ME 563-Fall 2023

Test 2

***Name*____________________________________**

***Pledge*___________________________________**

I have neither given nor received aid on this examination.

Instructions:

• This is a closed-book, closed-notes exam.

• Please read the question on this exam carefully and answer only the questions I ask. Don't waste your time doing extra work, and don't skip any of the smaller questions within a problem. I can't grade what I can't see and I can't give partial credit for something in your head.

Newton Euler

$$\sum \vec{F} = m\vec{a}_g$$
$$\sum \vec{M}_A = I_A\vec{\alpha}$$

Where A is a fixed point or center of gravity

SDOF Response

$$m\ddot{x} + c\dot{x} + kx = 0,$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$
$$2\zeta\omega_n = \frac{c}{m},\ \omega_n^2 = \frac{k}{m}$$

$$0 \leq \zeta \leq 1,$$
$$x(t) = \exp -\zeta\omega_n t\,(C\cos\omega_d t + S\sin\omega_d t)$$

Power Equation

$$T + U = T_o + U_o + W^{(nc)}$$
$$Power = \frac{dW^(nc)}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

Eigenvalue Problem

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0},$$
$$\vec{x}(t) = \vec{X}\exp i\omega t,$$
$$\left(-\omega^2[M] + [K]\right)\vec{X} = \vec{0}$$

MDOF Response

$$\vec{x}(t) = \sum_{j=1}^{N} \vec{X}^(j)[c_j\cos\omega_j t + s_j\sin\omega_j t]$$
$$c_j = \frac{\vec{X}^{(j)T}[M]\vec{x}(0)}{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$
$$sj = \frac{\vec{X}^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$

Linearized Lagrange's Equations

$$[M]\ddot{\overrightarrow{z}} + [C]\dot{\overrightarrow{z}} + [K]\overrightarrow{z} = \overrightarrow{0}$$

$$\overrightarrow{z}(t) = \overrightarrow{q}(t) - \overrightarrow{q}_0$$
$$M_{ik} = (m_{ik})_{\overrightarrow{q}_0} = M_{ki}$$
$$C_{ik} = \left(\frac{\partial^2 R}{\partial\dot{q}_i\partial\dot{q}_k}\right)_{\overrightarrow{q}_0} = C_{ki}$$
$$K_{ik} = \left(\frac{\partial^2 U}{\partial q_i\partial q_k}\right)_{\overrightarrow{q}_0} = K_{ki}$$

Mass Normalized Eigenvectors

$$\alpha_j = \frac{1}{\sqrt{\vec{X}^{(j)T}[M]\vec{X}^j}}$$
$$\vec{X}_m^j = \alpha_j\vec{X}^j$$
$$c_j = \vec{X}_m^{(j)T}[M]\vec{x}(0)$$
$$s_j = \frac{\vec{X}_m^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j}$$

Orthogonality of Mass Normalized Eigenvectors

$$\vec{X}_m^{(i)T}[M]\vec{X}_m^j = \delta_{ij}$$

Log Decrement SDOF

$$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right)$$
$$\zeta = \frac{\delta/2\pi}{\sqrt{1 + (\delta/2\pi)^2}}$$
$$\zeta << 1 \rightarrow \zeta = \frac{\delta}{2\pi}$$

Second-Order Continuous Systems

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right)+f(x,t)=r(x)\frac{\partial^2 u}{\partial t^2}$$

Solution form

$$u(x,t)=\sum U_n(x)(C_n\cos\omega_n t+S_n\sin\omega_n t)$$

$$C_n=\frac{\int_0^L r(x)U_n(x)u(x,0)dx}{\int_0^L r(x)U_n^2(x)dx}$$

$$S_n=\frac{1}{\omega_n}\frac{\int_0^L r(x)U_n(x)\frac{\partial u}{\partial t}(x,0)dx}{\int_0^L r(x)U_n^2(x)dx}$$

Fourth-Order Continuous Systems

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)+\rho(x)A(x)\frac{\partial^2 w}{\partial t^2}=0$$

Solution form

$$w(x,t)=\sum W_n(x)(C_n\cos\omega_n t+S_n\sin\omega_n t)$$

$$C_n=\frac{\int_0^L \rho(x)A(x)W_n(x)w(x,0)dx}{\int_0^L \rho(x)A(x)W_n^2(x)dx}$$

$$S_n=\frac{1}{\omega_n}\frac{\int_0^L \rho(x)A(x)W_n(x)\frac{\partial w}{\partial t}(x,0)dx}{\int_0^L \rho(x)A(x)W_n^2(x)dx}$$

Properties of Even and Odd functions

The product of two even functions is an even function.
The product of two odd functions is an odd function.
The product of an even and an odd function is an odd function

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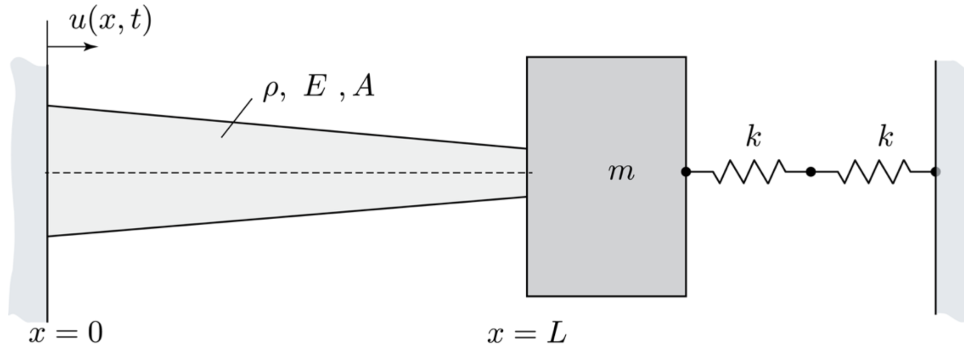
Test Problem 1 – 20 points

The longitudinal motion of the rod shown below is described by $u(x,t)$. The rod has length L, mass density ρ, cross sectional area A. Where the cross sectional

$$A(x) = \left(2A_o - \frac{A_o}{L}x\right),$$

at x=L a lumped mass, m, is attached to beam. Attached to the lumped mass are two springs of stiffness k.

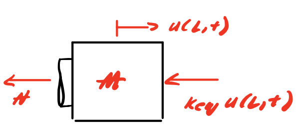


a) State the geometric boundary conditions at x=0, and derive the natural boundary conditions at x=L. For the natural boundary conditions an **accurate Freebody diagram must be provided and used in the derivations**.

b) Write governing partial differential equation of motion for the system. No FBD diagram is needed answer.

c) Assuming a spatiotemporal solution for $u(x,t) = U(x)T(t)$, Convert the boundary conditions found in a) to spatial boundary in terms of $U(x)$ and its associated spatial derivatives. Do not find the characteristic equation.

a) $u(x,0) = 0$



$$\overset{+}{\rightarrow} \Sigma F_x: \; -N + k_{eq}u(L,t) = m\frac{\partial^2 u(L,t)}{\partial t^2}$$

$$k_{eq} = \frac{kk}{k+k} = k/2 \quad \text{and} \quad N = EA\frac{\partial u(L,t)}{\partial x}$$

$$\therefore \; -EA_o\frac{\partial u(L,t)}{\partial x} - k_{eq}u(L,t) = m\frac{\partial^2 u(L,t)}{\partial t^2}$$

b) $$\frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) = \rho A(x)\frac{\partial^2 u}{\partial t^2} \longrightarrow \frac{\partial}{\partial x}\left(E(2A_o - A_o x/L)\frac{\partial u}{\partial x}\right) = \rho(2A_o - A_o x/L)\frac{\partial^2 u}{\partial t^2}$$

$$-\frac{EA_o}{L}\frac{\partial u}{\partial x} + E(2A_o - A_o x/L)\frac{\partial^2 u}{\partial x^2} = \rho(A_o - A_o x/L)\frac{\partial^2 u}{\partial t^2}$$

c) $u(x,t) = \bar{U}(x)T(t)$

$$\frac{\partial u}{\partial x} = \bar{U}'(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \bar{U}(x)T(t)$$

$$u(0,t) = \bar{U}(0)T(t) = 0 \longrightarrow \bar{U}(0) = 0$$

$$-EA_0\frac{\partial u(L,t)}{\partial x} - k_{eq}u(L,t) = m\frac{\partial^2 u(L,t)}{\partial t^2}$$

$$-EA\,\bar{U}'(L)T\cancel{(t)} - k_{eq}\bar{U}(L)T\cancel{(t)} = -m\omega^2\,\bar{U}(L)\,T\cancel{(t)}$$

$$-EA_0\bar{U}'(L) \quad \tfrac{k}{2}\bar{U}(L) = -m\omega^2\bar{U}(L)$$

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Test Problem 2 -40 points

A 2-DOF system has the following equations of motion

$$m\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}.$$

where m and k have units of kg and N/m, respectively. The system is given a set of initial conditions of $x_1(0) = x_2(0) = x_0$ and $\dot{x}_1(0) = \dot{x}_2(0) = 0$. Find $\vec{x}(t)$.

Bonus (3 points): If damping is added to the system the equation of motion can be written as

$$m\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + c\begin{bmatrix}6 & -2\\ -2 & 4\end{bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Is the system proportionally damped? You must justify your answer to receive credit.

assume $\vec{x} = \vec{X} e^{-i\omega t}$, 1st find natural frequencies and mode shapes

rewrite

$$m\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \vec{0} \qquad \left(-\omega^2\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} + \frac{k}{m}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\right)\begin{Bmatrix}X_1\\ X_2\end{Bmatrix} = \vec{0}$$

$$\left(\begin{bmatrix}-2\omega^2 + 2k/m & -k/m\\ -k/m & -2\omega^2 + 2k/m\end{bmatrix}\right)\begin{Bmatrix}X_1\\ X_2\end{Bmatrix} = \vec{0},$$ next set determinant to 0.

$$(-2\omega^2 + 2k/m)(-2\omega^2 + 2k/m) - 3k^2/m^2 = 0$$
$$4\omega^4 - 4\omega^2 k/m - 4\omega^2 k/m - 3k^2/m^2 = 0$$
$$4\omega^4 - 8\omega^2 k/m - k^2/m^2 = 0$$

$$\longrightarrow \quad \omega^2 = \frac{-(-8)k/m \pm \sqrt{(-8k/m)^2 - 4(4)(3)k^2/m^2}}{2(4)} =$$
$$\omega^2 = 1k/m \pm 1/8\sqrt{16}\,k/m = (1 \pm 1/2)k/m$$

$$\omega^2 = 1/2\,k/m,\ 3/2\,k/m \longrightarrow \omega_1 = \sqrt{1/2}\sqrt{k/m} \text{ and } \omega_2 = \sqrt{3/2}\sqrt{k/m}$$

$$\left(\begin{bmatrix}-2\omega^2 + 2k/m & -k/m\\ -k/m & -2\omega^2 + 2k/m\end{bmatrix}\right)\begin{Bmatrix}X_1\\ X_2\end{Bmatrix} = \vec{0} \longrightarrow (-2\omega^2 + 2k/m)X_1 - (k/m)X_2 = 0$$

$$\frac{X_1}{X_2} = \frac{k/m}{2\omega^2 + 2k/m} \longrightarrow$$

at $\omega = \omega_1 = \sqrt{1/2}\sqrt{k/m}$

$$\frac{X_1^1}{X_2^1} = \frac{k/m}{-2(1/2)k/m + 2k/m} = \frac{1}{1} \longrightarrow \vec{X}^1 = \begin{Bmatrix}1\\ 1\end{Bmatrix}$$

at $\omega = \omega_2 = \sqrt{3/2}\sqrt{k/m}$

$$\frac{X_1^2}{X_2^2} = \frac{k/m}{-2(3/2)k/m + 2k/m} = \frac{-1}{1} \longrightarrow \vec{X}^2 = \begin{Bmatrix}-1\\ 1\end{Bmatrix}$$

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Test Problem 2 Additional Page

Evaluate c and s constants

$$c_1 = \frac{\bar{\bar{X}}^{1T}[m]\bar{x}(0)}{\bar{\bar{X}}^{1T}[m]\bar{\bar{X}}^{1}} = \frac{\{1 \;\; 1\}m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}x_0\\x_0\end{Bmatrix}}{\{1 \;\; 1\}m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}} = \frac{\{1 \;\; 1\}\begin{Bmatrix}2x_0\\2x_0\end{Bmatrix}}{\{1 \;\; 1\}\begin{Bmatrix}2\\2\end{Bmatrix}} = \frac{4x_0}{4} = x_0$$

$$c_2 = \frac{\bar{\bar{X}}^{2T}[m]\bar{x}(0)}{\bar{\bar{X}}^{2T}[m]\bar{\bar{X}}^{1}} = \frac{\{-1 \;\; 1\}m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}x_0\\x_0\end{Bmatrix}}{\{-1 \;\; 1\}m\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}-1\\1\end{Bmatrix}} = \frac{\{-1 \;\; 1\}\begin{Bmatrix}2x_0\\2x_0\end{Bmatrix}}{\{-1 \;\; 1\}\begin{Bmatrix}-2\\2\end{Bmatrix}} = \frac{0}{4} = 0$$

s_1 and s_2 are zero since $\dot{\bar{x}}(0) = \bar{0}$

$$\bar{x}(t) = c_1 \bar{\bar{X}}^{1} \cos \omega_1 t = x_0 \begin{Bmatrix}1\\1\end{Bmatrix} \cos\left(\sqrt{1/2}\sqrt{k/m}\, t\right)$$

Bonus: $[C] = \alpha[m] + \beta[k]$

$6 = \alpha(2) + \beta(2)$

$-2 = \alpha(0) + \beta(-1) \longrightarrow \beta = 2$ and $\alpha = 1$

✓✓ with $4 = \alpha(2) + \beta(2)$

$4 \neq 1(2) + 2(2) = 6$

Not proportionally damped

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Test Problem 3 -30 points

Name_______________

A 2-DOF system has the following mass matrix:

$$m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

with modal/natural frequencies of

$$\omega_1 = 0, \text{ rads/s} \quad \text{and} \quad \omega_2 = \sqrt{3}\ \text{rads/s}.$$

In addition, the modes shapes are given by:

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \text{and} \quad \vec{X}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

a) Find the mass normalized modes shapes of the system.

b) Use the mass normalized mode shapes to find the stiffness matrix of the system.

a) $$\alpha_1 = \frac{1}{\sqrt{\bar{X}^{1T}[m]\bar{X}^1}} = \frac{1}{\sqrt{\{1\ 1\}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}}} = \frac{1}{\sqrt{\{1\ 1\}\begin{Bmatrix}1\\2\end{Bmatrix}}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\bar{\bar{X}}_1 = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{\bar{X}^{2T}[m]\bar{X}^2}} = \frac{1}{\sqrt{\{1\ -1/2\}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{Bmatrix}1\\-1/2\end{Bmatrix}}} = \frac{1}{m\sqrt{\{1\ -1/2\}\begin{Bmatrix}1\\-1\end{Bmatrix}}} = \frac{1}{\sqrt{1+1/2}} = \frac{1}{\sqrt{3/2}}$$

$$\bar{\bar{X}}_2 = \frac{1}{\sqrt{3/2}}\begin{Bmatrix}1\\-1/2\end{Bmatrix}$$

b) $$\bar{\bar{X}}^{1T}[k]\bar{\bar{X}}^1 = \omega_1 \quad \text{and} \quad \bar{\bar{X}}^{1T}[k]\bar{\bar{X}}^2 = 0$$

$$\bar{\bar{X}}^{2T}[k]\bar{\bar{X}}^2 = \omega_2 \qquad [k] = \begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}$$

Expand $\bar{\bar{X}}_1^{T}[k]\bar{\bar{X}}_1 = \omega_1^2$

$$\frac{1}{\sqrt{3}\sqrt{m}}\{1\ 1\}\begin{bmatrix}k_{11} & k_{12}\\ k_{12} & k_{22}\end{bmatrix}\frac{1}{\sqrt{3}\sqrt{m}}\begin{Bmatrix}1\\1\end{Bmatrix} = \frac{1}{3m}\{1\ 1\}\begin{Bmatrix}k_{11}+k_{12}\\ k_{12}+k_{22}\end{Bmatrix}$$

$$k_{11} + 2k_{12} + k_{22} = 0 \qquad = \frac{1}{3}(k_{11} + 2k_{12} + k_{22}) = 0$$

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Test Problem 3 Additional Page

Expand $\vec{X}_2^T [K] \vec{X}_2 = \omega_2^2$

$$\frac{1}{\sqrt{3/2}} \left\{ 1 \ -1/2 \right\} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \frac{1}{\sqrt{3/2}} \begin{Bmatrix} 1 \\ -1/2 \end{Bmatrix} = 3$$

$$2/3 \left\{ 1 \ -1/2 \right\} \begin{Bmatrix} K_{11} & -K_{12}/2 \\ K_{12} & -K_{22}/2 \end{Bmatrix} = 3$$

$$2/3 \left\{ K_{11} - K_{12}/2 \ -K_{12}/2 \ +K_{22}/4 \right\} = 3$$

$$K_{11} - K_{12} + K_{22}/4 = 9/2$$

Expand $\vec{X}_2' [K] \vec{X}_1 = 0$

$$\frac{1}{\sqrt{3/2}} \left\{ 1 \ -1/2 \right\} \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 0$$

$$\frac{1}{\sqrt{9/2}} \left\{ 1 \ -1/2 \right\} \begin{Bmatrix} K_{11} + K_{12} \\ K_{12} + K_{22} \end{Bmatrix} = 0$$

$$K_{11} + K_{12} - K_{12}/2 - K_{22}/2 = 0$$

$$K_{11} + 3K_{12}/2 - K_{22}/2 = 0$$

You have 3 equations and 3 unknowns

$$k_{11} + 2k_{12} + k_{22} = 0$$

$$k_{11} - k_{12} + k_{22}/4 = 9/2$$

$$k_{11} + 3k_{12}/2 - k_{22}/2 = 0$$

$k_{11} = 2$ $k_{12} = -2$ $k_{22} = 2$