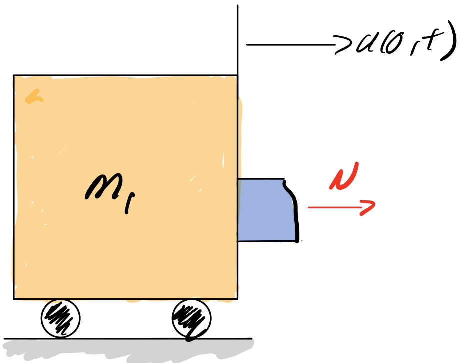
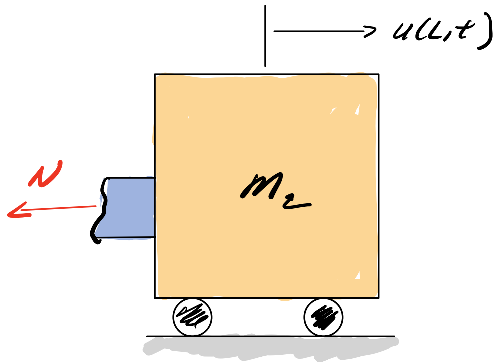
Homework 4.1







$\overset{+}{\rightarrow} \Sigma F: N = m\ddot{u}(0,t)$

$\overset{+}{\rightarrow} \Sigma F: -N = m\ddot{u}(L,t)$

① $$EA\frac{\partial u}{\partial x}(0,t) = m_1\ddot{u}(0,t)$$

② $$-EA\frac{\partial u}{\partial x}(L,t) = m_2\ddot{u}(L,t)$$

The boundary conditions

① $$EA\frac{\partial u}{\partial x}(0,t) = m_1\ddot{u}(0,t)$$

② $$-EA\frac{\partial u}{\partial x}(L,t) = m_2\ddot{u}(L,t)$$

The governing equation: $$EA\frac{\partial u}{\partial x} = \rho A\frac{\partial^2 u}{\partial t^2}$$

Separable soln. $u(x,t) = U(x)\,T(t)$

$$E\,U''(x)\,T(t) = \rho\,U(x)\,\ddot{T}(t) \longrightarrow \frac{E\,U''(x)}{\rho\,U(x)} = \frac{\ddot{T}(t)}{T(t)} = -\omega^2$$

① $U''(x) + \frac{\omega^2 \rho}{E} U(x) = 0 \longrightarrow U''(x) + \beta U(x) = 0$

$\beta = \omega\sqrt{\rho/E}$

② $\ddot{T}(t) + \omega^2 T(t) = 0$

The solutions are of the form:

$T(t) = C\cos\omega t + S\sin\omega t, \quad U(x) = a\cos\beta x + b\sin\beta x$

Now evaluate boundary conditions

$U(x) = a\cos\beta x + b\sin\beta x$

$U'(x) = -a\beta\sin\beta x + b\beta\cos\beta x$

$U(0) = a \quad , \quad U(L) = a\cos\beta L + b\sin\beta$

$U'(0) = b\beta \quad , \quad U'(L) = \beta(-a\sin\beta L + b\cos\beta L)$

$$EA \frac{\partial u(0,t)}{\partial x} = m_1 \frac{\partial^2 u(0,t)}{\partial t^2}$$

$$EA\, U'(0) T(t) = +m_1 U(0) \ddot{T}(t) = -m_1 \omega^2 U(0) T(t)$$

$$EA\, U'(0) = -m_1 \omega^2 U(0)$$

$$EA \frac{\partial u(L,t)}{\partial x} = -m_1 \frac{\partial^2 u(L,t)}{\partial t^2}$$

$$EA\, U'(L) T(t) = -m_1 U(L) \ddot{T}(t) = m_1 \omega^2 U(L) T(t)$$

$$EA U'(L) = m_1 \omega^2 U(L)$$

$$EA U'(0) = -m_1 \omega^2 U(0)$$

$$EA\, a\beta = -m_1 \omega^2 b \qquad \omega = \beta \sqrt{E/\rho}$$

$$EA\, a\beta = -m_1 \beta^2 \frac{E}{\rho} b$$

$$A b \beta = \frac{-m_1 \beta^2 a}{\rho}$$

$$b \beta = \frac{-m_1 \beta^2 a}{\rho A} \rightarrow b = \frac{-m_1 \beta}{\rho A} a$$

$$b = \frac{-m_1 \beta L}{\rho A L} a$$

μ_1

$$\underline{b = -\mu_1 \beta L a}$$

$$EAU'(L) = m_2\omega^2 U(L)$$
$$U(L) = a\cos\beta L + b\sin\beta$$
$$U'(L) = \beta(-a\sin\beta L + b\cos\beta L)$$

$$EA\beta(-a\sin\beta L + b\cos\beta L) = m_2\omega^2(a\cos\beta L + b\sin\beta L)$$
$$EA\beta(-a\sin\beta L - \mu_1\beta L a\cos\beta L) = m_2\omega^2(a\cos\beta L - \mu_1\beta L a\sin\beta L)$$
$$\frac{EA\beta}{m_2\omega^2}(\sin\beta L + \mu_1\beta L\cos\beta L) = (-\cos\beta L + \mu_1\beta L\sin\beta L)$$

$$\Rightarrow \frac{EA\beta}{m\omega^2} = \frac{EA\beta}{m\beta^2 E/\rho} = \frac{\rho A\beta}{m\beta^2} = \frac{\overbrace{\rho AL}^{1/\mu_2}}{m\beta L} = 1/\beta L\mu_2$$

$$(\sin\beta L + \mu_1\beta L\cos\beta L) = \mu_2\beta L(-\cos\beta L + \mu_1\beta L\sin\beta L)$$

$$(\sin\beta L + \mu_1\beta L\cos\beta L) = -\mu_2\beta L\cos\beta L + \mu_1\mu_2(\beta L)^2\sin\beta L$$

$$(\mu_1\beta L + \mu_2\beta L)\cos\beta L = (\mu_1\mu_2(\beta L)^2 - 1)\sin\beta L$$

$$(\mu_1 + \mu_2)\beta L = (\mu_1\mu_2(\beta L)^2 - 1)\tan\beta L \qquad \eta = \beta L$$

$$\boxed{(\mu_1 + \mu_2)\eta = (\mu_1\mu_2\eta^2 - 1)\tan\eta}$$

Modeshape

$$\boxed{U_n = a_n\cos(\beta_n x) + \mu_1\eta_n a_n\sin\beta_n x}$$

Again exami

$\tan\beta L =$

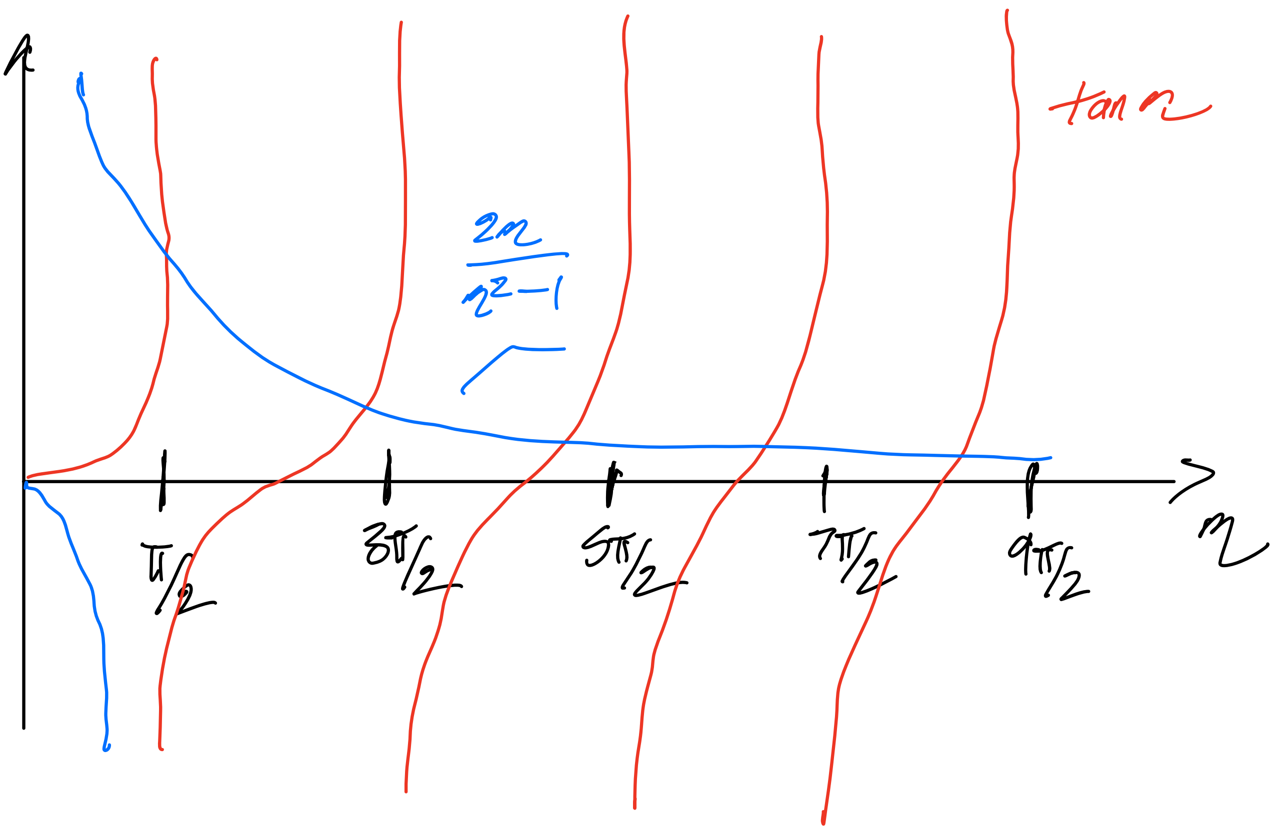
Again examine

$\tan\beta L = \frac{2\beta L}{(\beta L)^2}$

Graphical solution $u_1 = u_2$

Again examine

$$\tan \eta = \frac{2\eta}{(\eta)^2 - 1}$$ → goes to infinity at $\eta = \pm 1$



Rigid Body Motion

One root is $\eta = \beta L = 0 \longrightarrow \beta = 0, \ \omega = 0$

The BVP $\longrightarrow u''(x) + \cancel{\frac{\omega^2 E}{\rho}} u(x) = 0$

$u(x) = cx + d \quad , \quad u'(x) = c$

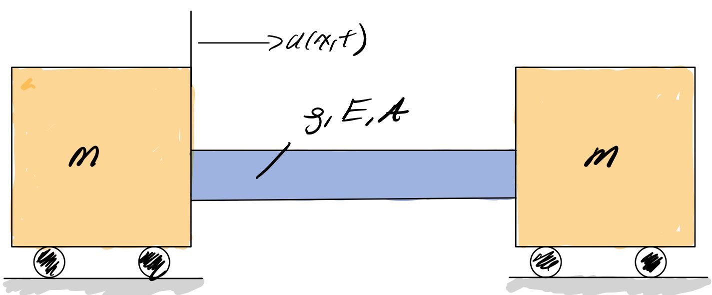
What are c and d

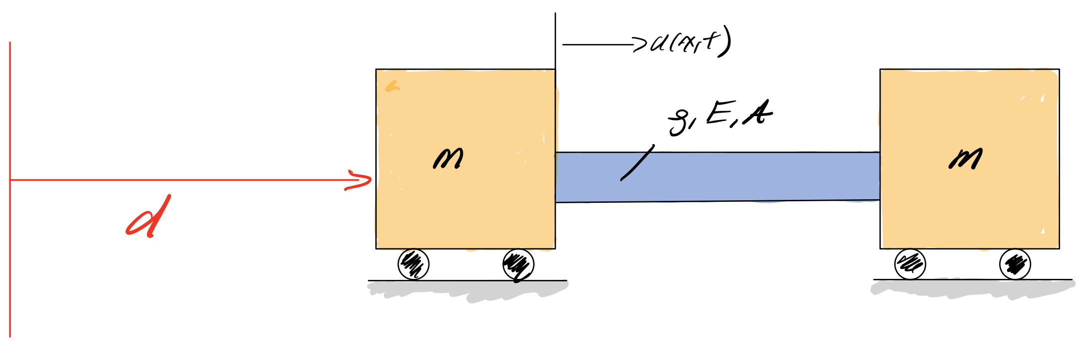
$$EA\,u'(0) = -m\,\cancel{\omega^2}^{\,0}\,u(0) = 0$$
$$EA\,u'(L) = m\,\cancel{\omega^2}^{\,0}\,u(L) = 0$$

$\therefore \quad u'(0) = 0 = c \qquad c = 0$

$\qquad u'(L) = 0 = cL$

$u(x) = d \longrightarrow$ is a constant motion





Rigid Body translation

$$\boxed{(\mu_1+\mu_2)\eta = (\mu_1\mu_2\eta^2-1)\tan\eta}$$

$\mu_1 = \dfrac{M_1}{m}$ $\mu_2 = \dfrac{M_2}{m}$ $m = gAL$

Rearrange $\tan\eta = \dfrac{(\mu_1+\mu_2)\eta}{(\mu_1\mu_2\eta^2-1)}$

a) $\mu_1, \mu_2 \to \infty$

$\tan\eta = 0$ ~ since RHS goes to zero as $\mu_1, \mu_2 \to \infty$

$\sin\eta = 0$ $\eta = n\pi$ $n = 1, 2, 3$

b) $\mu_1 = 0$ $\mu_2 = 1$

$\tan\eta = \eta$

c) $\tan\eta = 0$

$\sin\eta = 0$ $\eta = n\pi$

d) $\tan\eta = \dfrac{2\eta}{\eta^2-1}$

Vibration

Again examine

$$\tan \beta L = \frac{2\beta L}{(\beta L)^2 - 1}$$

From Matlab code

$\beta_1 L = 1.3065 \longrightarrow \omega_1 = 1.3065 \sqrt{E/gL^2}$

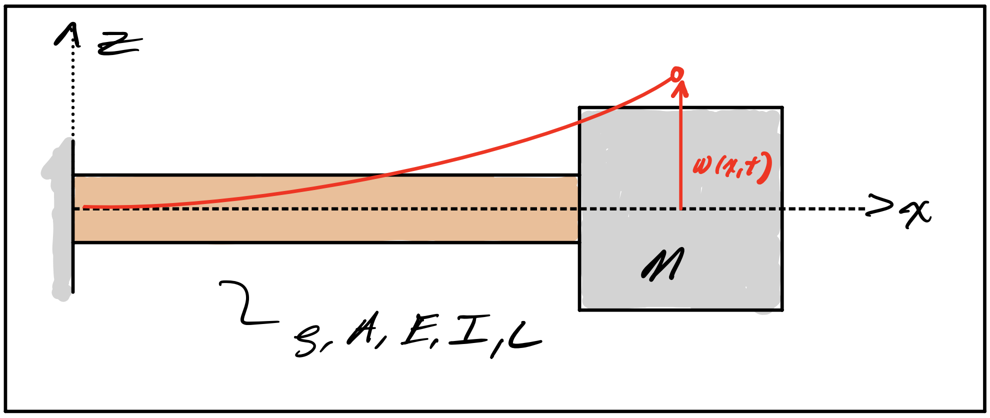
$\beta_2 L = 3.6732 \longrightarrow \omega_2 = 3.6732 \sqrt{E/gL^2}$

$\beta_3 L = 6.5846 \longrightarrow \omega_3 = 6.5846 \sqrt{E/gL^2}$

$\beta_4 L = 9.6795 \longrightarrow \omega_4 = 9.6795 \sqrt{E/gL^2}$

$\beta_5 L = 12.6796 \longrightarrow \omega_5 = 12.6796 \sqrt{E/gL^2}$

Homework 4.3



1st write the EOM

$$EI\frac{\partial^4 w}{\partial x^4} = -\rho A\frac{\partial^2 w}{\partial t^2}$$

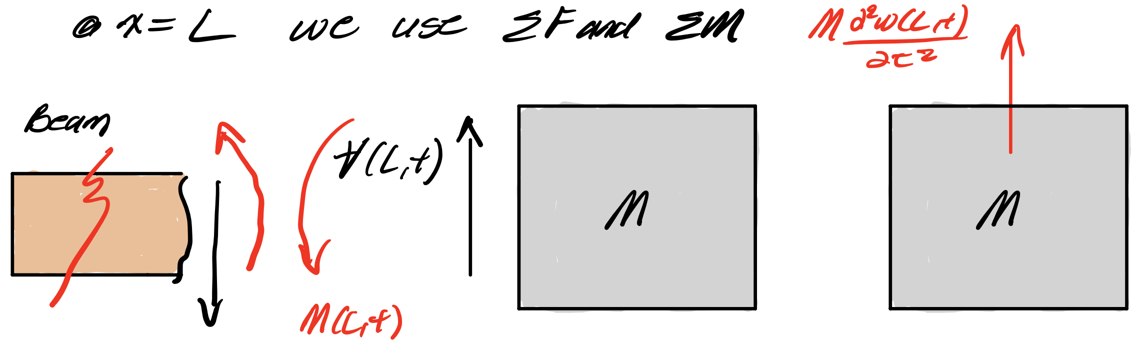
The boundary conditions can be written as

@ $x=0$

① $w(0,t) = W(0)\,T(t) = 0 \longrightarrow \boxed{W(0)=0}$

② $\frac{\partial w}{\partial x}(0,t) = W'(0)\,T(t) = 0 \longrightarrow \boxed{W'(0)=0}$

@ $x=L$ we use ΣF and ΣM



$\circlearrowleft \Sigma M$: $\quad M(L,t) = EI\frac{\partial^2 w(L,t)}{\partial x^2} = 0$

$+\uparrow \Sigma F_x$: $\quad +V(L,t) = M\frac{\partial^2 w(L,t)}{\partial t^2}$

③ $$EI\frac{\partial^2 w(L,t)}{\partial x^2} = EI\,W''(L)T(t) = 0 \rightarrow W''(L)=0$$

④ $$EI\frac{\partial^3 w(L,t)}{\partial x^3} = M\frac{\partial^2 w(L,t)}{\partial t^2}$$

$$EI\,W'''(L)T(t) = M\,W(L)\cancelto{-\omega^2 T(t)}{\ddot{T}(t)}$$

$$EI\,W'''(L) = -M\,W(L)\,\omega^2$$

$$W'''(L) = -\omega^2\frac{M}{EI}\,W(L)$$

The spatial solution is

$$W(x) = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$$

where $\beta^2 = \sqrt{\rho A/EI}\;\omega$

① $$w(0) = a\cancelto{1}{\cosh 0} + b\cancelto{0}{\sinh 0} + c\cancelto{1}{\cos 0} + d\cancelto{0}{\sin 0} = 0$$

② $$w'(0) = \beta(a\cancelto{0}{\sinh 0} + b\cancelto{1}{\cosh 0} - c\cancelto{0}{\sin 0} + d\cancelto{1}{\cos 0}) = 0$$

① $0 = a + c \longrightarrow c = -a$

② $0 = \beta(b+d) \longrightarrow d = -b \qquad \beta \neq 0$

Now,

$$W(x) = a(\cosh\beta x - \cos\beta x) + b(\sinh\beta x - \sin\beta x)$$

$$W''(x) = \beta^2(a\cosh\beta x + a\cos\beta x + b\sinh\beta x + b\sin\beta x)$$

$$W'''(x) = \beta^3(a\sinh\beta x - a\sin\beta x + b\cosh\beta x + b\cos\beta x)$$

③ $0 = \beta^2(a\cosh\beta L + a\cos\beta L + b\sinh\beta L + b\sin\beta L)$

$0 = a(\cosh\beta L + \cos\beta L) + b(\sinh\beta L + \sin\beta L)$

④ $0 = \beta^3(a\sinh\beta L - a\sin\beta L + b\cosh\beta L + b\cos\beta L)$
$+ \frac{w^2 M}{EI}(+a\cosh\beta L - a\cos\beta L + b\sinh\beta L$
$- b\sin\beta L)$

$w = \sqrt{\frac{EI}{\rho A}}\beta^2$ and $w^2 = \frac{EI}{\rho A}\beta^4$

$\frac{w^2 M}{EI} = \frac{EI}{\rho A}\frac{M}{EI}\beta^4 = \frac{M}{\rho A}\beta^4$

④ $0 = \beta^3(a\sinh\beta L - a\sin\beta L + b\cosh\beta L + b\cos\beta L)$
$+ \frac{M}{\rho A}\beta^4(a\cosh\beta L - a\cos\beta L + b\sinh\beta L - b\sin\beta L)$

$0 = (a\sinh\beta L - a\sin\beta L + b\cosh\beta L + b\cos\beta L)$
$+ \frac{M}{\rho A}\beta(a\cosh\beta L - a\cos\beta L + b\sinh\beta L - b\sin\beta L)$

multiply $\frac{M}{\rho A}\beta$ by L/L

$0 = (a\sinh\beta L - a\sin\beta L + b\cosh\beta L + b\cos\beta L)$
$+ \cancel{\frac{M}{\rho A L}}\,\beta L(a\cosh\beta L - a\cos\beta L + b\sinh\beta L - b\sin\beta L)$
μ

Put in matrix form

$$\underbrace{\begin{bmatrix} (\cosh \beta L + \cos \beta L) & (\sinh \beta L + \sin \beta L) \\ \sinh \beta L - \sin \beta L & \cosh \beta L + \cos \beta L \\ + \mu \beta L (\cosh \beta L - \cos \beta L) & + \mu \beta L (\sinh \beta L - \sin \beta L) \end{bmatrix}}_{D} \begin{Bmatrix} a \\ b \end{Bmatrix} = 0$$

Simplified these calculations in the Matlab code

$CE = \det(D)$

$$\boxed{CE = 1 + \cos \beta L \cosh \beta L + \mu \beta L \cos \beta L \sinh \beta L - \mu \beta L \cosh \beta L \sin \beta L}$$

From Matlab Code ... $\mu = 1$

βL_1	1.2479
βL_2	4.0311
βL_3	7.1341
βL_4	10.2566
βL_5	13.3878

Now, solve for modeshapes using 1st row of matrix equation

$$(\cosh \beta L - \cos \beta L) a + (\sinh \beta L - \sin \beta L) b = 0$$

$$b = \frac{(\cosh \beta L - \cos \beta L)}{(\sinh \beta L - \sin \beta L)} a$$

$$W(x) = a(\cosh \beta x - \cos \beta x) + b(\sinh \beta x - \sin \beta x)$$

$$\boxed{W(x) = a(\cosh \beta x - \cos \beta x) + \frac{a(\cosh \beta L - \cos \beta L)}{(\sinh \beta L - \sin \beta L)}(\sinh \beta x - \sin \beta x)}$$

Normalization

Rewrite

$$W_n(x) = a \hat{W}_n(x)$$

where $\hat{W}_n(x) = (\cosh \beta x - \cos \beta x) + \frac{(\cosh \beta L - \cos \beta L)}{(\sinh \beta L - \sin \beta L)}(\sinh \beta x - \sin \beta x)$

we need to find a

1) $W_n(L) = a \hat{W}_n(L) = 1 \longrightarrow a = 1/\hat{W}_n(L)$

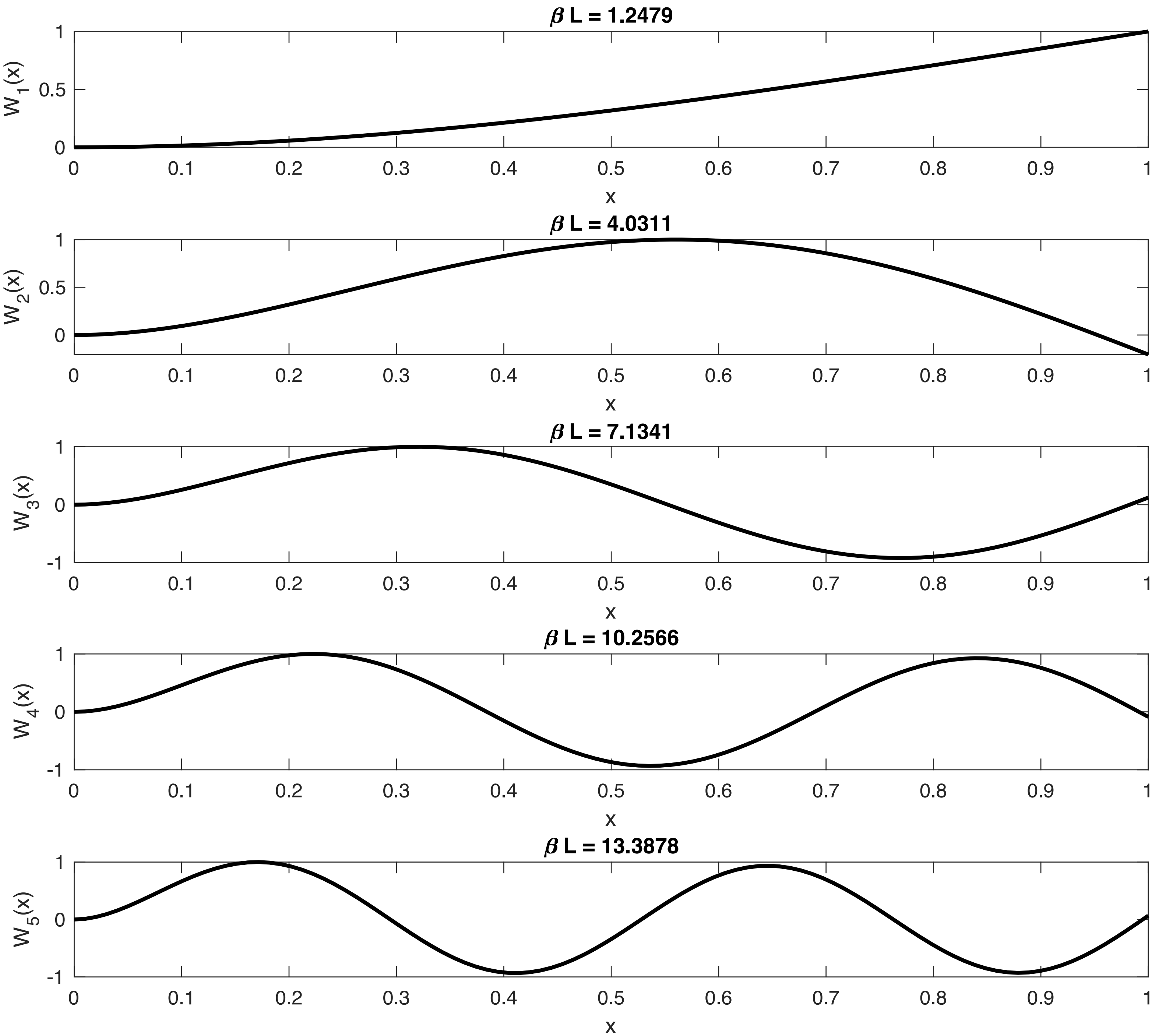
$$W_n(x) = \frac{\hat{W}_n(x)}{\hat{W}_n(L)}$$

2) $m_n = \rho A \int_0^L W_n^2(x)\,dx + M W_n(L)^2$

$$1 = \rho A \int_0^L a^2 \hat{W}_n(x)^2\,dx + M a^2 \hat{W}_n(L)^2$$

$$a = \frac{1}{\rho A \int_0^L \hat{W}_n(x)\,dx + M \hat{W}_n(L)^2}$$

$$W_n(x) = a \hat{W}_n(x)$$



β L = 1.2479
W1(x)
x
β L = 4.0311
W2(x)
x
β L = 7.1341
W3(x)
x
β L = 10.2566
W4(x)
x
β L = 13.3878
W5(x)
x

```
clc
clear
close all
fprintf(['\n\n\nStarting file >>' mfilename '<< at ' datestr(now,0) '\n\n']);
format long
% HW  5.3

% x= Beta*L
syms B x l a b c d M E I w rho L A BL

W    = a*cosh(B*x)-a*cos(B*x)+b*sinh(B*x)-b*sin(B*x)
Wp   = diff(W,x)
Wpp  = diff(Wp,x)
Wppp = diff(Wpp,x)

% Evaluate at endpoint x =L
x   = L;
WL    = eval(W)
WppL  = eval(Wpp)
WpppL = eval(Wppp)

E1 = WppL
E2 = WpppL+w^2*L/L*M/(E*I)*WL

% Math Simplification - this is a guide you need to look at the problem on
% paper to realize how to use Matlab to simlify this....

E1 = E1;
E2 = subs(E2,w^2,(E*I)/(rho* A)*B^4);
E2 = expand(E2/B^3);

% get rid of B*L and replace with y
E1 = subs(E1,B*L,BL);
E2 = subs(E2,B*L,BL);

E2 = subs(E2/L,B,  BL);
E2 = subs(E2, {M, rho, A, L}, {1, 1, 1, 1});
E1 = simplify(E1/B^2);

% Write as matrix pull out coefficents
BC11 = coeffs(E1,a);
BC12 = coeffs(E1,b);
BC21 = coeffs(E2,a);
BC22 = coeffs(E2,b);

% order from lowest to highest power- lowest is coefficient^0,
%highest is coefficent^1;
BC11 = BC11(2);
BC12 = BC12(2);
BC21 = BC21(2);
BC22 = BC22(2);
BC   = [BC11 BC12; BC21 BC22];

CE  = simplify(det(BC));
BL  = linspace(0,5*pi,10^4);
CEv = eval(CE);

% This CE does not have  plot that we can easily identify our iniitial
```

very cruel I asked you to plot

JG

```
% guess
% In order to get initial guess we look at places where the is a sign
% change in the CE
[Val1, loc1] = find(abs(diff(sign(CEv)))==2); % This time loc1 is indices
% Initial Guesses
x0v = BL(loc1);
options_all=[];
% set your length L
L=1;
for i =1:5
    x0 = x0v(i);
    fprintf(['\n\n\n >> Mode' num2str(i) '<<  \n\n']);
    % Fzero
    [xval, fval, exitflag] = fzero(@CEfunction,x0,options_all,CE);
    betaL(i) = xval;%This is Beta*L
    beta(i)  = betaL(i)/L; % THis is Beta

end
%Solve for Modeshapes
%solve fofr a constant
bsol=solve(E1==0,b)
%plug into W(x)
Wmode= simplify(subs(W,b,bsol));
pretty(Wmode)
%Plot mode shapes
x  =linspace(0,L,100);
figure
for i =1:5
    B  = beta(i);
    BL = betaL(i);
    a  = 1;
    WmodeP = eval(Wmode);
    Wmax   = max(abs(WmodeP));
    WmodeP = WmodeP/Wmax;
    subplot(5,1,i)
    line(x,WmodeP,'linewidth',2,'color','k')
    xlabel('x')
    ylabel(['W_',num2str(i),'(x)'])
    title(['\beta L = ', num2str(BL)])
    box on
end


function [F] = CEfunction(BL,CE)
% Pass the symbolic characteristic equation into a function
F =  eval(CE);
end
```

Homework 4.3

$$-EI\,\partial y^4/\partial x^4 + T\,\partial^2 y/\partial x^2 = \rho A \frac{\partial^2 y}{\partial t^2}$$

note from the paper there is a "-" sign

a) Non dimensionalize

$$\xi = x/L, \quad \frac{\partial()}{\partial x} = \frac{\partial()}{\partial \xi}\frac{\partial \xi}{\partial x}, \quad \frac{\partial \xi}{\partial x} = 1/L$$

Higher order derivatives in x: $\dfrac{\partial^n()}{\partial x^n} = \dfrac{\partial^n()}{\partial \xi^n}\left(1/L\right)^n$

Now, $-EI\dfrac{\partial^4 y}{\partial x^4} + T\dfrac{\partial^2 y}{\partial x^2} = \rho A\dfrac{\partial^2 y}{\partial t^2}$

$$-\frac{EI}{L^4}\frac{\partial^4 y}{\partial \xi^4} + \frac{T}{L^2}\frac{\partial^2 y}{\partial \xi^2} = \rho A\frac{\partial^2 y}{\partial t^2}$$

$$-\frac{EI}{L^2T}\frac{\partial^4 y}{\partial \xi^4} + \frac{\partial^2 y}{\partial \xi^2} = \frac{\rho A L^2}{T}\frac{\partial^2 y}{\partial t^2}$$

assume: $Y(\xi) = S_n \sin(n\pi)$

$Y_{\xi\xi}(\xi) = -n^2\pi^2 S_n \sin(n\pi)$ assume $y(\xi,t) = Y(\xi)e^{i\omega t}$

$Y_{\xi\xi\xi\xi}(\xi) = n^4\pi^4 S_n \sin(n\pi)$

$$\left(-\frac{EI}{L^2T}n^4\pi^4 S_n\sin(n\pi) - n^2\pi^2 S_n \sin(n\pi) = -\frac{\rho A L^2}{T}\omega^2 S_n \sin(n\pi)\right)e^{i\omega t}$$

$$-\frac{EI}{L^2T}n^4\pi^4 - n^2\pi^2 = -\frac{\rho A L^2\omega^2}{T} \quad \sim (1)$$

$$+\underbrace{\frac{ETn^2\pi^2}{L^2T}}_{\to \mathbb{B}} + 1 = +\frac{\rho A L^2\omega^2}{Tn^2\pi^2}$$

At small $\mathbb{B}$, tension dominates and the string behaves as an ideal spring.

At large B, the flexural stiffness is EI, the string behaves as an Euler beam.

b) Two ways to do this you can directly use equations.

The boundary function $U(x) = C_1 \sin n\pi x/L$ is the simplification of

$$U(x) = C_1 \sin(kx) + C_2 \cos(kx) + C_3 \sinh(kx) + C_4 \cosh(kx)$$

with boundary conditions

$U(0) = U(L) = 0$ ~ No movement ~ could stop here

$U_{xx}(0) = U_{xx}(L) = 0$ ~ No motion

See example III.2.1

Full Proof

$$-EI \frac{\partial^4 y}{\partial x^4} + T \frac{\partial^2 y}{\partial x^2} = \rho A \frac{\partial^2 y}{\partial t^2}$$

assume: $y(x,t) = Y(x)\overline{T}(t)$

$$-EI\, Y^4(x)\overline{T}(t) + T\, Y^2(x)\overline{T}(t) = \rho A\, Y(x)\ddot{\overline{T}}(t)$$

$$\frac{EI\, Y^4(x) + T\, Y^2(x)}{\rho A\, Y(x)} = \frac{\ddot{\overline{T}}(t)}{\overline{T}(t)} = -\omega^2$$

IVP: $\ddot{\overline{T}}(t) + \omega^2 \overline{T}(t) = 0$

BVP: $-EI\, Y^4(x) + T\, Y^2(x) + \omega^2 \rho A Y(x) = 0$

$$Y^4(x) - \frac{T}{EI} Y^2(x) - \frac{\omega^2 \rho A Y(x)}{EI} = 0$$

Name $\beta^4 = \frac{\omega^2 \rho A}{EI}$, $\gamma^2 = \frac{T}{EI}$

$Y^4(x) - \gamma^2 Y^2(x) - \beta^4 Y(x)$

$Y(x) = Ae^{kx}$

$k^4 - \gamma^2 k^2 - \beta^4 = 0$

$$k^2 = \frac{+\gamma^2 - \sqrt{\gamma^2 - 4\gamma^2 \beta^4}}{2} \qquad (i)$$

Evaluate BC's

$Y(0) = C_1 \underbrace{\cancel{\sin(0)}}_{0} + C_2 \underbrace{\cancel{\cos(0)}}_{1} + C_3 \underbrace{\cancel{\sinh(0)}}_{0} + C_4 \underbrace{\cancel{\cosh(0)}}_{1} = 0$

$0 = C_2 + C_4$

$Y_{xx}(0) = k^2(-C_1 \cancel{\sin(0)} - C_2 \cancel{\cos(0)} + C_3 \cancel{\sinh(0)} + C_4 \cancel{\cosh(0)}) = 0$

$0 = k^2(-C_2 + C_4)$

This means $C_2 = C_4 = 0$

$Y(x) = C_1 \sin kx + C_3 \sinh kx$

$Y(L) = C_1 \sin kL + C_3 \sinh kL = 0$

$Y_{xx}(x) = k^2(-C_1 \sin kx + C_3 \sinh kx)$

$Y_{xx}(L) = k^2(-C_1 \sin kL + C_3 \sinh kL) = 0$

Put in matrix form

$$\begin{bmatrix} \sin kL & \sinh kL \\ -\sin kL & \sinh kL \end{bmatrix} \begin{Bmatrix} C_1 \\ C_4 \end{Bmatrix} = 0$$

$2 \sin kL \sinh kL = 0$

CE $\sin kL = 0$ $\qquad k_n L = n\pi$

$\therefore$ Following the buckling example, III.2.1

we look at the boundary value problem

$$-EI\,Y^{4}(x) + T\,Y^{2}(x) + \omega^2 \rho A\, Y(x) = 0$$

$$Y(x) \longrightarrow Y_n(x) = \sin(k_n x)$$

$$(-EI\,k_n^4 - T k_n^2 + \omega_n^2 \rho A)\sin(k_n x) = 0$$

$$\rho A \omega_n^2 = T k_n^2 + EI k_n^4$$

$$\omega_n^2 = \frac{T}{\rho A} k_n^2 \left(1 + \frac{EI}{T} k_n^2\right) = \frac{T}{\rho A}\left(\frac{n\pi}{L}\right)^2\left(1 + \frac{EI}{T}\left(\frac{n\pi}{L}\right)^2\right)$$

$$\omega_n = \sqrt{\frac{T}{\rho A}\left(\frac{n\pi}{L}\right)^2\left(1 + \frac{EI}{T}\left(\frac{n\pi}{L}\right)^2\right)}$$

$$\omega_n = \underbrace{\frac{n\pi}{L}\sqrt{\frac{T}{\rho A}}}_{\omega_1}\ \underbrace{\sqrt{1 + \frac{EI}{T}\left(\frac{n\pi}{L}\right)^2}}_{\sqrt{1+\beta n^2}}$$

$$\omega_n = \omega_1\sqrt{1+\beta n^2} \longrightarrow f_n = f_1\sqrt{1+\beta n^2}$$

$$f_n = \frac{\omega_n}{2\pi} \qquad f_n = \frac{n}{2L}\sqrt{\frac{T}{\rho A}}\sqrt{1 + \frac{EI}{T}\left(\frac{n\pi}{L}\right)^2}$$

c) $E = 2.0 \times 10^{11}$ Pa $\rho = 7800$ kg/m³ $T = 600$ N

$d = 1.2/1000$ m $L = 0.80$ m

$$I = \pi d^4/64 = \pi (1.2/1000)^4 \, 1/64 = 1.131 \times 10^{-6} \; m^2$$

$$A = \pi d^2/4 = \pi (1.2/1000)^4 \, 1/4 = 1.0179 \times 10^{-13} \; m^4$$

$$\rho A = \rho \pi d^2/4 = 0.008 \text{ kg/m}$$

$$B = \pi^2 E I / T L^2 = \pi \frac{(2.0 \times 10^{11}) \, 1.31 \times 10^{-6}}{600 \cdot 0.80} = 0.000523$$

$$f_1 = 1/2L \sqrt{T/\rho A} = 2/0.80 \sqrt{\frac{600}{0.008}} = 163 \text{ Hz}$$

n	$n f_1$	f_n	$f_n/(n f_1)$
1	163.00	163.04	1.00026
2	326.00	326.34	1.00105
3	488.99	490.14	1.00235
4	651.99	654.72	1.00418
5	814.99	820.30	1.00652
6	977.99	987.16	1.00937

d) 1) How does the bending stiffness shift the higher overtones

$EI\, \partial^4\eta/\partial x^4 \simeq$ restoring force that is depend on the second derivative of the curvature of the beam $\sim \partial^2\eta/\partial x^2$. Modes with higher curvature ~ i.e. more n give and more restoring force. This is seen in the percentage deviation. At higher modes become inharmonic the frequency intervals are wider than perfect integer multiples.

2) Treble strings are long & thin thus B is reduced by d^4 & $\uparrow T$ & L

$$B = \frac{\pi^2 E I}{T L^2} = \frac{\pi^2 E \pi^2 d^4/64}{T L^2}$$

$B\downarrow \ \downarrow d,\quad B\downarrow\ T\uparrow,\quad B\downarrow\ L\downarrow \rightarrow nf_1 \rightarrow f_n$

These strings are nearly ideal.

The opposite is true of short, thick brass string add will lead to inharmonocity.

3) The pinned beam has BC $\eta|_{0,L} = 0$ $\quad \eta_{xx}|_{0,L} = 0$

The clamped beam has BC $\eta|_{0,L} = 0$ $\quad \eta_x|_{0,L} = 0$

It is generally a stiffer model of vibration

Wrapped bass string

assume $I_c = \pi d_c^4/64 = \pi(1/1000)^2/64 = 4.9087 \times 10^{-14}\ m^4$

is an approximation actually an over estimation

$EI_c = 0.009817\ N \cdot m^3$

$EI = 0.20358\ N \cdot m^3$

$\rho A_c = 0.0801\ kg/m$ $\quad B_c = 0.000252$

$\rho A = 0.088\ kg/m$ $\quad B = 0.000523$

$f_1 = 163\ Hz$ $\quad f_{1c} = 53.90\ Hz$

Wrapping increases mass without increasing flexural stiffness and therefore lowers frequency.