Test 1

Name ____________________________________

Pledge: I have neither given nor received aid on this examination.

Signature: ____________________________________

Instructions:

• This is a closed-book, closed-notes exam.

• Please read the question on this exam carefully and answer only the questions I ask. Don’t waste your time doing extra work, and don’t skip any of the smaller questions within a problem. I can’t grade what I can’t see, and I can’t give partial credit for something in your head.

EQUATION SHEET IS AT BACK

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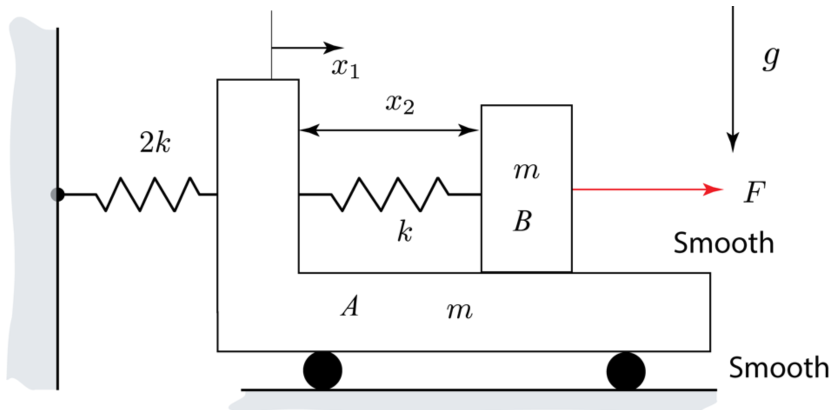
Test Problem 1 – 30 points

Consider the system below where all surfaces are smooth. Let x_1 denote the absolute position of mass *A* and x_2 is <u>the relative position</u> of mass *B* with respect to mass *A*. A force *F* is applied to mass *B*.



Find:

a) Write down an expression for the kinetic energy *T* in terms of the generalized coordinates x_1 and x_2 their time derivatives. From this expression, identify the elements m_{ij}, where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{q}_i\dot{q}_j$$

b) Determine the generalized force(s) associated with each generalized coordinate.

Solution:

$$\vec{r}_1 = x_1 \hat{\imath} \quad , \quad \vec{r}_2 = (x_1 + x_2)\hat{\imath}$$

$$\vec{v}_1 = \dot{x}_1 \hat{\imath} \quad , \quad \vec{v}_2 = (\dot{x}_1 + \dot{x}_2)\hat{\imath}$$

$$T = \tfrac{1}{2} A\, \vec{v}_1 \cdot \vec{v}_1 + \tfrac{1}{2} m\, \vec{v}_2 \cdot \vec{v}_2$$

$$T = \tfrac{1}{2} A\, \dot{x}_1^2 + \tfrac{1}{2} m(\dot{x}_2^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_1^2)$$

$$= \tfrac{1}{2}(m + A)\dot{x}_1^2 + \tfrac{1}{2}(2m)\dot{x}_1\dot{x}_2 + \tfrac{1}{2} m \dot{x}_2^2$$

$m_{11} = m + M_1 \quad m_{12} = m_{21} = m \ , \quad m_{22} = m$

$$[m] = \begin{bmatrix} m+M_1 & m \\ m & m \end{bmatrix}$$

$dW = F\hat{i} \cdot \delta\vec{r}_2\,\hat{i}$

$\delta r_2 = \delta x_1 \hat{i} + \delta x_2 \hat{i}$

$dW = F\hat{i} \cdot (\delta x_1 \hat{i} + \delta x_2 \hat{i})$

$= F\delta x_1 + F\delta x_2$

$Q_1 = F \quad$ and $\quad Q_2 = F$

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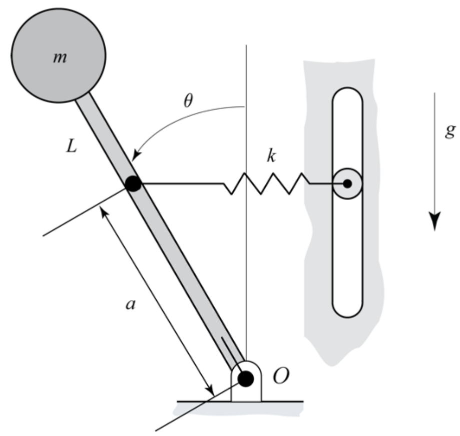
Test Problem 2 -30 points

A lumped mass m is a attached to a bar of negligible mass of length L. A spring of stiffness k is attached to lever at point A and the other end is allowed to slide on a smooth surface of a slot. Spring is unstretched at θ=0.



Find:

a) Use the power equation method and show that the equation of motion of the system can be written as

$$mL^2\ddot{\theta} + ka^2 \sin\theta \cos\theta - mgL \sin\theta = 0.$$

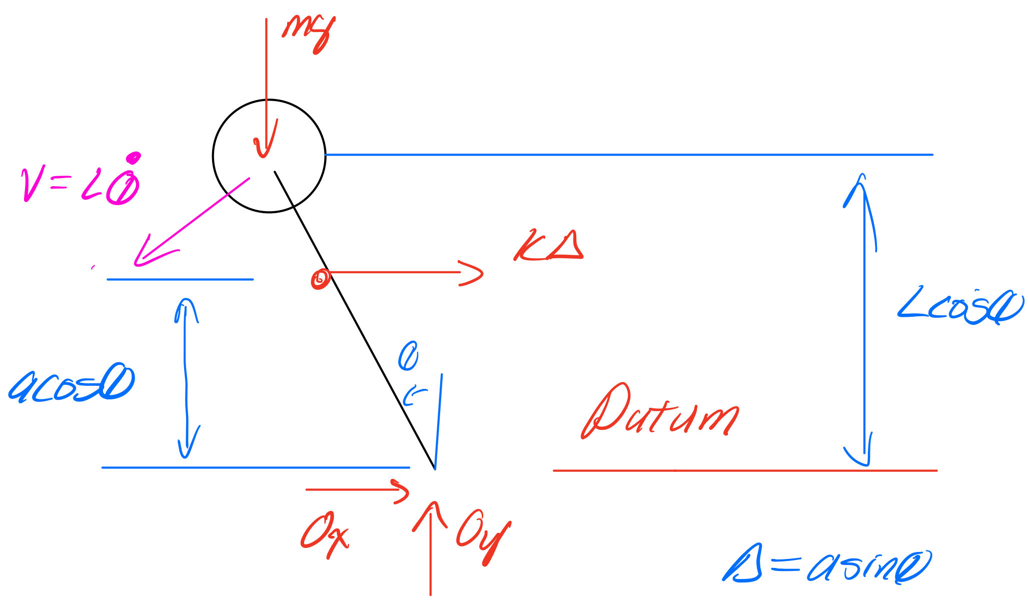
b) Use small angle approximations to derive the linear equation of motion around θ_e =0 (rad).

c) Bonus (2.5 points): The system has multiple equilibira . One set of equilibrium points θ_e =0, 2π, the other set corresponds to non-horizontal equilibirum points. Derive the equation that governs these non-horinzontal equilibrium points. Discuss the physics behind this phenomena.

d) Bonus (2.5 points): For what value of stiffness will the solution of the linear equations become unbounded. Discuss the physics behind this phenomena.

Solution:

a)



$$T = \tfrac{1}{2} m V^2 \qquad U = \tfrac{1}{2} k \Delta^2 + mgL\cos\theta$$

$$T = \tfrac{1}{2} m L^2 \dot{\theta}^2$$

$$U = \tfrac{1}{2} k a^2 \sin^2\theta + mgL\cos\theta$$

$$T + U = \tfrac{1}{2} m L^2 \dot{\theta}^2 + \tfrac{1}{2} k a^2 \sin^2\theta + mgL\cos\theta$$

$$\frac{d}{dt}(T+U) = mL^2\dot{\theta}\ddot{\theta} + ka^2\sin\theta\cos\theta\,\dot{\theta} - mgL\sin\theta\,\dot{\theta}$$

$$= 0$$

$$\left(mL^2\ddot{\theta} + ka^2\sin\theta\cos\theta - mgL\sin\theta\right)\dot{\theta} = 0$$

$$mL^2\ddot{\theta} + ka^2\sin\theta\cos\theta - mgL\sin\theta = 0$$

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b)

small ∠'s $\cos\theta \approx 1$, $\sin\theta \approx \theta$

$$mL^2\ddot{\theta} + ka^2\theta - mgL\theta = 0$$

$$mL^2\ddot{\theta} + (ka^2 - mgL)\theta = 0$$

c)

Equilibrium point

$\theta = \theta_{eq}$ $\ddot{\theta} = 0$

$$ka^2 \sin\theta_{eq}\cos\theta_{eq} - mgL\sin\theta_{eq} = 0$$

$$(ka^2\cos\theta_{eq} - mgL)\sin\theta_{eq} = 0$$

$\sin\theta_{eq} = 0$ $\theta_{eq} = 0, \pi, 2\pi, 3\pi, \ldots$

$$ka^2\cos\theta_{eq} = mgL$$

$$\theta_{eq} = \cos^{-1}\left(\frac{mgL}{ka^2}\right)$$

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d) $mL^2\ddot{\theta} + (ka^2 - mgL)\theta = 0$

all coefficients must have same sign for solution to be bounded

$ka^2 - mgL < 0$ to be unbounded

$ka^2 < mgL \longrightarrow k < mgL/a^2$

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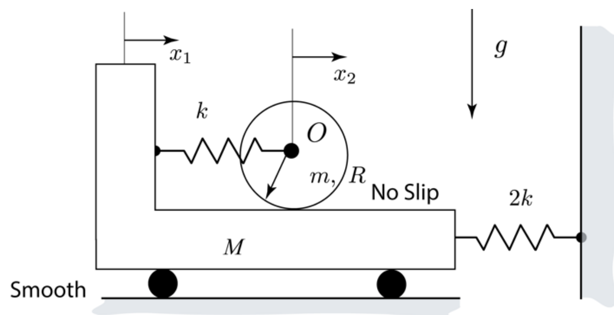
Test Problem 3-40 points

Consider the system below, whose motion is described by the absolute coordinates shown. The cylinder rolls without slip with respect to the cart.



Find:

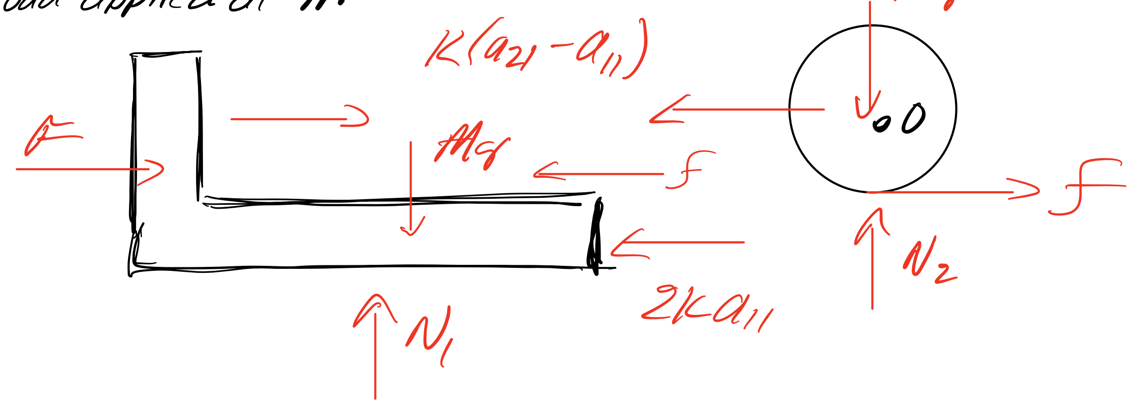
a) Use the **method of influence coefficients** to derive the flexibility matrix $[A] = [K]^{-1}$.

b) Write down the potential energy function U for this two-DOF system and use the following results from lecture to develop the stiffness matrix for the system:

$$K_{ij} = \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\mathbf{q}_0}$$

a) **Solution:**

load applied at ~~A~~



$$\xrightarrow{+} \Sigma F_x: \; F - 2ka_{11} + k(a_{21} - a_{11}) - f = 0$$

$$\xrightarrow{+} \Sigma F_x: \; -k(a_{21} - a_{11}) + f = 0$$

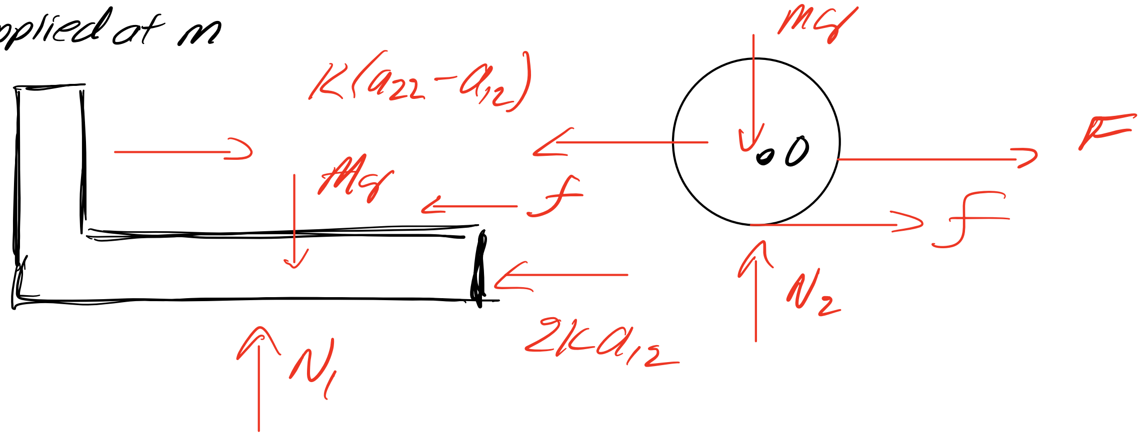
$$\circlearrowleft^{+} \Sigma M_O: \; fR = 0$$

$\theta = 0 \qquad a_{21} = a_{11}$

$a_{11} = F/2k = 1/2k$ since $F = 1$

$a_{11} = a_{21} = a_{12} = 1/2k$

load applied at m



$\overset{+}{\rightarrow} \Sigma F_x: \; k(a_{22}-a_{12}) - 2ka_{12} - f = 0$

$\overset{+}{\rightarrow} \Sigma F_x: \; f + F - k(a_{22}-a_{12}) = 0$

$\circlearrowleft_{+} \Sigma M_O: \; fR = 0$

$k(1/2k - 1/2k$

$f = 0$

$\overset{1}{\cancel{F}} - k(a_{22} - \underset{1/2k}{\cancel{a_{12}}}) = 0$

$1 - ka_{22} + k/2k = 0$

$ka_{22} = 3/2k \qquad a_{22} = 3/2k$

$$[a] = \frac{1}{k}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

b) $$U = \frac{1}{2}(2k)(x_1)^2 + \frac{1}{2}k(x_2 - x_1)^2$$

$$\frac{\partial U}{\partial x_1} = 2kx_1 + k(x_1 - x_2) = 3kx_1 - kx_2$$

$$\frac{\partial U}{\partial x_2} = k(x_2 - x_1)$$

$$k_{11} = \frac{\partial^2 U}{\partial^2 x_1} = 3k \quad , \quad k_{12} = \frac{\partial U}{\partial x_1 \partial x_2} = -k = k_{12}$$

$$k_{22} = \frac{\partial^2 U}{\partial x_2^2} = k$$

$$[k] = \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} = k\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[a][k] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cancelto{1}{3/2 - 1/2} & \cancelto{0}{-1/2 + 1/2} \\ \cancelto{0}{3/2 - 3/2} & \cancelto{1}{-1/2 + 3/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Newton Euler

$$\sum \vec{F} = m\vec{a}_g$$
$$\sum \vec{M}_A = I_A\vec{\alpha}$$

A is a fixed point or center of gravity

Power Equation

$$T + U = T_o + U_o + W^{(nc)}$$
$$Power = \frac{dW^(nc)}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

$$R_i = \frac{1}{2}c_i\dot{\Delta}_i^{\ 2}$$

$$U = \sum(U_{sp})_i + \sum(U_{gr})_i$$
$$(U_{sp})_i = \frac{1}{2}k_i\Delta_i^2$$
$$(U_{gr})_i = m_i g h_i$$

Linearized Lagrange's Equations

$$[M]\ddot{\vec{z}} + [C]\dot{\vec{z}} + [K]\vec{z} = \vec{0}$$

$$\vec{z}(t) = \vec{q}(t) - \vec{q}_o$$
$$M_{ik} = (m_{ik})_{\vec{q}_o} = M_{ki}$$
$$C_{ik} = \left(\frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_k}\right) = C_{ki}$$
$$K_{ik} = \left(\frac{\partial^2 R}{\partial q_i \partial q_k}\right) = K_{ki}$$