ME 563 - Fall 2025

Homework No. 3

Due: November 2, 2025

ME 563 - Fall 25
Homework Problem 3.1

For this problem, you may use AI tools to review relevant background concepts, clarify definitions, and check your reasoning after you have completed the derivation on your own. AI can also be used to provide simple illustrative examples not directly tied to this assignment. However, you may not ask AI to solve the problem or generate step-by-step solutions. Your submitted work must represent your own reasoning and understanding. If AI is used as a study aid, please acknowledge it briefly in your submission (for example, "I consulted AI to check my interpretation of the rigid-body mode").

The vertical vibration of an airplane and its wings is modeled as a **three-degree-of-freedom system**. One mass corresponds to the right wing (m), one to the left wing (m), and one to the fuselage (βm), where $\beta > 1$. The generalized coordinates are the absolute vertical displacements $x_1(t)$, $x_2(t)$, and $x_3(t)$. Each wing is connected to the fuselage by a bending stiffness

$$k = \frac{3EI}{\ell^3},$$

where E is the modulus of elasticity of the wing and ℓ is the distance from fuselage to wing tip. To avoid an unbounded rigid-body mode, assume the fuselage is also connected vertically to the ground through a soft spring k_0 (representing landing gear compliance).

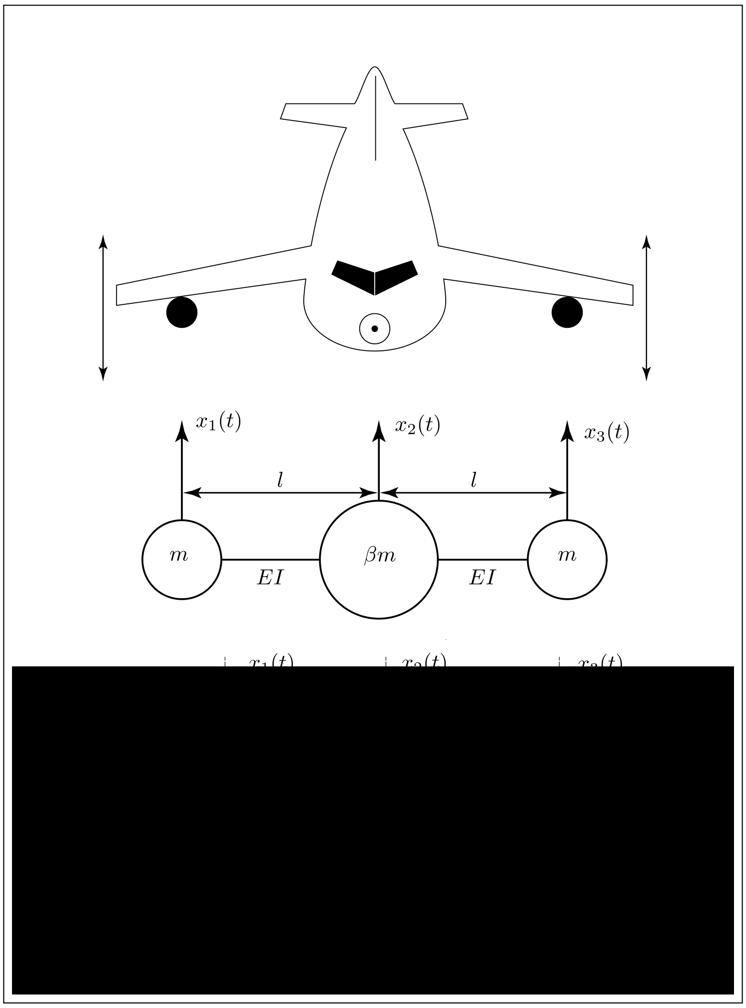

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Figure 1: Schematic of the three-DOF wing–fuselage system with fuselage spring k_0 representing landing gear.

Assume the following:

- Motions are small and restricted to vertical translation.

- Springs behave linearly; no geometric nonlinearities are considered.
- Aerodynamic and damping forces are neglected unless otherwise specified.
- The fuselage spring k_0 may be set to a small value to approximate free-flight conditions.

Learning Goals

By completing this problem, you should be able to:

1. Derive the equations of motion for a coupled multi-DOF system using **Lagrange's equations**.
2. Compute **natural frequencies** and **mode shapes**, and interpret the presence or absence of rigid-body modes.
3. Apply the concept of **initial condition projection** into modal space to identify conditions that excite specific modes.

Complete the following:

(a) Use Lagrange's equations to derive the equations of motion of the spring–mass model.

(b) Calculate the natural frequencies and mode shapes of the system with k_0=0.

(c) Calculate the natural frequencies and mode shapes of the system with $k_0 \neq 0$.

(d) Discuss how the soft fuselage spring k_0 modifies the zero-frequency rigid-body mode and interpret the physical meaning.

(c) Compute the total time response of the system. Then, determine sets of initial displacement conditions such that:

   (a) The system responds approximately as a rigid body (limit $k_0 \to 0$).

   (b) The system responds only in the first non-rigid mode of vibration.

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Homework Problem 3.2

AI Use Policy for This Problem You may use AI tools to review background concepts, clarify terminology, or check your derivation once you have completed it yourself. AI may also be used to provide simple illustrative examples that are not tied directly to this problem. However, you may not ask AI to solve this problem, provide step-by-step solutions, or generate work that substitutes for your own reasoning and derivation. Your submission must reflect your own understanding and process. If you use AI as a study aid, you should acknowledge it (for example, "I used AI to check my understanding of centrifugal softening").

Ground resonance is a potentially dangerous instability associated with rotary-wing aircraft on flexible supports (landing gear). For a visual demonstration of helicopter ground resonance, see this YouTube video. A simplified model is shown below.

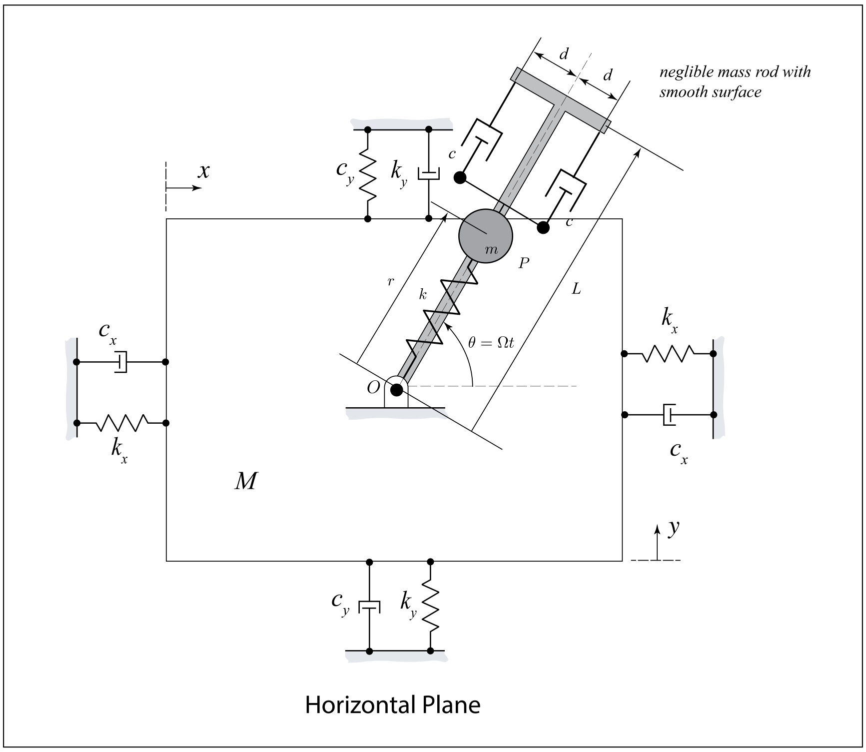

Figure 2: Simplified three-DOF model for ground resonance.

The system consists of:

- A fuselage modeled as a lumped mass M, attached to the ground by flexible supports (springs k_x, k_y, dampers c_x, c_y).
- A massless rod of length L rotating at constant angular speed Ω about the fuselage center. The rod angle is $\theta(t) = \Omega t$.
- A small slider mass m that can move radially a distance $r(t)$ along the rod. The slider is attached to the fuselage by a spring (k_r) and damper (c_r) aligned along the rod.

The generalized coordinates are $q = [x, y, r]^\top$, where x, y are the fuselage translations and r is the slider radial coordinate.

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Learning Goals

By solving this problem, you should learn to:

1. Apply Lagrange's equations to derive equations of motion for a multi-DOF system with rotating coordinates.
2. Recognize Coriolis ($\propto \Omega\dot{r}$) and centrifugal ($\propto \Omega^2 r$) terms in the equations.
3. Interpret how rotor speed Ω changes the *effective stiffness* seen by the slider.
4. Identify conditions under which the system becomes unstable and link them to physical intuition.

This problem is intended to build your intuition about rotor–fuselage coupling. Focus on recognizing *how the rotation introduces new terms* in the equations of motion. Instability here means the restoring force changes sign (negative effective stiffness). You do not need advanced math (e.g., Rayleigh quotients or Floquet theory). Stick to energy reasoning and careful bookkeeping of terms. —

Complete the following:

(a) **Derive the equations of motion.** Use Lagrange's method with the given coordinates. Clearly show how the velocity of the slider includes both $\dot{r}$ along the rod and additional terms due to the rotating frame. *Hint:* Write the absolute position of the slider as

$$\mathbf{p}(t) = \begin{bmatrix} x \\ y \end{bmatrix} + r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \theta = \Omega t.$$

Differentiate carefully to obtain $\dot{\mathbf{p}}$ before squaring.

(b) **Show how Ω changes the effective stiffness.** Assume the fuselage is fixed ($x = y = 0$). Simplify the radial equation of motion to show that the radial restoring force is

$$(k_r - m\Omega^2)\, r.$$

Explain why this means that if $\Omega > \sqrt{k_r/m}$ the radial motion becomes unstable, no matter what the fuselage supports are.

(c) **Partition constant and time-dependent parts.** Write the mass, damping, and stiffness matrices as

$$[M]\ddot{q} + [C(t)]\dot{q} + [K(t)]q = f,$$

and show which parts are constant and which vary with $\cos\theta$ or $\sin\theta$. *Guidance:* Keep the constant "centrifugal softening" $-m\Omega^2$ in the radial equation inside K_0, and put all $\sin\theta, \cos\theta$ terms in matrixes $[\Delta K(t)], [\Delta C(t)]$. Where $[K(t)] = [K_0] + [\Delta K(t)]$, and $[C(t)] = [C_0] + [\Delta C(t)]$

(d) **Ignore the time-dependent parts (approximation).** With only the constant parts retained, compute the three approximate natural frequencies given below and the their associated mode-shapes:

$$\omega_x = \sqrt{\tfrac{2k_x}{M+m}}, \quad \omega_y = \sqrt{\tfrac{2k_y}{M+m}}, \quad \omega_r = \sqrt{\tfrac{k_r - m\Omega^2}{m}}.$$

Interpret physically: what happens to ω_r as Ω increases?

(e) **Worst-case "frozen" coefficients.** Because not all entries of $[\Delta K(t)], [\Delta C(t)]$ reach maxima at the same time, consider two extreme rod orientations: (i) $\theta = 0$ ($\cos\theta = 1,\ \sin\theta = 0$) and (ii) $\theta = \pi/2$ ($\cos\theta = 0,\ \sin\theta = 1$). For each case, write down the approximate coupled stiffness submatrix. Which fuselage direction (x or y) couples most strongly with the radial slider?

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Homework Problem 3.3

For this problem, you may use AI tools to refresh background concepts such as Fourier series, wave equations, and mode shapes, or to clarify the meaning of energy conservation in vibrating systems. AI can also be consulted for general study examples unrelated to the specific triangular initial condition given here. However, you may not ask AI to compute the Fourier coefficients, perform the series expansion, or carry out the energy check for you. Your submitted work must be based on your own derivations and reasoning. If AI is used as a study aid, please acknowledge it briefly in your submission (for example, "I consulted AI to review how sine series expansions handle piecewise functions").

Consider a uniform bar of length L, cross-sectional area A, modulus of elasticity E, and density ρ. The bar is fixed at both ends ($x = 0$ and $x = L$). An axial force F_0 is applied at the midpoint and suddenly removed at $t = 0$. The effect of the applied force is that the bar has an initial displacement field shaped like a triangle:

$$u(x,0) = \begin{cases} \varepsilon\, x, & 0 \le x \le L/2, \\ \varepsilon\,(L-x), & L/2 \le x \le L, \end{cases} \qquad \text{where} \quad \varepsilon = \frac{F_0}{2AE}.$$

The initial velocity is zero: $u_t(x,0) = 0$.

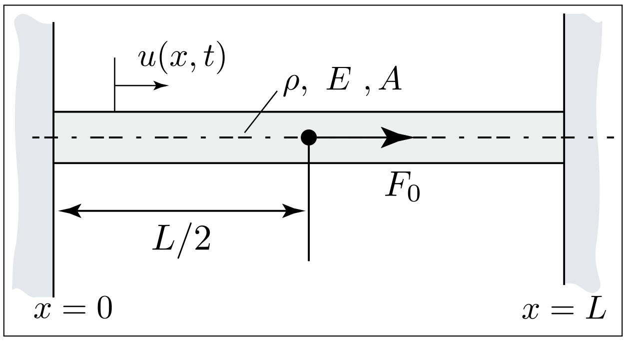

Figure 3: Fixed–fixed bar with triangular initial displacement due to suddenly removed midspan force.

Learning Goals

By solving this problem, you should learn to:

1. Derive the governing PDE for axial vibrations of a fixed–fixed bar.
2. Apply separation of variables to obtain eigenfunctions and natural frequencies.
3. Expand a piecewise initial condition into a Fourier sine series and interpret why only certain modes appear.
4. Relate symmetry of the initial displacement to modal content.
5. Connect the initial strain energy to total vibration energy (energy conservation).

Complete the following:

(a) **Governing PDE.** Write down the axial wave equation and the appropriate boundary conditions at $x = 0, L$.

(b) **Separation of variables.** Assume $u(x,t) = U(x)T(t)$ and derive the ordinary differential equations for X and T.

(c) **Characteristic equation.** Apply the fixed–fixed boundary conditions to obtain the characteristic values β_n and natural frequencies ω_n.

(d) **Mode shapes.** Write down the eigenfunctions $U_n(x)$ and sketch the first three modes.

(e) **Modal superposition.** Express the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} (C_n \cos(\omega_n t) + S_n \sin(\omega_n t))\, U_n(x).$$

Derive the expression for the Fourier coefficients a_n using the given triangular initial displacement. *Hint:* Symmetry implies only odd modes will remain in the series.

(f) **Energy check.** Compute the initial strain energy in the bar (from $u(x,0)$) and show that it equals the total energy of the vibration (kinetic + strain) at later times.

(g) **Conceptual Questions** 1) Why does symmetry eliminate the even modes? 2) Which mode dominates the motion, and why? 3) How would the response differ if the force were removed slowly rather than suddenly?

(h) **Numerical exploration.** Using MATLAB/Python, approximate $u(x,t)$ with the first 5 odd modes. Plot $u(x,0)$ reconstructed from the series and compare it to the exact triangular profile. Discuss convergence.

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Homework Problem 3.4

For this problem, you may use AI tools to review background material such as wave equations, boundary conditions, or properties of transcendental equations. You may also consult AI to clarify the meaning of dimensionless parameters (e.g., μ, κ) or to check simple sketches of $\tan\eta$ *versus* $\eta/(\mu\eta^2 - \kappa)$*. However, you may not ask AI to derive the governing boundary condition, to generate the step-by-step solution of the characteristic equation, or to compute the transcendental roots* η_n *for you. Your submitted work must represent your own reasoning and calculations.*

A uniform elastic rod of length L, cross-sectional area A, modulus of elasticity E, and density ρ is fixed at the left end ($x = 0$). At the free end ($x = L$), the rod is attached to a concentrated tip mass M in parallel with a chain of two linear springs in series. Let $u(x,t)$ denote the axial displacement of the rod in the positive x direction.

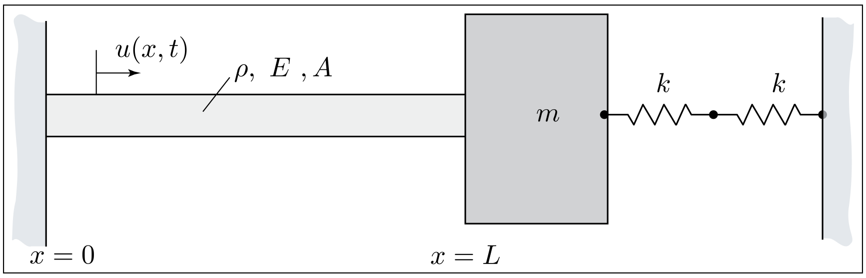

Figure 4: Fixed–end rod with tip mass and series springs. Positive displacement u is to the right.

Complete the following:

(a) Governing PDE

Derive the governing partial differential equation (PDE) for axial vibrations of the rod.

(b) Force Balance at the Free End

State the boundary conditions. At $x = 0$, the displacement is zero. At $x = L$, use free-body diagrams (FBDs) of the rod and tip system to derive the dynamic boundary condition. Make sure to include both the mass and the series spring contribution.

(c) Characteristic Equation

Assume harmonic motion $u(x,t) = U(x)e^{i\omega t}$. Show that the spatial solution is

$$U(x) = B\sin(\beta x), \qquad \beta = \omega\sqrt{\frac{\rho}{E}}.$$

Apply the boundary condition at $x = L$ to derive

$$\tan\eta = \frac{\eta}{\mu\eta^2 - \kappa},$$

where

$$\eta = \beta L, \quad \mu = \frac{M}{m}, \quad \kappa = \frac{K_{\text{eq}}}{k_{\text{rod}}},$$

with $m = \rho AL$ and $k_{\text{rod}} = EA/L$.

(d) Special Case

For $M = m$ and $K_{\text{eq}} = k_{\text{rod}}$, simplify the characteristic equation to

$$\tan \eta = \frac{\eta}{\eta^2 - 1}.$$

Use Matlab or Python to locate the first four roots $\eta_1, \eta_2, \eta_3, \eta_4$.

(e) Natural Frequencies

Show that the natural frequencies are

$$\omega_n = \frac{c\,\eta_n}{L} = \eta_n \sqrt{\frac{EA}{L^2 m}}, \qquad c = \sqrt{\frac{E}{\rho}}.$$

Report the first four frequencies in terms of the corresponding roots η_n.

(f) Modal Functions

Write down the corresponding mode shapes

$$U_n(x) = \sin\Big(\eta_n \frac{x}{L}\Big).$$

(g) Extensions and Interpretation

1. Discuss the limiting cases $M \to 0$, $K_{\text{eq}} \to 0$, and $M \to \infty$. What familiar systems do these limits represent?
2. Suppose the two springs were arranged in parallel instead of in series. How would the boundary condition at $x = L$ be modified?

(h) Complex Eigenvalues with a Tip Damper

Suppose the tip spring is replaced by a viscous damper with coefficient C. The PDE for the rod remains unchanged, but the boundary condition at $x = L$ becomes

$$EA\, u_x(L,t) = M\, u_{tt}(L,t) - C\, \dot{u}(L,t).$$

1. Explain why assuming $u(x,t) = U(x)e^{i\omega t}$ (with ω real) is not sufficient when damping is present. What contradiction arises if you enforce $s = i\omega$?
2. Instead, assume the more general form

   $$u(x,t) = U(x)e^{st}, \qquad s \in \mathbb{C},$$

   where $s = \sigma + i\omega$ has both real and imaginary parts. Show that the spatial solution remains

   $$U(x) = B\sin(\beta x), \qquad \beta = \frac{s}{c}, \qquad c = \sqrt{\frac{E}{\rho}}.$$

3. Derive the transcendental characteristic equation

   $$EA\,\beta \cot(\beta L) = Ms^2 - Cs,$$

   and rewrite it in nondimensional form as

   $$\eta \cot \eta = \mu\eta^2 - \gamma\eta, \qquad \eta = \frac{sL}{c}, \quad \mu = \frac{M}{m}, \quad \gamma = \frac{C}{\sqrt{m k_{\text{rod}}}}.$$

4. Discuss the physical meaning of $\Re(s) = \sigma < 0$. What does it imply about the long-time behavior of the vibrations?

Note: When damping is present, the modes are complex. The exponential-in-time ansatz e^{st}, with $s = \sigma + i\omega$, is the only consistent choice: ω represents the oscillation frequency while $\sigma < 0$ represents the decay rate.