Test 1

***Name*____________________________________**

Pledge: I have neither given nor received aid on this examination.

***Signature:*____________________________________**

Instructions:
• This is a closed-book, closed-notes exam.

• Please read the question on this exam carefully and answer only the questions I ask. Don't waste your time doing extra work, and don't skip any of the smaller questions within a problem. I can't grade what I can't see, and I can't give partial credit for something in your head.

EQUATION SHEET IS AT BACK

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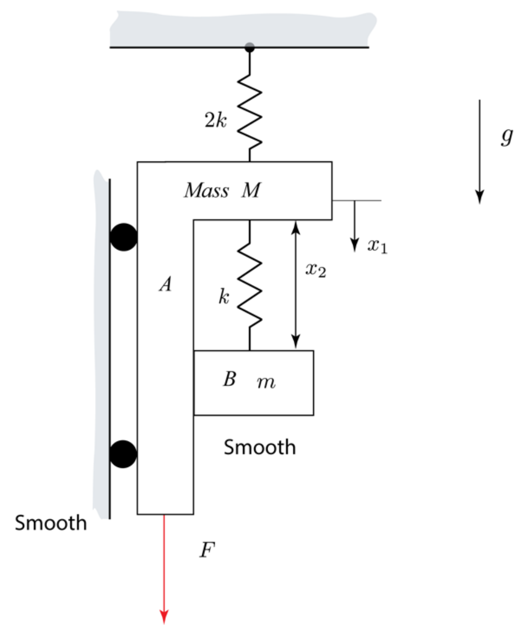
Test Problem 1 – 30 points

Consider the system below where all surfaces are smooth. Let x_1 denote the absolute position of mass A and x_2 is <u>the relative position</u> of mass B with respect to mass A. A force F is applied to mass B.



Find:

a) Write down an expression for the kinetic energy T in terms of the generalized coordinates x_1 and x_2 their time derivatives. From this expression, identify the elements m_{ij}, where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{q}_i\dot{q}_j$$

b) Determine the generalized force(s) associated with each generalized coordinate.

Solution:

$\vec{r}_1 = x_1 \hat{\imath}$, $\vec{r}_2 = (x_1 + x_2)\hat{\imath}$

$\vec{v}_1 = \dot{x}_1 \hat{\imath}$, $\vec{v}_2 = (\dot{x}_1 + \dot{x}_2)\hat{\imath}$

$T = \frac{1}{2} M \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m \vec{v}_2 \cdot \vec{v}_2$

$T = \frac{1}{2} M \dot{x}_1^2 + \frac{1}{2} m(\dot{x}_2^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_1^2)$

$= \frac{1}{2}(m + M)\dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_1\dot{x}_2 + \frac{1}{2} m \dot{x}_2^2$

$m_{11} = m + M_1 \quad m_{12} = m_{21} = m \ , \quad m_{22} = m$

$$[m] = \begin{bmatrix} m+M_1 & m \\ m & m \end{bmatrix}$$

$dW = F\hat{i} \cdot \delta\vec{r}_1 \ \hat{i}$

$\delta r_1 = \delta x_1 \hat{i} +$

$dW = F\hat{i} \cdot (\delta x_1 \hat{i})$

$= F \delta x_1$

$Q_1 = F$ and $Q_2 = 0$

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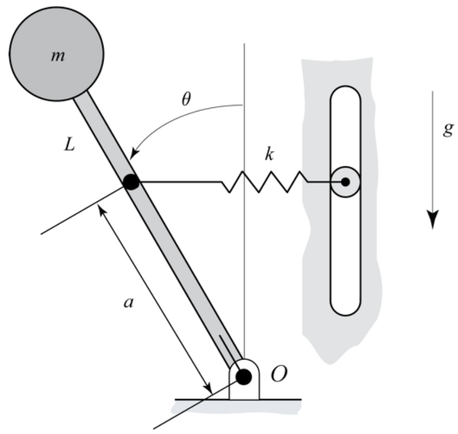
Test Problem 2 -30 points

A lumped mass m is a attached to a bar of negligible mass of length L. A spring of stiffness k is attached to lever at point A and the other end is allowed to slide on a smooth surface of a slot. Spring is unstretched at θ=0.



Find:

a) Use the power equation method and show that the equation of motion of the system can be written as

$$mL^2\ddot{\theta} + ka^2 \sin\theta\cos\theta - mgL\sin\theta = 0.$$

b) Use small angle approximations to derive the linear equation of motion around θ_e =0 (rad).

c) Bonus (2.5 points): The system has multiple equilibira . One set of equilibrium points θ_e =0, 2π, the other set corresponds to non-horizontal equilibirum points. Derive the equation that governs these non-horinzontal equilibrium points. Discuss the physics behind this phenomena.

d) Bonus (2.5 points): For what value of stiffness will the solution of the linear equations become unbounded. Discuss the physics behind this phenomena.

Solution:

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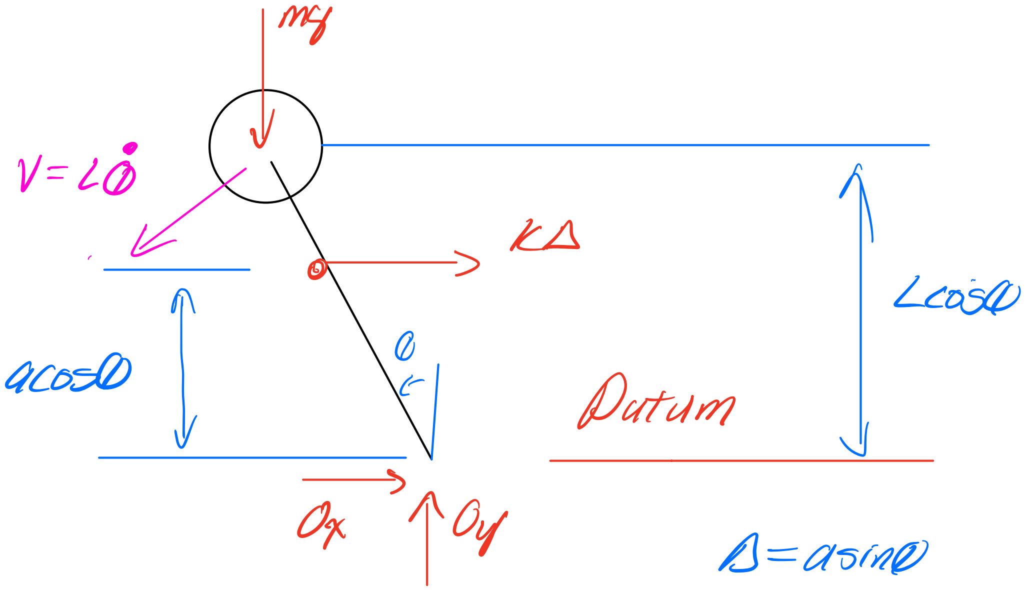
Test Problem 2 Additional Page

a)



$$T = \tfrac{1}{2} m V^2 \qquad U = \tfrac{1}{2} k \Delta^2 + mgL\cos\theta$$

$$T = \tfrac{1}{2} m L^2 \dot{\theta}^2$$

$$U = \tfrac{1}{2} k a^2 \sin^2\theta + mgL\cos\theta$$

$$T + U = \tfrac{1}{2} m L^2 \dot{\theta}^2 + \tfrac{1}{2} k a^2 \sin^2\theta + mgL\cos\theta$$

$$\frac{d}{t}(T+U) = mL^2\dot{\theta}\ddot{\theta} + ka^2\sin\theta\cos\theta\,\dot{\theta} - mgL\sin\theta\,\dot{\theta}$$

$$= 0$$

$$\left(mL^2\ddot{\theta} + ka^2\sin\theta\cos\theta - mgL\sin\theta\right)\dot{\theta} = 0$$

$$mL^2\ddot{\theta} + ka^2\sin\theta\cos\theta - mgL\sin\theta = 0$$

b) small ∠'s $\cos\theta \approx 1$, $\sin\theta \approx \theta$

$$mL^2\ddot{\theta} + ka^2\theta - mgL\theta = 0$$

$$mL^2\ddot{\theta} + (ka^2 - mgL)\theta = 0$$

c) Equilibrium point

$\theta = \theta_{eq}$ $\ddot{\theta} = 0$

$$ka^2 \sin\theta_{eq} \cos\theta_{eq} - mgL\sin\theta_{eq} = 0$$

$$(ka^2\cos\theta_{eq} - mgL)\sin\theta_{eq} = 0$$

$\sin\theta_{eq} = 0$ $\theta_{eq} = 0, \pi, 2\pi, 3\pi, \ldots$

$$ka^2\cos\theta_{eq} = mgL$$

$$\theta_{eq} = \cos^{-1}\left(\frac{mgL}{ka^2}\right)$$

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d) $mL^2\ddot{\theta} + (ka^2 - mgL)\theta = 0$

all coefficients must have same sign for solution to be bounded

$ka^2 - mgL < 0$ to be unbounded

$ka^2 < mgL \longrightarrow k < mgL/a^2$

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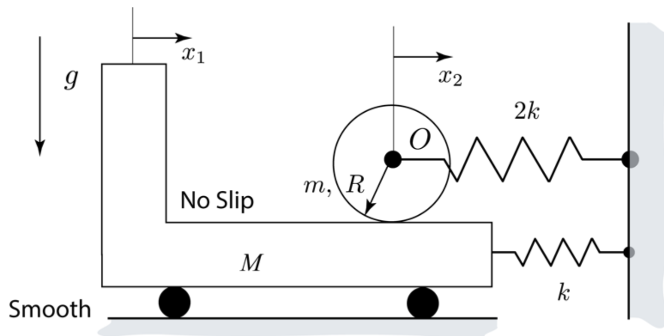
Test Problem 3-40 points

Consider the system below, whose motion is described by the absolute coordinates shown. The cylinder rolls without slip with respect to the cart.



Find:

a) Use the **method of influence coefficients** to derive the flexibility matrix $[A] = [K]^{-1}$.

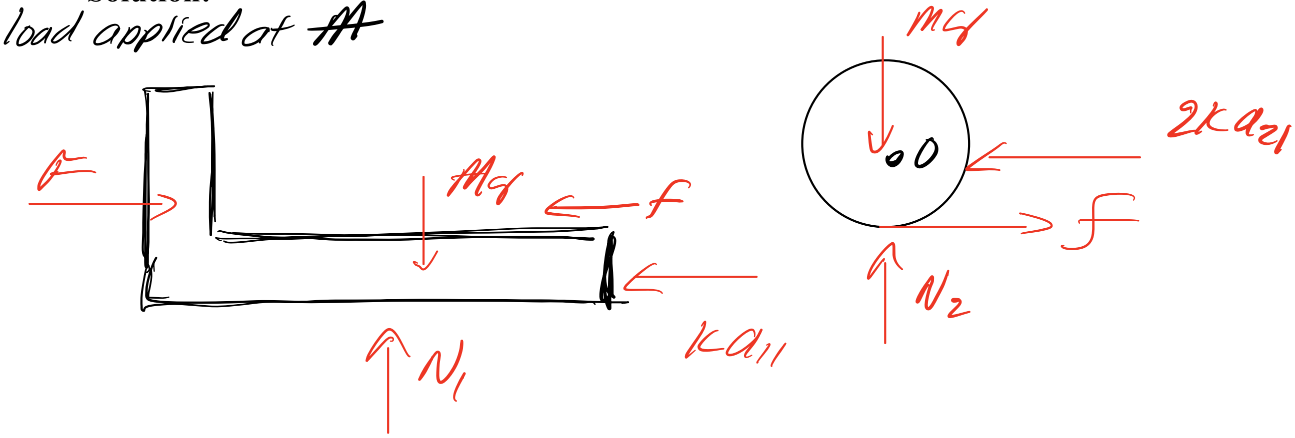
b) Write down the potential energy function U for this two-DOF system and use the following results from lecture to develop the stiffness matrix for the system:

$$K_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\mathbf{q}_0}$$

a)

Solution:

load applied at ~~A~~



$\overset{+}{\rightarrow} \Sigma F_x$: $F - f - ka_{11} = 0$

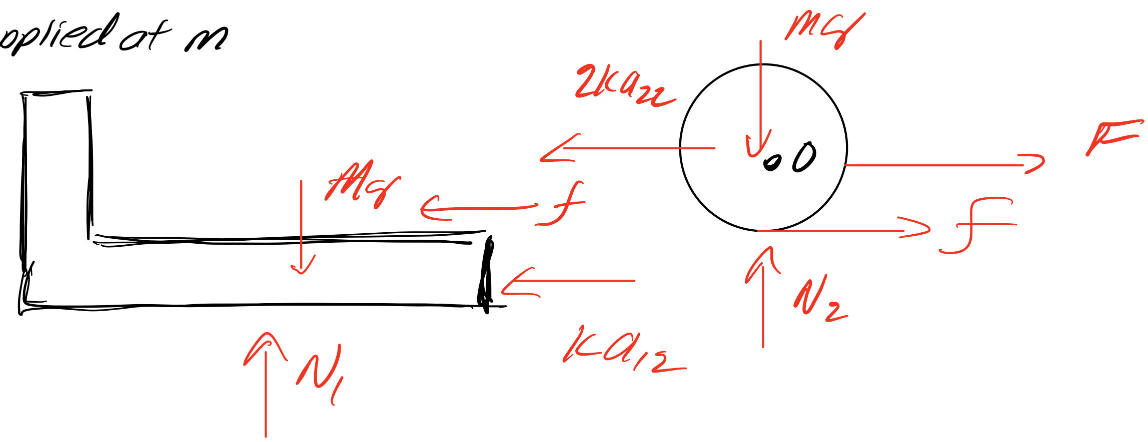
$\overset{+}{\rightarrow} \Sigma F_x$: $-2ka_{21} + f = 0$

$\overset{+}{\circlearrowleft} \Sigma M_O$: $fR = 0$

$a_{21} = 0 \qquad a_{12} = 0$

$a_{11} = 1/k$

load applied at m



$\xrightarrow{+} \sum F_x: \quad -ka_{12} - f = 0$

$\xrightarrow{+} \sum F_x: \quad -2ka_{22} + f + F = 0$

$\circlearrowleft^{+} \sum M_O: \quad fR = 0$

$f = 0 \qquad a_{12} = 0$

$a_{22} = 1/2k$

$$[a] = \begin{bmatrix} 1/k & 0 \\ 0 & 1/2k \end{bmatrix}$$

$$U = \tfrac{1}{2} k x_1^2 + \tfrac{1}{2}(2k) x_2^2$$

$$\frac{\partial U}{\partial x_1} = k x_1 \qquad \frac{\partial U}{\partial x_2} = 2k x_2$$

$$k_{11} = \frac{\partial^2 U}{\partial x_1^2} = k \qquad k_{12} = k_{21} = \frac{\partial^2 U}{\partial x_1 \partial x_2} = 0$$

$$k_{22} = \frac{\partial^2 U}{\partial x_2} = 2k$$

$$[K] = \begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix}$$

$$[a] = \begin{bmatrix} 1/k & 0 \\ 0 & 1/2k \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & 2k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Newton Euler

$$\sum \vec{F} = m\vec{a}_g$$
$$\sum \vec{M}_A = I_A\vec{\alpha}$$

A is a fixed point or center of gravity

Power Equation

$$T + U = T_o + U_o + W^{(nc)}$$
$$Power = \frac{dW^(nc)}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$R_i = \frac{1}{2}c_i\dot{\Delta}_i^{\,2}$$

$$U = \sum (U_{sp})_i + \sum (U_{gr})_i$$
$$(U_{sp})_i = \frac{1}{2}k_i\Delta_i^2$$
$$(U_{gr})_i = m_i g h_i$$

Linearized Lagrange's Equations

$$[M]\ddot{\vec{z}} + [C]\dot{\vec{z}} + [K]\vec{z} = \vec{0}$$

$$\vec{z}(t) = \vec{q}(t) - \vec{q}_o$$
$$M_{ik} = (m_{ik})_{\vec{q}_o} = M_{ki}$$
$$C_{ik} = \left(\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_k}\right) = C_{ki}$$
$$K_{ik} = \left(\frac{\partial^2 R}{\partial q_i \partial q_k}\right) = K_{ki}$$