Test 1

Name**______________________________________**

Pledge**______________________________________**

I have neither given nor received aid on this examination.

Instructions:

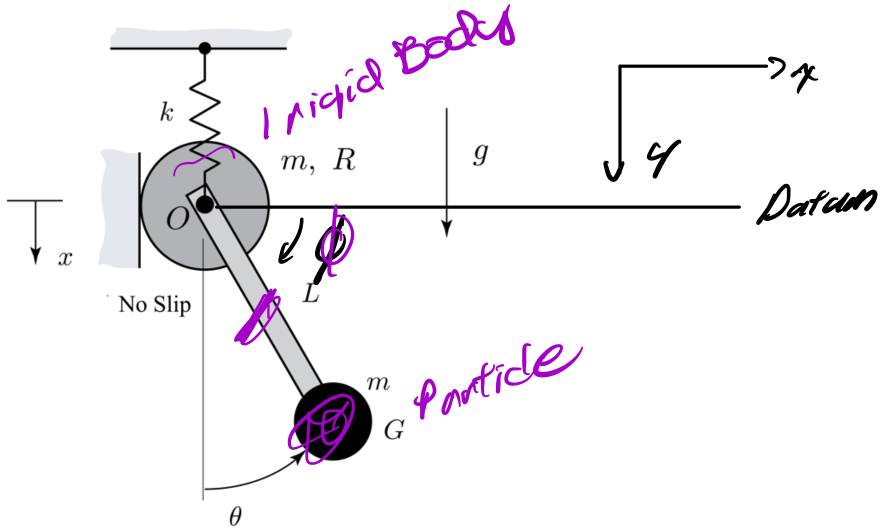
• This is a closed-book, closed-notes exam.

• You are NOT allowed to use a programmable calculator during the exam.

• Please read the question on this exam carefully and answer only the questions I ask. Don't waste your time doing extra work, and don't skip any of the smaller questions within a problem. I can't grade what I can't see and I can't give partial credit for something in your head.

ME 563 - Fall 2022 **Name_______________**
Test Problem 1 - 30points

A bar with an end mass m at G is attached to a wheel pt O. The end mass may be considered a particle. The wheel rolls without slip along a wall. A spring is attached to the wheel and deforms purely in verical (x) direction. The bar has neglible mass. The wheel has mass m and radius R and mass moment of inertia about its center of gravity $I^G = 1/2mR^2$. The coordinate x denotes the absolute position of the wheel's center at O and θ the angular position of the bar.

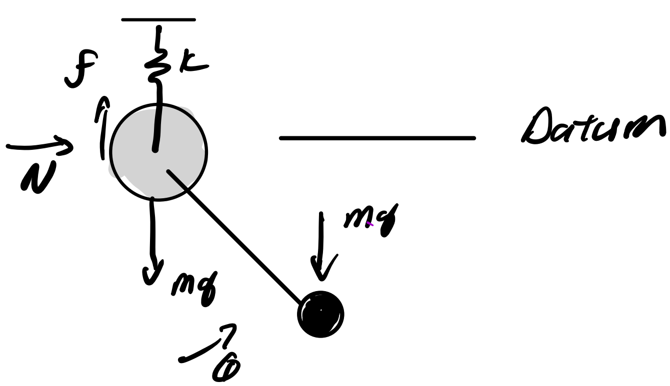


a) Determine the expression for potential energy U in terms of the generalized coordinates x and θ and determine the equilibrium positions of the system.

b) Write down an expression for the kinetic energy T in terms of the generalized coordinates x and θ and their time derivatives. From this expression, identify the elements m_{ij}, where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{q}_i\dot{q}_j$$

c) Determine the mass matrix $[M]$ and the stiffness matrix $[K]$ corresponding small oscillations about the equilibrium state.



a) $U = -mgx - mg(x + L\cos\theta) + \frac{1}{2}kx^2$

$$\frac{\partial U}{\partial x} = -mg - mg + kx = -2mg + kx \sim F$$

$$\frac{\partial U}{\partial \theta} = +mgL\sin\theta \sim M$$

$$\left.\frac{\partial U}{\partial q}\right|_{q_0} = 0 \longrightarrow \quad -2mg + kx_c \longrightarrow x_c = 2mg/k$$

$$mgL\sin\theta_c = 0 \longrightarrow \theta_c = n\pi$$

$n = 0, 1, 2$

b) $r_0 = x\hat{i}$ $\quad \vec{r}_c = \vec{r}_0 + \vec{r}_{c/o} = x\hat{i} + L\cos\theta\hat{i} + L\sin\theta\hat{j}$

$v_0 = \dot{x}\hat{i}$ $\quad \vec{r}_c = (x + L\cos\theta)\hat{i} + L\sin\theta\hat{j}$

$$\vec{v}_c = (\dot{x} - \dot{\theta}L\sin\theta)\hat{i} + \dot{\theta}L\cos\theta\hat{j}$$

$$T = \frac{1}{2}m\vec{v}_o\cdot\vec{v}_o + \frac{1}{2}m\vec{v}_c\cdot\vec{v}_c + \frac{1}{2}I^G\dot{\phi}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}L\sin\theta + \dot{\theta}^2L^2\sin^2\theta + \dot{\theta}^2L^2\cos^2\theta) + \frac{1}{2}I^G\dot{\phi}^2$$

$$\dot{x} = R\dot{\phi}$$

$$T = m(\dot{x}^2 - \dot{x}\dot{\theta}L\sin\theta + \frac{1}{2}\dot{\theta}^2L^2) + \frac{1}{2}\frac{I_G}{R^2}\dot{x}^2$$

$$m_{11} = 2m + I_o/R^2$$

$$m_{12} = m_{12} = mL\sin\theta$$

$$m_{22} = mL^2$$

$$[m] = \begin{bmatrix} 2m + I_o/R^2 & -mL\sin\theta \\ -mL\sin\theta & mL^2 \end{bmatrix}$$

$$[m]\Big|_{\bar{q}_0} = \begin{bmatrix} 2m + I_o/R^2 & 0 \\ 0 & mL^2 \end{bmatrix}$$

$$k_{11} = k \qquad k_{12} = k_{21} = 0$$

$$k_{22} = mgL\cos\theta$$

$$[k] = \begin{bmatrix} k & 0 \\ 0 & mgL\cancel{\cos}\theta \end{bmatrix}$$

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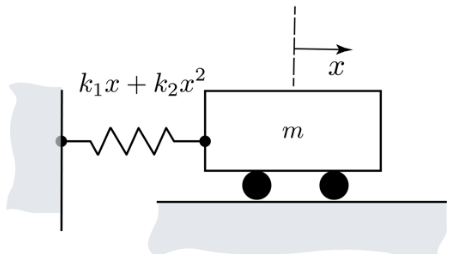
Test Problem 2 -30 points

A nonlinear linear single degree of freedom oscillator is attached to spring whose force is given $F_s = k_1x + k_2x^2$ where the stiffness's $k_1 = 4000$ N/m, and $k_2 = 20$ N/m^2 and a mass $m = 1000$ kg and $x(0)=1$ mm and ~~v(0)= 2.34 mm/s.~~ 0

a) Derive the linear equation of motion for the system small oscillations about an near equilibrium point.

$F_s = \frac{dV}{dx}$



b) Plot the solution for at least two periods. Determine numerical values for and label the initial conditions, amplitude, and period on the plot. Recall the solution to undamped oscillator can be written as

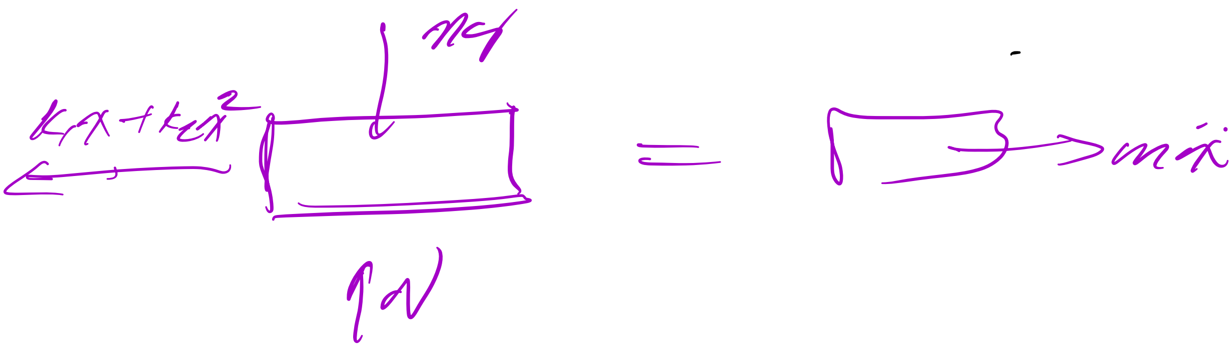
$$\cancel{x(t) = C\cos\omega_n t + S\sin\omega_n t},$$

or

$$\cancel{x(t) = A\sin(\omega_n t + \phi)},$$

~~where~~ $\cancel{A = \sqrt{C^2 + S^2}}$ ~~and~~ $\cancel{\phi = \tan^{-1}\left(\frac{C}{S}\right)}$.

c) If a damper is attached to the system with damping coefficient of $c = 200$ kg/s. Determine the damping ratio. Is the linear system underdamped, overdamped, or critically damped. If the system is underdamped find the damped natural frequency. If the system is overdamped or critically damped report the eigenvalues of the system.



$\Sigma F_x: \; -k_1x - k_2x^2 = m\ddot{x}$

$m\ddot{x} + k_1x + k_2x^2 = 0$

$x = x_c + \hat{x}$ $\qquad x = x_c$ $\qquad \ddot{x} = \dot{x} = 0$

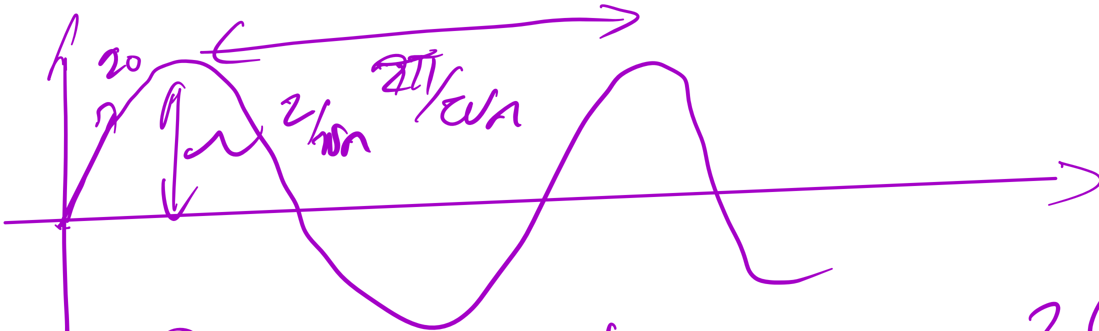
$k_1 x_c + k_2 x_c^2 = 0$

$x_c(k_1 + k_2 x_c) = 0$ $\qquad x_c = 0$ $\qquad x_c = -\frac{k_1}{k_2}$

$F_s = k_1 x + k_2 x^2 \Big|_{x_c = 0} + (k_1 + 2k_2 x)\Big|_{x_c} (\overset{\hat{x}}{\cancel{x - x_c}})$

$F_s \cong k_1 \hat{x}$

$m\ddot{\hat{x}} + k_1 \hat{x} = 0$ $\qquad \omega_n = \sqrt{k_1/m}$



$c = 0$

$\hat{x} = C\cos\omega_n t + D\sin\omega_n t$

$\dot{\hat{x}} = -\omega_n C\sin\omega_n t + \omega_n D\cos\omega_n t$

$D = \frac{v_0}{\omega_n}$

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Test Problem 3-30 points

A 2-DOF system has the follow equations of motion.

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\\ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}\alpha+\frac{1}{2} & \alpha-1\\\alpha-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\x_2\end{Bmatrix}=\vec{0}$$

mass normalized

a) Find the value of α that admits a rigid body mode where α>0.
b) Using a value of α found in a) to determine the natural frequencies and mode shapes of the system.
c) ~~Given that the system is at rest find the response when $x_1(0)=0$ and $x_2(0)=2$.~~
d) ~~Can you find initial conditions~~

a) $\det[K]=0$

$$\Delta_K = 2\alpha+1-(\alpha-1)^2=0$$

$$=2\alpha+1-(\alpha^2-2\alpha+1)=0$$

$$=2\alpha+1-\alpha^2+2\alpha-1=0$$

$$=4\alpha-\alpha^2=0$$

$$\alpha=0 \quad \alpha=4$$

$$m\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\ddot{\bar{x}}+k\begin{bmatrix}9/2 & 3\\3 & 2\end{bmatrix}\bar{x}=0$$

$$\begin{bmatrix}-\omega^2 m+9/2k & 3k\\3k & -\omega^2 m+2k\end{bmatrix}\begin{Bmatrix}X_1\\X_2\end{Bmatrix}=\vec{0}$$

$$(-\omega^2 m+9/2k)(-\omega^2 m+2k)-9k^2=0$$

$$\omega^4 m^2 - 2km\omega^2 - \tfrac{9}{2}km\omega^2 + 9k^2 - 9k^2 = 0$$

$$\omega^4 m^2 - \frac{13}{2} km\omega^2 = 0$$

$$\omega^4 - \frac{13}{2} \frac{k}{m} \omega^2 = 0$$

$$\left(\omega^2 - \frac{13}{2}\frac{k}{m}\right)\omega^2 = 0$$

$$\omega = \pm\sqrt{\tfrac{13}{2} k/m} \qquad \omega = 0$$

Modeshapes

$$(-\omega^2 m + \tfrac{9}{2}k)X_1 + 3k X_2 = 0$$

$$\frac{X_2}{X_1} = \frac{-\omega^2 m + \tfrac{9}{2}k}{-3k} = \frac{+\omega^2 m}{3k} - \frac{3}{2}$$

$$\frac{X_2}{X_1} = \frac{\omega^2 m}{3k} - \frac{3}{2}$$

$\omega = 0$

$$\frac{X_2}{X_1} = -\frac{3}{2} \qquad \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix}$$

$$\omega = \sqrt{13/2}\sqrt{k/m}$$

$$\frac{X_2}{X_1} = \frac{13}{6} - \frac{3}{2} = \frac{2}{3}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix} =$$

$$\alpha_1 = \frac{1}{\sqrt{x'^T [m] x}} = \frac{1}{\sqrt{\{1\ {-3/2}\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix} m}}$$

$$\alpha_1 = \frac{1}{\sqrt{m\,(1^2 + 9/4)}} = \frac{1}{\sqrt{m\,13/4}}$$

$$\alpha_1 = \frac{2}{\sqrt{13m}}$$

$$\alpha_2 = \frac{1}{\sqrt{\{1\ \ 2/3\} \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix} m}} = \frac{1}{\sqrt{m\,(1 + 4/9)}}$$

$$\frac{1}{\sqrt{m\,13/9}}$$

$$\alpha_2 = \frac{3}{\sqrt{13m}}$$

$$\bar{X} = (c_1 + s_1 t)\,\bar{\bar{X}}' + c_2 \bar{\bar{X}}_2 \cos$$

$+ s_2 \bar{X}_2 \sin$

Tnw

$\dot{x}(0) = 0 \qquad s_1 = s_2$

$\vec{x}(0) = c_1 \vec{X} + c_2 \vec{X}_2$

$\vec{x}(0) = c_1$

$$X = c_1 X_1 + c_2 X_2 \cos \omega_2 t$$

$$c_2 = X^T [m] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$0 = \frac{3}{\sqrt{13m}} \begin{pmatrix} 1 & 2/3 \end{pmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\left(\frac{3\sqrt{m}}{\sqrt{13}} \right) \left(x_1 + 2/3\, x_2 \right) = 0$$

$$x_1 = -2/3\, x_2$$

Newton Euler

$$\sum \vec{F} = m\vec{a}_g$$
$$\sum \vec{M}_A = I_A\vec{\alpha}$$

A is a fixed point or center of gravity

SDOF Response

$$m\ddot{x} + c\dot{x} + kx = 0,$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$
$$2\zeta\omega_n = \frac{c}{m},\ \omega_n^2 = \frac{k}{m}$$

$$0 \le \zeta \le 1,$$
$$x(t) = \exp -\zeta\omega_n t\,(C\cos\omega_d t + S\sin\omega_d t)$$

Power Equation

$$T + U = T_o + U_o + W^{(nc)}$$
$$Power = \frac{dW^(nc)}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

Eigenvalue Problem

$$[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0},$$
$$\vec{x}(t) = \vec{X}\exp i\omega t,$$
$$\left(-\omega^2[M] + [K]\right)\vec{X} = \vec{0}$$

MDOF Response

$$\vec{x}(t) = \sum_{j=1}^{N}\vec{X}^(j)[c_j\cos\omega_j t + s_j\sin\omega_j t]$$
$$c_j = \frac{\vec{X}^{(j)T}[M]\vec{x}(0)}{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$
$$sj = \frac{\vec{X}^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$

Linearized Lagrange's Equations

$$[M]\ddot{\vec{z}} + [C]\dot{\vec{z}} + [K]\vec{z} = \vec{0}$$

$$\vec{z}(t) = \vec{q}(t) - \vec{q}_0$$
$$M_{ik} = (m_{ik})_{\vec{q}_0} = M_{ki}$$
$$C_{ik} = \left(\frac{\partial^2 R}{\partial q_i\partial q_k}\right)_{\vec{q}_0} = C_{ki}$$
$$K_{ik} = \left(\frac{\partial^2 U}{\partial q_i\partial q_k}\right)_{\vec{q}_0} = K_{ki}$$

Mass Normalized Eigenvector

$$\alpha_j = \frac{1}{\sqrt{\vec{X}^{(j)T}[M]\vec{X}^j}}$$
$$\vec{X}_m^j = \alpha_m\vec{X}^j$$
$$c_j = \vec{X}_m^{(j)T}[M]\vec{x}(0)$$
$$s_j = \frac{\vec{X}_m^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j}$$

Log Decrement SDOF

$$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right)$$
$$\zeta = \frac{\delta/2\pi}{\sqrt{1 + (\delta/2\pi)^2}}$$
$$\zeta << 1 \rightarrow \zeta = \frac{\delta}{2\pi}$$

Test 1

Name KEY

Pledge______________________________

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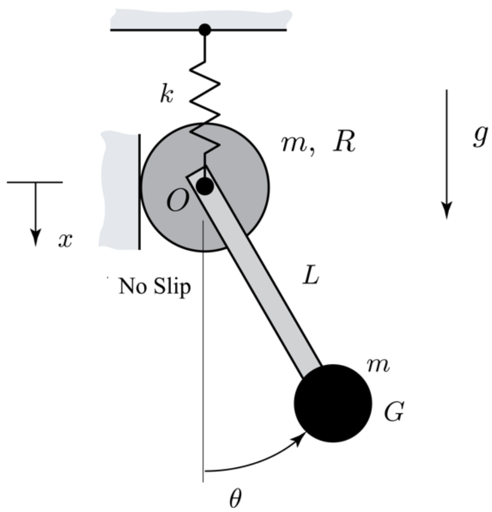
ME 563 - Fall 2022 ***Name_______________***

Test Problem 1 – 40 points

A bar with an end mass *m* at *G* is attached to a wheel pt *O*. The end mass may be considered a particle. The *wheel rolls without slip* along a wall. A spring is attached to the wheel and deforms purely in verical direction, *x* measures the distance point *O* moves. The bar has neglible mass. The wheel has mass *m* and radius *R* and mass moment of inertia about its center of gravity $I^G = 1/2mR^2$. The coordinate *x* denotes the absolute position of the wheel's center at *O* and θ the angular position of the bar.

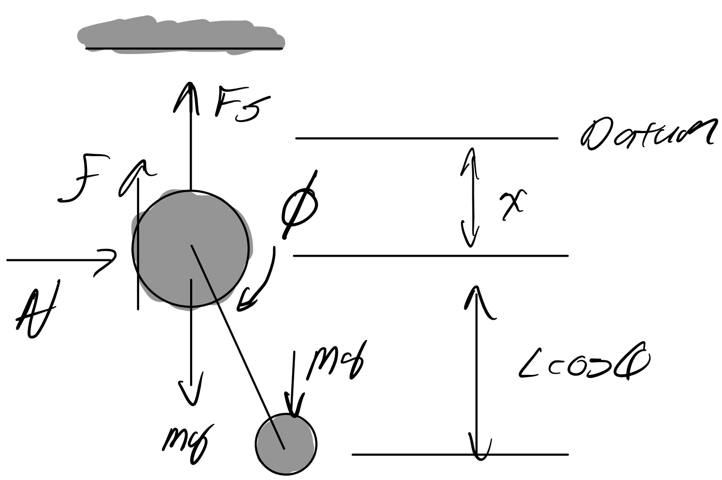


a) Determine the expression for potential energy *U* in terms of the generalized coordinates *x* and θ and determine the equilibrium positions of the system.

b) Write down an expression for the kinetic energy *T* in terms of the generalized coordinates *x* and θ and their time derivatives. From this expression, identify the elements m_{ij}, where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{q}_i\dot{q}_j$$

c) Determine the mass matrix [*M*] and the stiffness matrix [*K*] corresponding small oscillations about the equilibrium state.



Position & Velocity Vectors

$\vec{r}_O = x\hat{i}$

$\vec{r}_G = (x + L\cos\theta)\hat{i} + L\sin\theta\hat{j}$

$\vec{v}_O = \dot{x}\hat{i}$

$\vec{v}_G = (\dot{x} - \dot{\theta}L\sin\theta)\hat{i} + \dot{\theta}L\cos\theta\hat{j}$

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Test Problem 1 Additional Page

a) $U = \frac{1}{2}kx^2 - mgx - mg(x + L\cos\theta)$

$= \frac{1}{2}kx^2 - 2mgx - mgL\cos\theta$ ~Ans

$\frac{\partial U}{\partial x} = kx - 2mg \qquad \frac{\partial U}{\partial \theta} = +mgL\sin\theta$

$\left.\frac{\partial U}{\partial x}\right|_{x_e} = kx_e - 2mg = 0 \longrightarrow x_e = +\frac{2mg}{k}$

$\left.\frac{\partial U}{\partial \theta}\right|_{\theta_e} = +mgL\sin\theta_e = 0 \longrightarrow \theta_e = n\pi \quad n = 0, 1, 2 \ldots$

$\vec{q}_e = \begin{Bmatrix} 2mg/k \\ n\pi \end{Bmatrix}, \; n = 0, 1, 2 \ldots$ ~Ans

b) $T = \frac{1}{2}m\vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2}I^G\dot{\phi}^2 + \frac{1}{2}m\vec{v}_G \cdot \vec{v}_G$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I^G\dot{\phi}^2 + \frac{1}{2}m(\dot{x}^2 - 2\dot{x}\dot{\theta}L\sin\theta$

$+ \dot{\theta}^2L^2\sin^2\theta + \dot{\theta}L^2\cos^2\theta)$

$T = \frac{1}{2}(2m)\dot{x}^2 + \frac{1}{2}I^G\dot{\phi}^2 + \frac{1}{2}m\dot{\theta}^2L^2 + \frac{1}{2}(-2mL\sin\theta)\dot{x}\dot{\theta}$

Kinematics $\dot{x} = R\dot{\phi} \longrightarrow \dot{\phi} = \dot{x}/R$

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Test Problem 1 Additional Page

$$T = \frac{1}{2}\left(2m + \cancel{\frac{I^G}{R^2}}\right)\dot{x}^2 + \frac{1}{2} m L^2 \dot{\theta}^2 + \frac{1}{2}(-2mL\sin\theta)\dot{x}\dot{\theta}$$

$\frac{1}{2} mR^2/R^2$

$$T = \frac{1}{2}\underbrace{\left(\frac{5}{2}m\right)}_{m_{11}}\dot{x}^2 + \frac{1}{2}\underbrace{(-2mL\sin\theta)}_{2m_{12}=2m_{21}}\dot{x}\dot{\theta} + \frac{1}{2}\underbrace{mL^2}_{m_{22}}\dot{\theta}^2 \sim \text{Ans}$$

c)

$$[m] = m\begin{bmatrix} 5/2 & -L\sin\theta \\ -L\sin\theta & L^2 \end{bmatrix}_{\bar{q}_e} = m\begin{bmatrix} 5/2 & 0 \\ 0 & L^2 \end{bmatrix} \sim \text{Ans}$$

$$k_{11} = \left.\frac{\partial^2 U}{\partial x_1^2}\right|_{\bar{q}_e} = k \qquad k_{12} = \frac{\partial^2 U}{\partial x \partial \theta} = k_{21} = 0$$

$$k_{22} = \left.\frac{\partial^2 U}{\partial \theta^2}\right|_{\bar{q}_e} = mgL\cos\theta_e = mgL$$

$$[k] = \begin{bmatrix} k & 0 \\ 0 & mgL \end{bmatrix} \sim \text{Ans.}$$

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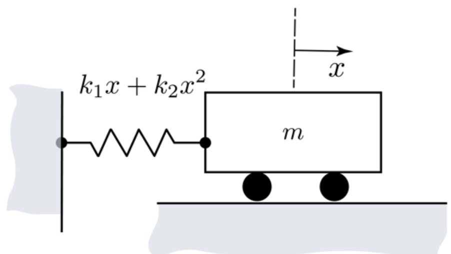
Test Problem 2 -30 points

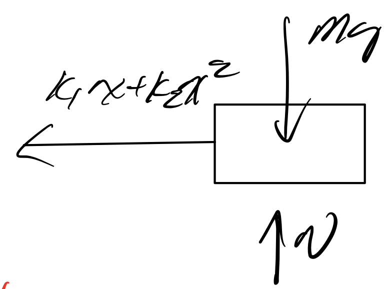
A nonlinear single degree of freedom oscillator is attached to spring whose resistive force is given $F_s = k_1x + k_2x^2$ where the stiffness's $k_1 = 4000$ N/m, and $k_2 = 20$ N/m^2 and a mass $m = 1000$ kg and $x(0) = 0$ mm and $v(0) = 2$ mm/s.

a) Determine the equilibrium points of the system. Derive the linear equation of motion of the system for small oscillations around equilibrium point. Hint there are two, one may not be physically possible.



b) Plot the solution of the equation derived in a) for at least two periods. Determine numerical values for the initial conditions, amplitude, and period and label them on the plot.

c) If a damper is attached to the system with damping coefficient of $c = 200$ kg/s. Determine the damping ratio. Is the linear system underdamped, overdamped, or critically damped. If the system is underdamped find the damped natural frequency. If the system is overdamped or critically damped report the eigenvalues of the system.



Can use Newton or Lagrange

a)

Newton Euler

$\Rightarrow \sum F_x: -k_1x - k_2x^2 = m\ddot{x}$

$m\ddot{x} + k_1x + k_2x^2 = 0$

$x_e = x + \tilde{x}$

$k_1x_e + k_2x_e^2 = 0$

Lagrange

$U = \frac{1}{2}k_1x^2 + \frac{1}{3}k_2x^3$

$\frac{\partial U}{\partial x_e} = k_1x_e + k_2x_e^2 = 0$

$x_e(k_1 + k_2x_e) = 0 \rightarrow x_e = 0$ or $x_e = -\frac{k_1}{k_2} = \frac{4000}{20} = -200m$

Name________________

Newton Euler

Taylor Series

$$k_1 x + k_2 x^2 \cong \left. k_1 x + \cancel{k_2 x^2} \right|_{x_e}^{\to 0} + \left. (k_1 + 2k_2 x) \right|_{x_e} \hat{x}$$

$$m\ddot{\hat{x}} + (k_1 + 2k_2 x_e)\hat{x} = 0$$

~ Ans

$k_{Lin} = 8000$

Lagrange

$T = \frac{1}{2} m \dot{x}^2$

$$k = \left. \frac{\partial^2 U}{\partial x_e} \right|_{x_e} = k_1 + 2k_2 x_e$$

$$m\ddot{x} + (k_1 + 2k_2 x_e)x = 0$$

~ Ans

Same answer by both methods

b)

The equation depends on equilibrium points

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \begin{cases} \sqrt{k_1/m} = 2\,\text{rad/s} & x_e = 0 \\ \sqrt{\dfrac{k_1 - 2\frac{k_1}{k_2}}{m}} = 3.46\,\text{rad/s} & x_e = -\dfrac{k_1}{k_2} \end{cases}$$

You only need to analyze one equilibrium point.

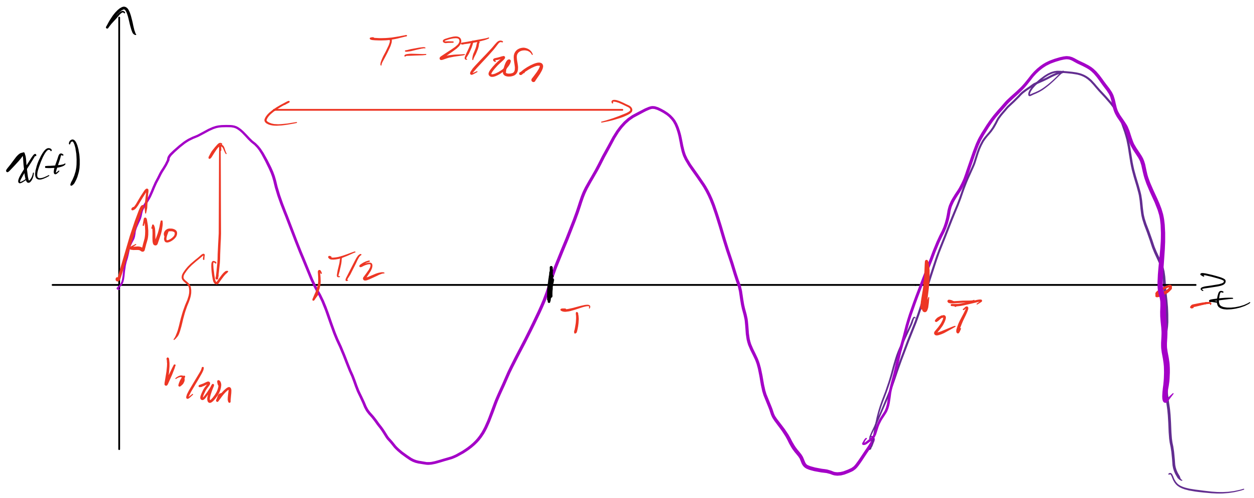
The solution is of the form

$$x(t) = C\cos\omega_n t + S\sin\omega_n t \qquad \dot{x}(0) = 2\,\text{mm/s} = V_0$$

$$\dot{x}(t) = -\omega_n C\sin\omega_n t + \omega_n S\cos\omega_n t$$

$$0 = C(1) + S(0) \qquad C = 0 \qquad x(t) = \frac{V_0}{\omega_n}\sin\omega_n t$$

$$V_0 = -\omega_n C(0) + \omega_n S(1) \qquad S = V_0/\omega_n$$



$x_c = 0$ m $\qquad \omega_n = 2$ rad/s $=$

$x_c = -0.0050$ m $\qquad \omega_n = 3.46$ rad/s

c) $c/m = 2\zeta\omega_n$

$\zeta = \frac{c}{2m\omega_n} =$

$x_c = 0.0 \qquad \omega_n = 2$ rad/s $\qquad \zeta = 0.05$

$x_c = -200$ m $\qquad \omega_n = 1.999$ rad/s $\qquad \zeta \cong 0.0289$

both are underdamped.

You only needed to do the analysis for one equilibrium point

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Test Problem 3-30 points

A 2-DOF system has the follow equations of motion.

$$m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}\alpha+\frac{1}{2} & \alpha-1\\ \alpha-1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\vec{0}$$

a) Find the value of α that admits a rigid body mode where α>0.
b) Using a value of α found in a) to determine the natural frequencies and mass normalized mode shapes of the system.
c) If possible, find initial conditions such that the only the rigid body mode is present in the response.

a) $\det[K]=0$

$k\left(2(\alpha+\tfrac{1}{2})-(\alpha-1)^2\right)=0$

$2\alpha+1-(\alpha^2-2\alpha+1)=0$

$2\alpha+2\alpha+1-1+\alpha^2=0$

$\alpha^2+4\alpha=0 \qquad \alpha=0 \qquad \underline{\alpha=4}$ ~ Ans

b) $m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}9/2 & 3\\ 3 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\vec{0}$

$x=\bar{X}e^{i\omega t}$

$\left(-\omega^2[M]+[K]\right)\bar{X}=0$

$$\begin{bmatrix}-\omega^2 m+9/2k & 3k\\ 3k & -\omega^2 m+2k\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\bar{0}$$

$$|D| = (-\omega^2 m + \tfrac{9}{2}k)(-\omega^2 m + 2k) - 9k^2 = 0$$

$$= \omega^4 m^2 - 2\omega^2 mk - \tfrac{9}{2}\omega^2 mk + 9k^2 - 9k^2 = 0$$

$$= \omega^4 m^2 - \tfrac{13}{2}\omega^2 mk = 0$$

$$= \omega^4 m - \tfrac{13}{2}\omega^2 k = 0$$

$$= (\omega^2 m - \tfrac{13}{2}k)\omega^2 = 0$$

$\omega_1 = 0$

$\omega_2 = \sqrt{13/2}\sqrt{k/m}$ ~ Ans

Pick 1st row of D

$$(-\omega^2 m + \tfrac{9}{2}k)X_1 + 3k X_2 = 0$$

$$\left(\frac{X_2}{X_1}\right)^1 = \frac{\omega_i^2 m - \tfrac{9}{2}k}{3k} = \frac{\omega_i^2 m}{3k} - \frac{3}{2}$$

$\omega_1 = 0$

$$\left(\frac{X_2}{X_1}\right)^1 = -3/2$$

$$\bar{X}^1 = \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix}$$ ~ Ans

$\omega_2 = \sqrt{13/2}\sqrt{k/m}$

$$\left(\frac{X_2}{X_1}\right)^2 = \frac{\left((\sqrt{13/2})\sqrt{k/m}\right)^2 m}{3k} - 3/2$$

$$\left(\frac{X_2}{X_1}\right)^2 = \frac{\frac{13}{2}\,{}^{k}\!/\!{}_{m}\; m}{3k} - \frac{3}{2}$$

$$= \frac{13}{6} - \frac{3}{2} = \frac{13}{6} - \frac{9}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\bar{X}^2 = \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}$$

Calculate α_i's for mode shapes

$$\alpha_1 = \frac{1}{\sqrt{\{1 \;\; -3/2\}\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix}}} = \frac{1}{\sqrt{m}\sqrt{1 + 9/4}} = \frac{1}{\sqrt{13/4\, m}}$$

$$= \frac{2}{\sqrt{13m}}$$

$$\alpha_2 = \frac{1}{\sqrt{\{1 \;\; 2/3\}\, m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}}} = \frac{1}{\sqrt{m}\sqrt{1 + 4/9}} = \frac{1}{\sqrt{13/9\, m}}$$

$$= \frac{3}{\sqrt{13m}}$$

$$\vec{X}^1 = \frac{2}{\sqrt{13m}} \begin{Bmatrix} 1 \\ -3/2 \end{Bmatrix} \qquad \vec{X}^2 = \frac{3}{\sqrt{13m}} \begin{Bmatrix} 1 \\ 2/3 \end{Bmatrix}$$ ~ Ans

c)

$$\vec{x}(t) = (c_1 + s_1 t)\vec{X}^1 + (c_2 \cos\omega_2 t + s_2 \sin\omega_2 t)\vec{X}^2$$

$$c_1 = \vec{X}^{1T}[m]\vec{x}(0) \qquad c_2 = \vec{X}^{2T}[m]\vec{x}(0)$$

$$s_1 = \vec{X}^{1T}[m]\dot{\vec{x}}(0) \qquad s_2 = \vec{X}^{2T}[m]\dot{\vec{x}}(0)$$

Several options

1) $\vec{x}(t) = c_1 \vec{X}^1$ ~ No motion ~ no strain no movement

$$\dot{\vec{x}}(0) = 0 \longrightarrow s_1 = s_2 = 0$$

$$c_2 = \frac{3}{\sqrt{13m}} \{1 \;\; 2/3\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} x_{10} \\ x_{20} \end{Bmatrix} = 0$$

$$c_2 = \frac{3\sqrt{m}}{\sqrt{13}} \left(x_{10} + 2/3\, x_{20} \right) = 0 \longrightarrow x_{10} = -2/3\, x_{20}$$ ~ Ans

✓ c_1 $\quad x_{20} = 1 \quad x_{10} = -2/3$

$$c_1 = \frac{2}{\sqrt{13}\sqrt{m}} \{1 \;\; -3/2\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -2/3 \end{Bmatrix}$$

$$c_1 = \frac{2\sqrt{m}}{\sqrt{13}} \{1 \;\; -3/2\} \begin{Bmatrix} 1 \\ -2/3 \end{Bmatrix} = \frac{2\sqrt{m}}{\sqrt{13}}(2) = \frac{4\sqrt{m}}{\sqrt{13}}$$

2) $\vec{x}(t) = s_1 t \bar{\bar{X}}'$

$\vec{x}(0)=0 \rightarrow c_1 = c_2 = 0$

$$s_2 = \frac{3}{\sqrt{13m}} \{1 \quad 2/3\} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \dot{x}_{10} \\ \dot{x}_{20} \end{Bmatrix} = 0$$

$$= \frac{3}{\sqrt{13}} \sqrt{m} \{\dot{x}_{10} + 2/3\, \dot{x}_{20}\} = 0 \qquad \dot{x}_{10} = -2/3\, \dot{x}_{20}$$ ✓Ans

following the previous analysis

$$s_2 = \frac{4\sqrt{m}}{\sqrt{13}}$$

3) $\vec{x}(t) = c_1 \bar{\bar{X}}' + s_1 t \bar{\bar{X}}'$

then from previous

$$c_2 = \frac{3\sqrt{m}}{\sqrt{13}} (x_{10} + 2/3\, x_{20}) = 0 \longrightarrow x_{10} = -2/3\, x_{20}$$

$$s_2 = \frac{3\sqrt{m}}{\sqrt{13}} \{\dot{x}_{10} + 2/3\, \dot{x}_{20}) = 0 \rightarrow \dot{x}_{10} = -2/3\, \dot{x}_{20}$$

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Newton Euler

$$\sum \vec{F} = m\vec{a}_g$$
$$\sum \vec{M}_A = I_A\vec{\alpha}$$

A is a fixed point or center of gravity

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SDOF Response

$$m\ddot{x} + c\dot{x} + kx = 0,$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$
$$2\zeta\omega_n = \frac{c}{m},\ \omega_n^2 = \frac{k}{m}$$

$0 \leq \zeta \leq 1,$
$x(t) = \exp -\zeta\omega_n t\,(C\cos\omega_d t + S\sin\omega_d t)$

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Power Equation
$T + U = T_o + U_o + W^{(nc)}$
$$Power = \frac{dW^(nc)}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

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Eigenvalue Problem
$[M]\ddot{\vec{x}} + [K]\vec{x} = \vec{0},$
$\vec{x}(t) = \vec{X}\exp i\omega t,$
$\left(-\omega^2[M] + [K]\right)\vec{X} = \vec{0}$

MDOF Response
$$\vec{x}(t) = \sum_{j=1}^{N} \vec{X}^(j)[c_j\cos\omega_j t + s_j\sin\omega_j t]$$
$$c_j = \frac{\vec{X}^{(j)T}[M]\vec{x}(0)}{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$
$$sj = \frac{\vec{X}^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$

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Linearized Lagrange's Equations
$[M]\ddot{\vec{z}} + [C]\dot{\vec{z}} + [K]\vec{z} = \vec{0}$

$\vec{z}(t) = \vec{q}(t) - \vec{q}_0$
$$M_{ik} = (m_{ik})_{\vec{q}_0} = M_{ki}$$
$$C_{ik} = \left(\frac{\partial^2 R}{\partial q_i \partial q_k}\right)_{\vec{q}_0} = C_{ki}$$
$$K_{ik} = \left(\frac{\partial^2 U}{\partial q_i \partial q_k}\right)_{\vec{q}_0} = K_{ki}$$

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Mass Normalized Eigenvector
$$\alpha_j = \frac{1}{\sqrt{\vec{X}^{(j)T}[M]\vec{X}^j}}$$
$$\vec{X}_m^j = \alpha_m\vec{X}^j$$
$$c_j = \vec{X}_m^{(j)T}[M]\vec{x}(0)$$
$$s_j = \frac{\vec{X}_m^{(j)T}[M]\dot{\vec{x}}(0)}{\omega_j}$$

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Log Decrement SDOF
$$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right)$$
$$\zeta = \frac{\delta/2\pi}{\sqrt{1 + (\delta/2\pi)^2}}$$
$$\zeta << 1 \rightarrow \zeta = \frac{\delta}{2\pi}$$

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