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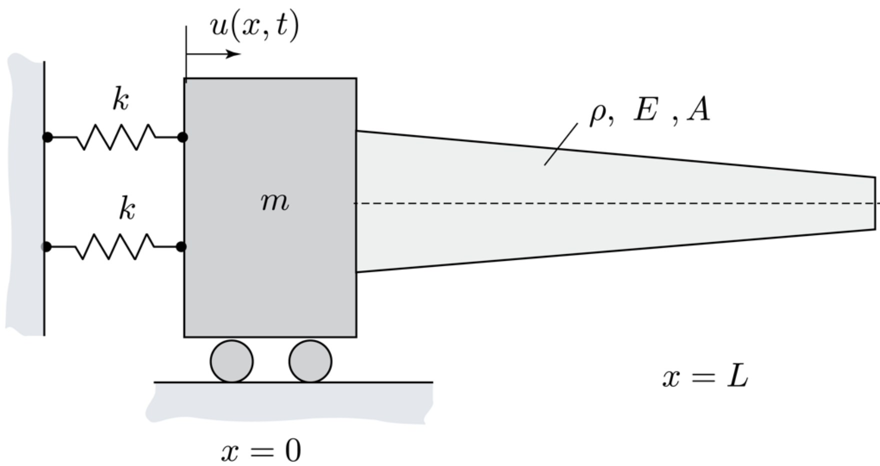
Test Problem 1 – 20 points

The longitudinal motion of the rod shown below is described by $u(x,t)$. The rod has length L, mass density ρ, cross sectional area A. Where the cross sectional varies as

$$A(x) = \left(2A_o - \frac{A_o}{L}x\right),$$

at $x=L$ a lumped mass, m, is attached to beam. Attached to the lumped mass are two springs of stiffness k.

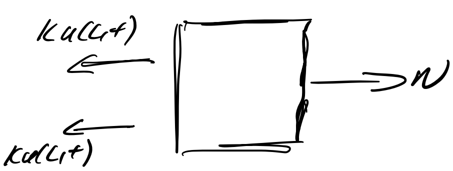


a) State the geometric boundary conditions at $x=0$, and derive the natural boundary conditions at $x=L$. For the natural boundary conditions an **accurate Freebody diagram must be provided and used in the derivations**.

b) Write governing partial differential equation of motion for the system. No FBD diagram is needed answer.

c) Assuming a spatiotemporal solution for $u(x, t) = U(x)T(t)$, Convert the boundary conditions found in a) to spatial boundary in terms of $U(x)$ and its associated spatial derivatives. Do not find the characteristic equation.

a) $\dfrac{\partial u(L,t)}{\partial x} = 0$



$\overset{+}{\rightarrow} \sum F_x \quad -2ku(0,t) + EA\dfrac{\partial u(0,t)}{\partial x} = m\dfrac{\partial^2 u(0,t)}{\partial t^2}$

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b) $$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) = \rho A(x)\frac{\partial^2 u}{\partial t^2} \longrightarrow \frac{\partial}{\partial x}\left(E(2A_0 - A_0 x/L)\frac{\partial u}{\partial x}\right) = \rho\,(2A_0 - A_0 x/L)\frac{\partial^2 u}{\partial t^2}$$

$$-\frac{EA_0}{L}\frac{\partial u}{\partial x} + E(2A_0 - A_0 x/L)\frac{\partial^2 u}{\partial x^2} = \rho\,(A_0 - A_0 x/L)\frac{\partial^2 u}{\partial t^2}$$

c) $u(x,t) = \bar{U}(x)T(t)$

$$\frac{\partial u}{\partial x} = \bar{U}'(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \bar{U}(x)T(t)$$

$$\frac{\partial u}{\partial x}(0,t) = U'(0)\,T(t) = 0 \qquad U'(0) = 0$$

$$\pm EA_0\frac{\partial u(L,t)}{\partial x} - k_{eq}u(L,t) = m\frac{\partial^2 u(L,t)}{\partial t^2}$$

$$\pm EA\,\bar{U}'(L)\cancel{T(t)} - k_{eq}\bar{U}(L)\cancel{T(t)} = -m\omega^2\,\bar{U}(L)\cancel{T(t)}$$

$$\pm EA_0\bar{U}'(L) - 2k\bar{U}(L) = -m\omega^2\bar{U}(L)$$

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Test Problem 2 -40 points

A 2-DOF system has the following equations of motion

$$m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}.$$

where m and k have units of kg and N/m, respectively. The system is given a set of initial conditions of $x_1(0) = x_2(0) = x_0 = 0$ and $\dot{x}_1(0) = \dot{x}_2(0) = v_0$. Find $\vec{x}(t)$.

Bonus (3 points): If damping is added to the system the equation of motion can be written as

$$m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix}+c\begin{bmatrix}6 & -2\\-2 & 4\end{bmatrix}+k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Is the system proportionally damped? You must justify your answer to receive credit.

assume $\vec{x} = \vec{X} e^{-i\omega t}$, 1st find natural frequencies and mode shapes

rewrite

$$m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\vec{0} \qquad \left(-\omega^2\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}+\frac{k}{m}\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}\right)\begin{Bmatrix}X_1\\X_2\end{Bmatrix}=\vec{0}$$

$$\left(\begin{bmatrix}-2\omega^2+2k/m & -k/m\\-k/m & -2\omega^2+2k/m\end{bmatrix}\right)\begin{Bmatrix}X_1\\X_2\end{Bmatrix}=\vec{0},$$ next set determinant to 0.

$$(-2\omega^2+2k/m)(-2\omega^2+2k/m)-3k^2/m^2=0$$

$$4\omega^4-4\omega^2 k/m-4\omega^2 k/m-3k^2/m^2=0$$

$$4\omega^4-8\omega^2 k/m-k^2/m^2=0$$

$$\longrightarrow \omega^2=\frac{-(-8)k/m\pm\sqrt{(-8k/m)^2-4(4)(3)k^2/m^2}}{2(4)}=$$

$$\omega^2=1k/m\pm 1/8\sqrt{16}\,k/m=(1\pm 1/2)k/m$$

$$\omega^2=1/2\,k/m,\ 3/2\,k/m \longrightarrow \omega_1=\sqrt{1/2}\sqrt{k/m} \text{ and } \omega_2=\sqrt{3/2}\sqrt{k/m}$$

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$$\left(\begin{bmatrix} -2\omega^2 + 2k/m & -k/m \\ -k/m & -2\omega^2 + 2k/m \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \vec{0} \longrightarrow (-2\omega^2 + 2k/m)X_1 - (k/m)X_2 = 0$$

$$\frac{X_1}{X_2} = \frac{k/m}{2\omega^2 + 2k/m} \longrightarrow$$

at $\omega = \omega_1 = \sqrt{1/2}\sqrt{k/m}$

$$\frac{X_1^1}{X_2^1} = \frac{k/m}{-2(1/2)k/m + 2k/m} = \frac{1}{1} \longrightarrow \bar{\bar{X}}^1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

at $\omega = \omega_2 = \sqrt{3/2}\sqrt{k/m}$

$$\frac{X_1^2}{X_2^2} = \frac{k/m}{-2(3/2)k/m + 2k/m} = \frac{-1}{1} \longrightarrow \bar{\bar{X}}^2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Evaluate c and s constants

c_1 and c_2 are zero since $\dot{\vec{x}}(0) = \vec{0}$

$$s_1 = \frac{\bar{\bar{X}}^{1T}[m]\vec{x}(0)}{\omega_1 \bar{\bar{X}}^{1T}[m]\bar{\bar{X}}^1} = \frac{\{1 \;\; 1\}m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix}}{\omega_1\{1 \;\; 1\}m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}} = \frac{\{1 \;\; 1\}\begin{Bmatrix} 2v_0 \\ 2v_0 \end{Bmatrix}}{\omega\{1 \;\; 1\}\begin{Bmatrix} 2 \\ 2 \end{Bmatrix}} = \frac{4x_0}{4\omega_1} = \frac{x_0}{\omega_1}$$

$$s_2 = \frac{\bar{\bar{X}}^{2T}[m]\vec{x}(0)}{\omega_2 \bar{\bar{X}}^{2T}[m]\bar{\bar{X}}^1} = \frac{\{-1 \;\; 1\}m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix}}{\omega_2\{-1 \;\; 1\}m\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}} = \frac{\{1 \;\; 1\}\begin{Bmatrix} 2v_0 \\ 2v_0 \end{Bmatrix}}{\{-1 \;\; 1\}\begin{Bmatrix} -2 \\ 2 \end{Bmatrix}} = \frac{0}{4\omega_1} = 0$$

$$\vec{x}(t) = c_1\bar{\bar{X}}^1 \sin\omega_1 t = \frac{v_0}{\omega_1}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\sin\left(\sqrt{1/2}\sqrt{k/m}\,t\right)$$

Bonus: $[C] = \alpha[m] + \beta[k]$

$6 = \alpha(2) + \beta(2)$

$-2 = \alpha(0) + \beta(-1) \longrightarrow \beta = 2$ and $\alpha = 1$

VV with $4 = \alpha(2) + \beta(2)$

$4 \neq 1(2) + 2(2) = 6$

Not proportionally damped

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Test Problem 3 -30 points

A 2-DOF system has the following mass matrix:

$$m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

with modal/natural frequencies of

$$\omega_1 = 0, \text{ rads/s} \quad \text{and} \quad \omega_2 = \sqrt{3} \text{ rads/s}.$$

In addition, the modes shapes are given by:

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \text{and} \quad \vec{X}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

a) Find the mass normalized modes shapes of the system.

b) Use the mass normalized mode shapes to find the stiffness matrix of the system.

a) $\alpha_1 = \dfrac{1}{\sqrt{\bar{X}^{1T}[m]\bar{X}^1}} = \dfrac{1}{\sqrt{\{1\ 1\}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}} = \dfrac{1}{\sqrt{\{1\ 1\}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}} = \dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\bar{\bar{X}}_1 = \dfrac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$\alpha_2 = \dfrac{1}{\sqrt{\bar{X}^{2T}[m]\bar{X}^2}} = \dfrac{1}{\sqrt{\{1\ -\frac{1}{2}\}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}}} = \dfrac{1}{m\sqrt{\{1\ -\frac{1}{2}\}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}} = \dfrac{1}{\sqrt{1+\frac{1}{2}}} = \dfrac{1}{\sqrt{3/2}}$

$\bar{\bar{X}}_2 = \dfrac{1}{\sqrt{3/2}}\begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$

b) $\bar{\bar{X}}^{1T}[k]\bar{\bar{X}}^1 = \omega_1$ and $\bar{\bar{X}}^{1T}[k]\bar{\bar{X}}^2 = 0$

$\bar{\bar{X}}^{2T}[k]\bar{\bar{X}}^2 = \omega_2$ $\quad [k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$

Expand $\bar{\bar{X}}_1^{T}[k]\bar{\bar{X}}_1 = \omega_1^2$

$\dfrac{1}{\sqrt{3}\sqrt{m}}\{1\ 1\}\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\dfrac{1}{\sqrt{3}\sqrt{m}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{1}{3m}\{1\ 1\}\begin{Bmatrix} k_{11}+k_{12} \\ k_{12}+k_{22} \end{Bmatrix}$

$k_{11} + 2k_{12} + k_{22} = 0$ $\quad = \frac{1}{3}(k_{11} + 2k_{12} + k_{22}) = 0$

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Expand $\vec{\vec{X}}_2^T [K] \vec{\vec{X}}_2 = \omega_2^2$

$$\frac{1}{\sqrt{3/2}} \{1 \ -1/2\} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \frac{1}{\sqrt{3/2}} \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} = 3$$

$$2/3 \ \{1 \ -1/2\} \begin{Bmatrix} K_{11} & -K_{12}/2 \\ K_{12} & -K_{22}/2 \end{Bmatrix} = 3$$

$$2/3 \ \{K_{11} - K_{12}/2 \ -K_{12}/2 \ +K_{22}/4\} = 3$$

$$K_{11} - K_{12} + K_{22}/4 = 9/2$$

Expand $\vec{X}_2' [K] \vec{X}_1 = 0$

$$\frac{1}{\sqrt{3/2}} \{1 \ -1/2\} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0$$

$$\frac{1}{\sqrt{9/2}} \{1 \ -1/2\} \begin{Bmatrix} K_{11} + K_{12} \\ K_{12} + K_{22} \end{Bmatrix} = 0$$

$$K_{11} + K_{12} - K_{12}/2 - K_{22}/2 = 0$$

$$K_{11} + 3K_{12}/2 - K_{22}/2 = 0$$

You have 3 equations and 3 unknowns

$$k_{11} + 2k_{12} + k_{22} = 0$$

$$k_{11} - k_{12} + k_{22}/4 = 9/2$$

$$k_{11} + 3k_{12}/2 - k_{22}/2 = 0$$

$k_{11} = 2$ $k_{12} = -2$ $k_{22} =$