MECHANICAL VIBRATIONS

A Lecturebook

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About the Authors:

Charles M. Krousgrill: Charles M. Krousgrill is a Professor in the School of Mechanical Engineering at Purdue University. He received his M.S. in mechanical engineering from Purdue University in 1975, and his M.S. and Ph.D. in applied mechanics from the California Institute of Technology in 1976 and 1980, respectively. During his more than 30 years at Purdue University, few, if any, individuals have had a greater impact on the university's undergraduate students and the institution's commitment to engineering education. Dr. Krousgrill's efforts in this regard have garnered numerous awards and international accolades. To date, these include the Purdue University School of Mechanical Engineering's Harry L. Solberg Best Teacher Award (eight times), the Purdue University College of Engineering's Potter Best Teacher Award (four times), the Purdue University Murphy Best Teacher Award, the 2010 Purdue University Helping Students Learn Award, the 2010 Purdue Alumni Association Special Boilermaker Award and the American Society of Engineering Education's 2011 Archie Higdon Distinguished Educator Award – the *de facto* lifetime achievement award for educational accomplishments in the field of mechanics. In January 2018, it was announced that Dr. Krousgrill had been awarded the named 150th Anniversary Professorship by Purdue University. Outside of the classroom, Dr. Krousgrill conducts research in the general areas of dynamics and mechanical vibration.

Jeffrey F. Rhoads: Jeffrey F. Rhoads is a Professor in the School of Mechanical Engineering at Purdue University and is affiliated with both the Birck Nanotechnology Center and Ray W. Herrick Laboratories at the same institution. He also serves as the Director of Practice for MEERCat Purdue: The Mechanical Engineering Education Research Center at Purdue University and the Associate Director of PERC: The Purdue Energetics Research Center. Dr. Rhoads received his B.S., M.S., and Ph.D. degrees, each in mechanical engineering, from Michigan State University in 2002, 2004, and 2007, respectively. Dr. Rhoads' current research interests include the predictive design, analysis, and implementation of resonant micro/nanoelectromechanical systems (MEMS/NEMS) for use in chemical and biological sensing, electromechanical signal processing, and computing; the dynamics of parametrically-excited systems and coupled oscillators; the thermomechanics of energetic materials (including explosives, pyrotechnics, and propellants); additive manufacturing; and mechanics education. Dr. Rhoads is a Member of the American Society for Engineering Education (ASEE) and a Fellow of the American Society of Mechanical Engineers (ASME), where he serves on the Design Engineering Division's Technical Committee on Vibration and Sound. Dr. Rhoads is a recipient of numerous research and teaching awards, including the National Science Foundation's Faculty Early Career Development (CAREER) Award; the Purdue University School of Mechanical Engineering's Harry L. Solberg Best Teacher Award (twice), the Robert W. Fox Outstanding Instructor Award, the and B.F.S. Schaefer Outstanding Young Faculty Scholar Award; the ASEE Mechanics Division's Ferdinand P. Beer and E. Russell Johnston, Jr. Outstanding New Mechanics Educator Award; and the ASME C. D. Mote Jr., Early Career Award. In 2014, Dr. Rhoads was included in ASEE Prism Magazine's 20 Under 40.

Chapter I

Equations of motion for discrete systems

Introduction

In this chapter we will discuss several different approaches for obtaining the equations of motion (EOM's) of systems for which the mass, damping and stiffness components appear in discrete components. These EOM's will be ordinary differential equations which describe the motion of the system. For the most part, we will be interested in describing small-amplitude motion. Because of this, we will be able to deal with a linearized form of these differential equations, where the linearization is performed about equilibrium states.

We will be considering a number of approaches in deriving the EOM's:

- *Newton-Euler formulation* – This is a vector-based approach beginning with forces and accelerations written in vector form. In doing so, we will need to deal with all of the forces acting on the system. In particular, we will include forces of reaction and forces of constraint along with the applied forces. It is often the case that we will not care about quantifying the reaction and constraint forces, and consequently, we will eliminate these forces from the EOM's before attempting to solve the EOM's. This elimination can be a tedious task.

- *Power equation* – The power equation, as we will see, is based on a work/energy (scalar) description of the motion. An important consequence of this is that many of the forces of reaction and constraint will naturally not appear in the energy equation. However, since we will start with a single work/energy equation for the system of interest, we will obtain only a single EOM regardless of the number of coordinates needed to completely describe the motion. Hence, the power equation will be useful to us only for systems having a single degree of freedom (DOF), that is, systems for which only one coordinate is needed to describe the motion.

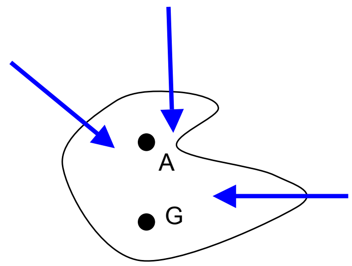
- *Lagrange's equations* – The Lagrangian formulation is a means by which we will be able to separate out from the power equation the correct number of EOM's needed to describe the motion of the system. That is, for a system having N DOFs, the Lagrangian formulation will produce N EOM's.

- *Linearized equations of motion* – For small amplitude oscillations about an equilibrium state, we will be able to replace nonlinear terms appearing in the EOM's obtained from the Lagrangian approach by their linearized approximations. This will produce a set of linear ordinary differential equations that we will use later on for determining the response of the system. We will be able to obtain these linear EOM's directly from the kinetic energy, potential energy and Rayleigh dissipation functions without directly using Lagrange's equations. This formulation will allow us to observe symmetry properties of the mass and stiffness matrices. These symmetry properties will prove useful to use later on in the course in when we use orthogonality properties of the modal vectors of the problem.

- *Flexibility matrix and influence coefficients* – Later on in the course we will need to use the inverse of the stiffness matrix (called the flexibility matrix) for some of the numerical eigenvalue extraction methods. Here we will discuss a direct approach for finding the flexibility matrix using the so-called 'influence coefficients. These influence coefficients are generally easier to find than the inverse of the stiffness and will provide some physical insight into the problem.

I.1 EOM's: Newton-Euler equations

Background



For a single rigid body executing planar motion, we have the following set of Newton-Euler equations:

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_A = I_A\vec{\alpha}$$

where

$G =$ center of mass of the body

$A =$ EITHER the center of mass OR a fixed point on the rigid body

$\sum \vec{F} =$ the resultant force acting on the body

$\sum \vec{M}_A =$ the resultant moment about point A acting on the body

$I_A =$ the mass moment of inertia of the body about point A

$\vec{a}_G =$ the acceleration vector for the center of mass G

$\vec{\alpha} =$ the angular acceleration vector for the body

Remarks:

- ALWAYS draw appropriate free body diagrams (FBD's) before you attempt to develop the EOM's for the system.

- Carefully consider the coordinates used (such as sign conventions, whether measured from fixed reference or moving reference, whether measured from equilibrium or unstretched positions) before drawing FBD's.

- You need to include ALL of the forces acting on the body when writing down the EOM's (including forces of reaction). Later on when we use the power equation and Lagrange's equations we will be able to ignore forces that do no virtual work when deriving EOM's.

- In the end, you should have exactly N EOM's for a system having N degrees of freedom (we will discuss the concept of number of degrees of freedom in a later lecture). Typically you will have more than N Newton/Euler equations at the start; however, you will be able to eliminate forces of constraint (reaction forces) and enforce kinematic constraints to reduce this number of EOM's to N.

- Never attempt to write down the EOM's by inspection.

Example I.1.1

A homogeneous cylinder having a mass of M and radius r rolls without slipping on an inclined surface. A spring having a stiffness of k is attached between the center G of the cylinder and ground.

- Determine the equation of motion (EOM) corresponding to the coordinate x, where x describes the position on the incline for point G **with x being measured from the position of G when the spring is unstretched**.

- Determine the EOM corresponding to the coordinate z, **where z describes the position on the incline for point G with z being measured from the equilibrium position of G.**

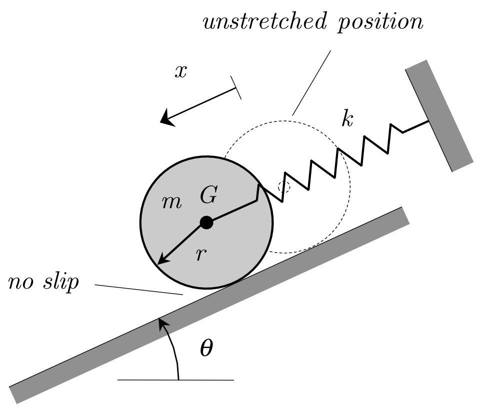


figure a)

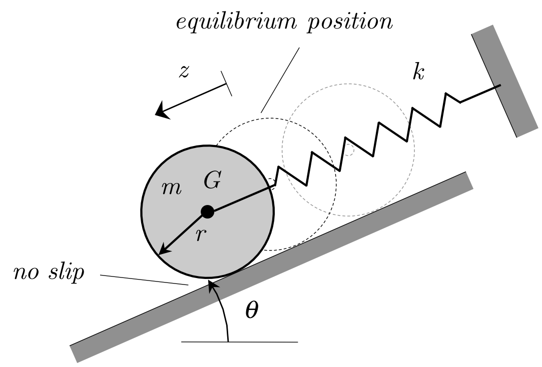


figure b)

Example I.1.2

Two particles, 1 and 2, (each of mass m) are interconnected by a set of three springs (each of stiffness k) and three dashpots (each having a damping constant of c) and slide on a smooth horizontal surface. A force f acts on particle 2.

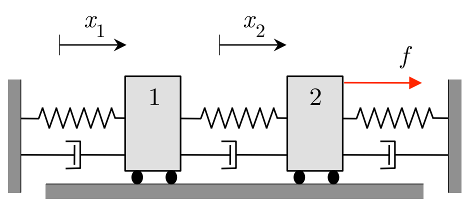
Find the equations of motion (EOM's) for this problem corresponding to three different sets of coordinates:

- Using x_1 and x_2 with both x_1 and x_2 being measured positive to the right from fixed reference points with the springs being unstretched when $x_1 = x_2 = 0$.

- Using y_1 and y_2 with y_1 being measured positive to the right from a fixed reference point, y_2 being measured positive to the left from a fixed reference point and with the springs being unstretched when $y_1 = y_2 = 0$.

- Using z_1 and z_2 with z_1 being measured positive to the right from a fixed reference point, z_2 representing the stretch/compression of the middle spring ($z_2 > 0$ when the spring is stretched) and with the springs being unstretched when $z_1 = z_2 = 0$.

figure a)



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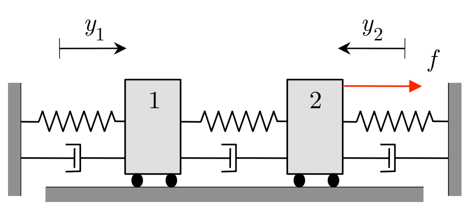

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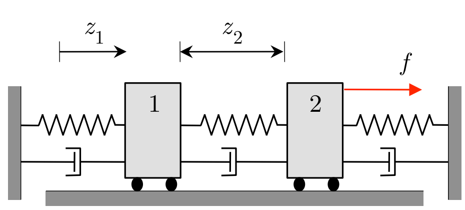
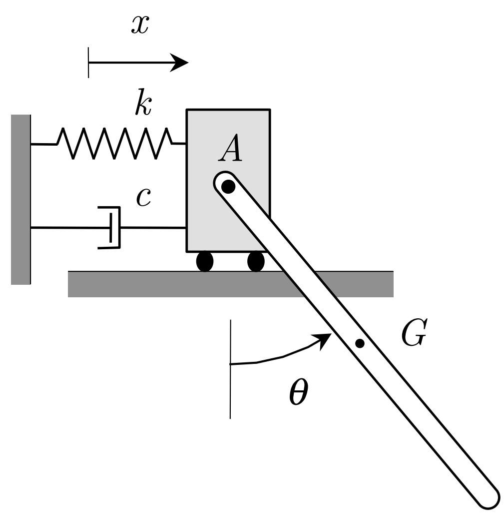


Example I.1.3

Particle A (having a mass of m) slides on a smooth horizontal surface with a thin homogeneous bar (having a length of L and mass M) being attached to A with a smooth pin, as shown. A spring and dashpot connect A to ground. The coordinate x describes the motion of A (x is measured positive to the right and $x = 0$ when the spring is unstretched). The coordinate θ describes the orientation of the bar, with θ being positive for a counterclockwise rotation from the vertical.

Find the EOM's of the system using the coordinates of x and θ.



I.2 EOM's: Power equation

Objectives

It is desired to be able to obtain the differential equation of motion for a single degree-of-freedom system without having to account for "workless" forces acting on the system.

Background

The work-energy equation for a system of n rigid bodies is given by:

$$T + U = T_0 + U_0 + W^{(nc)}$$

where

$T_j =$ kinetic energy of the jth body $= \frac{1}{2} m {v_{Aj}}^2 + \frac{1}{2} I_{Aj} \omega^2$

$T =$ kinetic energy of the system $= \sum_{i=j}^{n} T_j$

$U =$ potential energy due to conservative forces

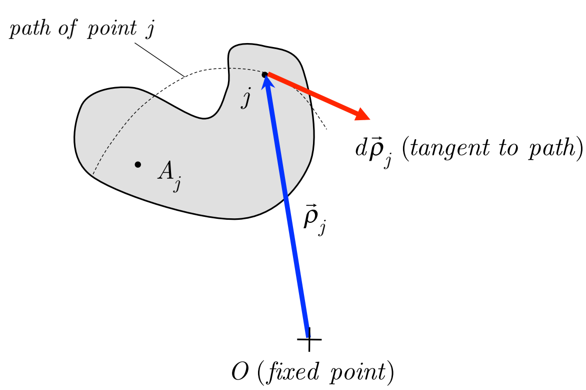
$W^{(nc)} =$ work done by nonconservative forces $= \sum_j \int_1^2 \vec{F}^{(j)} \cdot d\vec{\rho}_j$

$A_j =$ EITHER a fixed point OR the center of mass of the jth body

$v_{Aj} =$ speed of point A_j

$\omega =$ angular speed of the body

$T_0, U_0 =$ initial values for the kinetic and potential energies, respectively



Results

The "power equation" results from taking a time derivative of the work-energy equation:

$$Power = \frac{dW^{(nc)}}{dt} = \frac{dT}{dt} + \frac{dU}{dt}$$

Remarks

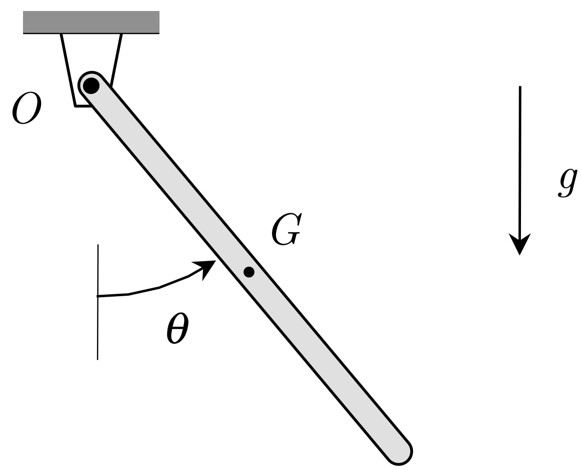
- As we will see in the following examples, the power equation will produce a second-order differential EOM for a system for which only a single coordinate is needed to describe its motion (i.e., systems having a single degree of freedom). The power equation has the advantage over the Newton-Euler approach in that one can ignore forces that do no nonconservative work on the system.

- A particle is a rigid body for which either the dimensions are small ($I_G \approx 0$) or is under pure translation ($\omega = 0$). Therefore, the kinetic energy for a particle is given by $T = \frac{1}{2} m {v_G}^2$.

- Note that the initial values for the kinetic and potential energies (T_0 and U_0, respectively) drop out when differentiating the work-energy with respect to time.

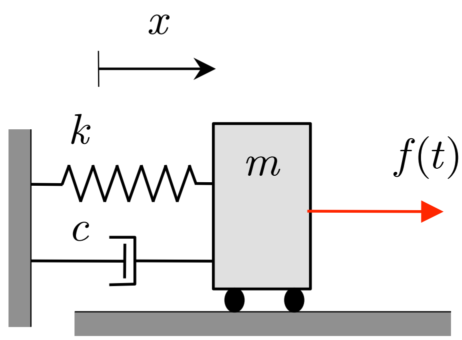
Example I.2.1

A homogeneous thin bar of mass M and length L is pinned to ground at point O. Using the power equation, find the equation of motion (EOM) for the bar corresponding to the coordinate of θ where θ is measured counterclockwise from the vertical.



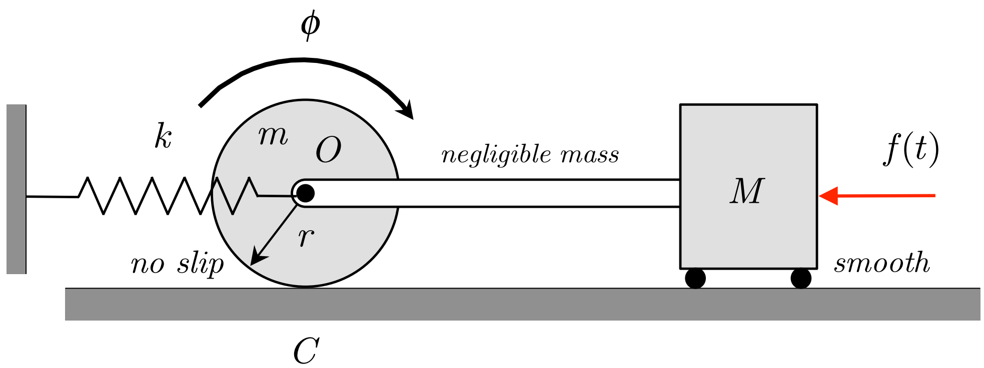
Example I.2.2

Use the power equation to find the EOM for the system shown corresponding to the coordinate x, where the coordinate x is defined such that $x = 0$ when the spring is unstretched.



Example I.2.3

Use the power equation to find the EOM of the system shown corresponding to the coordinate ϕ. The mass of the particle is M, and the mass of the homogeneous cylinder (outer radius r) is m. The bar connecting the particle and the center of the cylinder has a mass that is negligible. The spring is unstretched when $\phi = 0$.

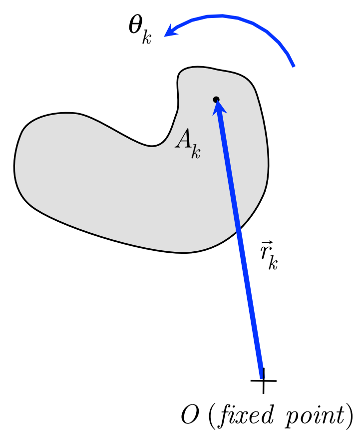


I.3 EOM's: Generalized coordinates, generalized forces and generalized mass coefficients

Objectives

Here we will rewrite the power equation from the last section in a way that the set of chosen coordinates and their time derivatives are explicitly apparent. In this form, sets of "generalized forces" and "generalized mass coefficients" are introduced. The results of this section will lead us directly in the Lagrangian formulation of the EOM's found in the next section.

Background



- For a system of n planar rigid bodies, let A_k be either the center of mass or a fixed point on the kth body. Assume that the position vectors, $\vec{r}_k$, and angular rotations, θ_k, are written in terms of a set of N generalized coordinates q_i $(i = 1, 2, ..., N)$. It is assumed that these generalized coordinates completely describe the configuration of the system for all time. It is also assumed that $\vec{r}_k$ and θ_k are not explicit functions of time. From this, we can write for $k = 1, 2, ..., n$:

$$\vec{r}_k = \vec{r}_k\left(q_1, q_2, ..., q_N\right)$$

$$\theta_k = \theta_k\left(q_1, q_2, ..., q_N\right)$$

- For a function $b = b\left(q_1, q_2, ..., q_N\right)$, we have the chain rule of differentials:

$$db = \frac{\partial b}{\partial q_1}dq_1 + \frac{\partial b}{\partial q_2}dq_2 + ... + \frac{\partial b}{\partial q_N}dq_N = \sum_{i=1}^{N}\frac{\partial b}{\partial q_i}dq_i$$

- For a function $b = b\left(q_1, q_2, ..., q_N\right)$, we have the chain rule of differentiation:

$$\frac{db}{dt} = \dot{b} = \frac{\partial b}{\partial q_1}\dot{q}_1 + \frac{\partial b}{\partial q_2}\dot{q}_2 + ... + \frac{\partial b}{\partial q_N}\dot{q}_N = \sum_{i=1}^{N}\frac{\partial b}{\partial q_i}\dot{q}_i$$

Results

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{ik}\dot{q}_i\dot{q}_k$$

$$dT = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] dq_i$$

$$dU = \sum_{i=1}^{N}\frac{\partial U}{\partial q_i} dq_i$$

$$dW^{(nc)} = \sum_{i=1}^{N} Q_i dq_i$$

where Q_i = "generalized force" corresponding the generalized coordinate q_i and m_{ik} = "generalized mass coefficient", each corresponding to the generalized coordinates q_i and q_k and given by:

$$Q_i = \sum_{j=1}^{M} \vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i}$$

$$m_{ik} = \sum_{j=1}^{n}\left[m_j \frac{\partial \vec{r}_j}{\partial q_i} \cdot \frac{\partial \vec{r}_j}{\partial q_k} + I_j \frac{\partial \theta_j}{\partial q_i} \cdot \frac{\partial \theta_j}{\partial q_k}\right]$$

where M is the number of applied forces on the system.

Derivation

Recall the following set of Newton-Euler equations for planar motion of a rigid body:

$$\vec{F}_j = m_j \vec{a}_{Gj} = m_j \ddot{\vec{r}}_j$$

$$M_j = I_j \alpha_j = I_j \ddot{\theta}_j$$

where

$\vec{F}_j =$ the resultant force acting on the jth body

$M_j =$ the resultant moment about point A_j acting on the jth body

$G_j =$ center of mass of the jth body

$A_j =$ EITHER the center of mass OR a fixed point on the jth rigid body

Say we take the dot product of the first equation with $\vec{r}_j$, multiply the second equation by $d\theta_j$, sum those results over all n bodies and add the results, producing:

$$\sum_{j=1}^{n} \left[m_j \ddot{\vec{r}}_j \cdot d\vec{r}_j + I_j \ddot{\theta}_j d\theta_j \right] = \sum_{j=1}^{n} \left[\vec{F}_j \cdot d\vec{r}_j + M_j d\theta_j \right]$$

We recognize the right hand side of the above equation as being the total differential work done on the system. Alternately, we can write this right hand side of the equation as the difference between the differential work done by nonconservative forces and the differential of the potential of conservative forces:

$$\sum_{j=1}^{n} \left[\vec{F}_j \cdot d\vec{r}_j + M_j d\theta_j \right] = dW^{(nc)} - dU$$

Therefore, the left hand side of this equation, according to the work-energy equation,

$$dT + dU = dW^{(nc)}$$

must be the differential kinetic energy for the system:

$$dT = \sum_{j=1}^{n} \left[m_j \ddot{\vec{r}}_j \cdot d\vec{r}_j + I_j \ddot{\theta}_j d\theta_j \right]$$

In this section of the notes, we will work to simplify the expressions for the three terms of dT, dU and $dW^{(nc)}$. Consider the following steps:

- From the chain rule of differentials:

$$d\vec{r}_j = \sum_{i=1}^{N} \frac{\partial \vec{r}_j}{\partial q_i} dq_i$$

$$d\theta_j = \sum_{i=1}^{N} \frac{\partial \theta_j}{\partial q_i} dq_i$$

Substituting these expressions into the expression for dT gives:

$$\begin{aligned} dT &= \sum_{j=1}^{n}\left[m_j\ddot{\vec{r}}_j\cdot\left(\sum_{i=1}^{N}\frac{\partial\vec{r}_j}{\partial q_i}dq_i\right)+I_j\ddot{\theta}_j\left(\sum_{i=1}^{N}\frac{\partial\theta_j}{\partial q_i}dq_i\right)\right] \\ &= \sum_{i=1}^{N}\left[\sum_{j=1}^{n}\left(m_j\ddot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i}+I_j\ddot{\theta}_j\frac{\partial\theta_j}{\partial q_i}\right)\right]dq_i \end{aligned}$$

Note that through the product rule of differentiation:

$$\begin{aligned} \ddot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i} &= \frac{d}{dt}\left(\dot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i}\right)-\dot{\vec{r}}_j\cdot\frac{d}{dt}\left(\frac{\partial\vec{r}_j}{\partial q_i}\right) \\ &= \frac{d}{dt}\left(\dot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i}\right)-\dot{\vec{r}}_j\cdot\frac{\partial\dot{\vec{r}}_j}{\partial q_i} \end{aligned}$$

It can be shown that (see Appendix I):

$$\frac{\partial\vec{r}_j}{\partial q_i}=\frac{\partial\dot{\vec{r}}_j}{\partial\dot{q}_i}$$

Substituting into the above gives:

$$\begin{aligned} \ddot{\vec{r}}_j\cdot\frac{\partial\vec{r}_j}{\partial q_i} &= \frac{d}{dt}\left(\dot{\vec{r}}_j\cdot\frac{\partial\dot{\vec{r}}_j}{\partial\dot{q}_i}\right)-\dot{\vec{r}}_j\cdot\frac{\partial\dot{\vec{r}}_j}{\partial q_i} \\ &= \frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_i}\left(\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j\right)\right]-\frac{1}{2}\frac{\partial}{\partial q_i}\left(\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j\right) \end{aligned}$$

In a similar way, it can be shown that:

$$\ddot{\theta}_j\frac{\partial\theta_j}{\partial q_i}=\frac{1}{2}\frac{d}{dt}\left[\frac{\partial\dot{\theta}_j^2}{\partial\dot{q}_i}\right]-\frac{1}{2}\frac{\partial\dot{\theta}_j^2}{\partial q_i}$$

Substituting these two results into the above equation for dT gives:

$$\begin{aligned} dT &= \sum_{i=1}^{N}\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_i}\sum_{j=1}^{n}\left(\frac{1}{2}m_j\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j+\frac{1}{2}I_j\dot{\theta}_j^2\right)\right]dq_i-\sum_{i=1}^{N}\left[\frac{\partial}{\partial q_i}\sum_{j=1}^{n}\left(\frac{1}{2}m_j\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j+\frac{1}{2}I_j\dot{\theta}_j^2\right)\right]dq_i \\ &= \sum_{i=1}^{N}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}\right]dq_i \end{aligned}$$

where:

$$T=\sum_{j=1}^{n}\left(\frac{1}{2}m_j\dot{\vec{r}}_j\cdot\dot{\vec{r}}_j+\frac{1}{2}I_j\dot{\theta}_j{}^2\right)$$

- Using the chain rule of differentiation:

$$
\begin{aligned}
T &= \frac{1}{2}\sum_{j=1}^{n} m_j \dot{\vec{r}}_j \cdot \dot{\vec{r}}_j + \frac{1}{2}\sum_{j=1}^{n} I_j \dot{\theta}_j^{\,2} \\
&= \frac{1}{2}\sum_{j=1}^{n} m_j \left(\sum_{i=1}^{N} \frac{\partial \vec{r}_j}{\partial q_i}\dot{q}_i\right) \cdot \left(\sum_{k=1}^{N} \frac{\partial \vec{r}_j}{\partial q_k}\dot{q}_k\right) + \frac{1}{2}\sum_{j=1}^{n} I_j \left(\sum_{i=1}^{N} \frac{\partial \theta_j}{\partial q_i}\dot{q}_i\right)\left(\sum_{k=1}^{N} \frac{\partial \theta_j}{\partial q_k}\dot{q}_k\right) \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} \left(\sum_{j=1}^{n} m_j \frac{\partial \vec{r}_j}{\partial q_i} \cdot \frac{\partial \vec{r}_j}{\partial q_k}\right)\dot{q}_i\dot{q}_k + \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} \left(\sum_{j=1}^{n} I_j \frac{\partial \theta_j}{\partial q_i}\frac{\partial \theta_j}{\partial q_k}\right)\dot{q}_i\dot{q}_k \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{ik}\dot{q}_i\dot{q}_k
\end{aligned}
$$

where

$$
m_{ik} = \sum_{j=1}^{n} \left[m_j \frac{\partial \vec{r}_j}{\partial q_i} \cdot \frac{\partial \vec{r}_j}{\partial q_k} + I_j \frac{\partial \theta_j}{\partial q_i}\frac{\partial \theta_j}{\partial q_k} \right]
$$

are the “generalized mass coefficients”. Two things to note from the above. First, the kinetic energy expression T is a function of both the N generalized coordinates q_j (through the m_{ik} coefficients) and their time derivatives $\dot{q}_j$. Secondly, the coefficients m_{ik} are “symmetric”; that is, $m_{ik} = m_{ki}$.

- The potential energy is a function of only the spatial configuration of the system. Therefore, U can be expressed completely in terms of the generalized coordinates q_i and will not involve their time derivatives $\dot{q}_i$: $U = U(q_1, q_2, ..., q_N)$. Using the chain rule of differentials, we can therefore write:

$$
dU = \sum_{i=1}^{N} \frac{\partial U}{\partial q_i} dq_i
$$

- Suppose that we have M nonconservative forces $\vec{F}_j$ $(j = 1, 2, ..., M)$ acting at the following M locations within the system: $\vec{\rho}_j = \vec{\rho}_j(q_1, q_2, ..., q_N)$. Using the chain rule of differentials gives us:

$$
d\vec{\rho}_j = \sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} dq_i
$$

Using this in the differential work term produces:

$$
\begin{aligned}
dW^{(nc)} &= \sum_{j=1}^{M} \vec{F}_j \cdot d\vec{\rho}_j \\
&= \sum_{j=1}^{M} \vec{F}_j \cdot \left(\sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} dq_i \right) \\
&= \sum_{i=1}^{N} \left(\sum_{j=1}^{M} \vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i} \right) dq_i \\
&= \sum_{i=1}^{N} Q_i dq_i
\end{aligned}
$$

where

$$
\begin{aligned}
Q_i &= \sum_{j=1}^{M} \vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i} \\
&= \text{“generalized force” corresponding to the generalized coordinate } q_i
\end{aligned}
$$

- In summary, the differential form of the power equation:

$$dT + dU = dW^{(nc)}$$

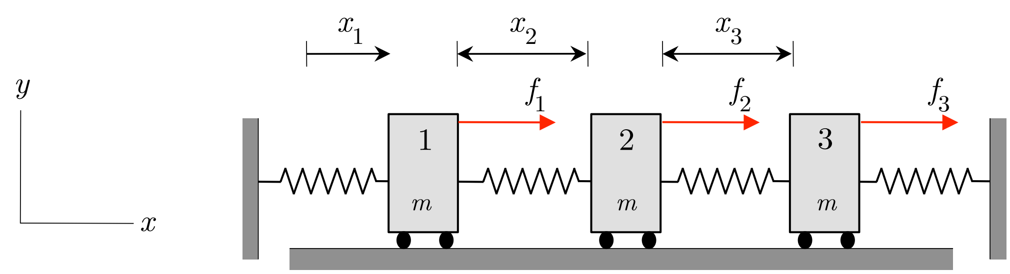
can be written in the following explicit form:

$$\sum_{i=1}^{N} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} - Q_i \right] dq_i = 0$$

Example I.3.1

Forces f_1 , f_2 and f_3 act on the three-particle system shown. Generalized coordinates x_1, x_2 and x_3 are used to describe the motion of these particles, where x_2 and x_3 are relative coordinates.

- Find the generalized forces corresponding to the generalized coordinates x_1, x_2 and x_3 for the forces f_1, f_2 and f_3 .
- Find the generalized mass coefficients corresponding to the generalized coordinates x_1 , x_2 and x_3 .



I.4 EOM's: Lagrange's equations

Objectives

Our goal here is to develop a systematic method for deriving the equations of motion for a system having N degrees of freedom. We will start with the explicit form of the power equation derived in the last section. From the power equation we will obtain a set of N EOM's for the system.

Background

In the last section of the notes, we saw that the power equation for a system described by N generalized coordinates q_i $(i = 1, 2, ..., N)$ can be written as:

$$\sum_{i=1}^{N} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} - Q_i \right] dq_i = 0$$

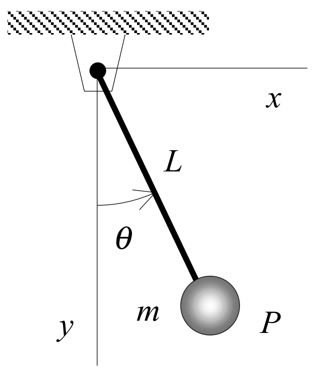
where T and U are the kinetic and potential energies for the system, and Q_i is the generalized force corresponding to the coordinate q_i:

$$Q_i = \sum_{j=1}^{M} \vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i}$$

Derivation and results

It is assumed that the generalized coordinates $q_i \, (i = 1, 2, ..., N)$ used above completely describe the configuration of the system at all instances in time. However, we have said nothing at this point as to whether the coordinates chosen are independent. That is, some of the coordinates could be related by constraints that have not yet been enforced, and as a result, the coordinates are not independent.

Consider an example of the simple pendulum shown below. The kinetic and potential energies for



this system in terms of the (x, y) coordinates for the particle P are given by:

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right)$$

$$U = -mgy = -mgq_2$$

where $q_1 = x$ and $q_2 = y$. Note that the chosen generalized coordinates are related by the constraint $q_1^2 + q_2^2 = L^2$, which by differentiating with respect to time produces: $\dot{q}_1^2 = q_2^2\dot{q}_2^2/\left(L^2 - q_2^2\right)$. Although q_1 and q_2 completely describe the configuration of the system, they are NOT independent. If we enforce the above constraint, T and U can be written in terms of q_2 alone as:

$$T = \frac{1}{2}m\left[\frac{q_2^2}{L^2 - q_2^2} + 1\right]\dot{q}_2^2$$

$$U = -mgq_2$$

This single degree-of-freedom system is now described in terms of a single coordinate.

An alternate (and better) choice of coordinates would be to describe the system in terms of the angle θ. From the figure we see that we have the following constraints: $x = L\sin\theta$ and $y = L\cos\theta$. Substituting these constraints into the original T and U gives:

$$T = \frac{1}{2}m\left[L^2\dot{\theta}^2cos^2\theta + L^2\dot{\theta}^2cos^2\theta\right] = \frac{1}{2}mL^2\dot{\theta}^2 = \frac{1}{2}mL^2\dot{q}_3^2$$

$$U = -mgLcos\theta = -mgLcosq_3$$

where $q_3 = \theta$. Again, we have used a single coordinate in describing the motion of a single degree-of-freedom system.

At this point, let us assume that all of the generalized coordinates are independent: any given coordinate cannot be expressed in terms of the remaining N-1 coordinates. With this being the case, N now represents the total number of degrees of freedom in the system. Since the coordinates are independent, the general form of our power equation:

$$\sum_{i=1}^{N} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} - Q_i \right] dq_i = 0$$

becomes N independent EOM's of the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i$$

for $i = 1, 2, ..., N$. The above are known as the set of Lagrange's equations for an N degree-of-freedom system.

Recall that the potential energy term U includes the work done by conservative forces (such as gravitational and spring forces). The generalized force terms Q_i includes the work done by all other forces. The contribution of damping terms naturally appears within the generalized force terms. It is possible to make the contribution of damping forces more explicit in Lagrange's equations through a so-called Rayleigh dissipation function R. The Rayleigh dissipation function for a single linear dashpot can be written as:

$$R_j = \frac{1}{2} c_j \dot{\Delta}_j^2$$

where c_j is the damping coefficient and $\dot{\Delta}_j$ is the relative speed across the two ends of the dashpot. If the system has r dashpots, then the total Rayleigh dissipation function is:

$$R = \sum_{j=1}^{r} R_j = \frac{1}{2} \sum_{j=1}^{r} c_j \dot{\Delta}_j^2$$

Including damping produces the following modified form of Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

for $i = 1, 2, ..., N$. Here, the generalized forces Q_j include all nonconservative forces except those from the damping terms included in R.

Steps in using Lagrange's equations

1. **Number of degrees of freedom (DOFs)** Determine the number of degrees of freedom (DOF's) in the problem. To do so, carefully consider the least number of generalized coordinates N that are needed to completely describe the configuration of the system at any time.

2. **Generalized coordinates.** Choose your set of generalized coordinates $q_j(t)$ for $j = 1, 2, ..., N$. Be sure that you have chosen an independent set of coordinates. (Ask the question: Can you change each coordinate individually while holding the other coordinates fixed and not violate any motion constraints on the system?) If your coordinates are not independent, there are constraints that must exist among your coordinates. Enforce the constraints at this point before proceeding. Also, reconsider your decision on the number of DOF's in 1. above before continuing: What is the correct number of DOFs?

3. **Free body diagrams.** Draw free body diagrams (FBD's) for all bodies in your system. These FBD's will be necessary later on when you derive the generalized forces acting on the system.

4. **Kinetic energy expression, T.**
   - Write down the velocity vector corresponding to point A_j (either center of mass or fixed point) for each body ($j = 1, 2, ..., n$):

     $$\vec{v}_j = \frac{d\vec{r}_j}{dt}$$

     If A_j is a fixed point, then, of course, $\vec{v}_j = 0$.
   - Write down the angular velocity corresponding to each body ($j = 1, 2, ..., n$):

     $$\omega_j = \frac{d\theta_j}{dt}$$

   - Form the kinetic energy expression for each body:

     $$T_j = \frac{1}{2} m_j \vec{v}_j \cdot \vec{v}_j + \frac{1}{2} I_j \omega_j^2$$

     (Note that the first term in the above expression is a dot product of the velocity vector with itself. Take care to write down the vector expression for velocity before writing down the kinetic energy expression. Also, I_j is the mass moment of inertia of the jth body for point the choice of point A_j. If the body is to be treated as a particle, then $I_j = 0$.)

   - Form the total kinetic energy for the system by summing up the kinetic energy expressions for the n rigid bodies:

     $$T = \sum_{j=1}^{n} T_j$$

- Expand the expression for T to explicitly show the appearance of the $\dot{q}_j$ terms:

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{ik}\dot{q}_i\dot{q}_k$$

  From this, identify the mass matrix elements m_{ik}. This step will help to simplify the differentiation process later on.

5. **Potential energy expression, U.**
   - Write down the potential energy for each spring in the system:

$$\left(U_{sp}\right)_j = \frac{1}{2}k_j\Delta_j^2$$

   - Write down the gravitational potential energy for each body in the system:

$$\left(U_{gr}\right)_j = m_j g h_j$$

     where h_j is the elevation of the center of mass of the body above the datum line that you have chosen. Please note that if $h_j > 0$, the center of mass is above the datum line, whereas $h_j < 0$ corresponds to the center of mass below the datum line.
   - Form the total potential energy for the system:

$$U = \sum \left(U_{sp}\right)_j + \sum_{j=1}^{n}\left(U_{gr}\right)_j$$

6. **Rayleigh dissipation expression, R.**
   - Write down the Rayleigh dissipation for each dashpot in the system:

$$R_j = \frac{1}{2}c_j\dot{\Delta}_j^{\,2}$$

     Form the total Rayleigh dissipation function for the system:

$$R = \sum_{j=1}^{r} R_j$$

7. **Generalized forces, Q_i.**
   - Reconsider your FBD's of the system. Determine which forces that do virtual work on the system. A reminder of some forces that you will NOT include:
     - Forces due to springs, gravitation attraction and dashpots (since you have already included these in U and R).
     - Contact forces at smooth interfaces. These will do no work on the system.
     - Frictional forces required for rolling without slipping. These do no work on the system.

- Constraint forces due to rigid connections between bodies. These do work on the individual bodies but NOT on the entire system.

- Write down position vectors for the point of application of forces that do virtual work on the system:

$$\vec{\rho}_j = \vec{\rho}_j(q_1, q_2, ..., q_N)$$

- Find the differential change in the position vectors for the point of application of the forces:

$$d\vec{\rho}_j = \sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} dq_i$$

- Find the differential work for each force:

$$dW_j = \vec{F}_j \cdot d\vec{\rho}_j = \vec{F}_j \cdot \sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} dq_i$$

- Find the differential work for entire system, expand and identify the generalized forces Q_i $(i = 1, 2, , N)$ as the coefficients of the individual dq_i terms in the following expression for dW:

$$dW = \sum_{j=1}^{M} W_j = \sum_{i=1}^{N} Q_i dq_i$$

8. **Equations of motion.**
   - Form the partial derivatives of T with respect to q_i and $\dot{q}_i$ $(i = 1, 2, , N)$. Recall that in forming the partial derivatives with respect to variables q_i and $\dot{q}_i$, only the EXPLICIT appearance of the variables are affected by the partial differentiation.
   - Form the total time derivative for the terms:

   $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$$

   Here, when finding the total time derivative, both EXPLICIT and IMPLICIT (through the q_i and $\dot{q}_i$ terms) appearance of time t is affected by the differentiation.
   - Form the partial derivatives of U and R with respect to q_i and $\dot{q}_i$, respectively
   - Combine results to form the N Lagrange's equations for the system $(i = 1, 2, ..., N)$:

   $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} + \frac{\partial U}{\partial q_i} = Q_i$$

   where N is the number of DOF's in the system.

Concluding remarks on Lagrange's equations: some advanced ideas

Recall that we made two important assumptions in deriving the preceding form of Lagrange's equations: i) the position vectors and rotation angles are not explicit functions of time ("scleronomic" systems), and ii) we have chosen an independent set of coordinates. The following discusses the consequences of these assumptions and what needs to be done when the assumptions are not valid for a particular system of interest.

Rheonomic systems. "Rheonomic" systems are those for which the position vectors and rotation angles are explicit functions of time: $\vec{r}_j = \vec{r}_j\left(q_1, q_2, ..., q_N, t\right)$ and $\theta_j = \theta_j\left(q_1, q_2, ..., q_N, t\right)$. With the explicit time dependence, the chain rule of differentials gives us:

$$d\vec{r}_j = \sum_{i=1}^{N} \frac{\partial \vec{r}_j}{\partial q_i} dq_i + \frac{\partial \vec{r}_j}{\partial t} dt$$

With this, we see that the differential work for an applied force includes an additional term due to this explicit time dependence:

$$dW_j = \vec{F}_j \cdot d\vec{\rho}_j = \vec{F}_j \cdot \left[\sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} dq_i + \frac{\partial \vec{\rho}_j}{\partial t} dt\right]$$

For rheonomic systems, Lagrange's equations are typically developed in terms of "virtual" displacements $\delta\vec{r}_j$, where:

$$\delta\vec{r}_j = \sum_{i=1}^{N} \frac{\partial \vec{r}_j}{\partial q_i} \delta q_i$$

Note that virtual displacements are differential displacements where the explicit time dependence is frozen while applying the chain rule. In fact, for scleronomic systems, virtual and differential displacements are exactly the same: $\delta\vec{r}_j = d\vec{r}_j$.

Using virtual displacements gives us "virtual work", δW_j:

$$\delta W_j = \vec{F}_j \cdot \delta\vec{\rho}_j = \vec{F}_j \cdot \left[\sum_{i=1}^{N} \frac{\partial \vec{\rho}_j}{\partial q_i} \delta q_i\right] = \sum_{i=1}^{N} \left[\vec{F}_j \cdot \frac{\partial \vec{\rho}_j}{\partial q_i}\right] \delta q_i = \sum_{i=1}^{N} Q_j \delta q_i$$

and, as a result, the generalized forces Q_i are the same as before. Therefore, the Lagrangian formulation developed in this section for scleronomic systems are still applicable for rheonomic systems provided that we use virtual work (rather than differential work) in finding our generalized forces.

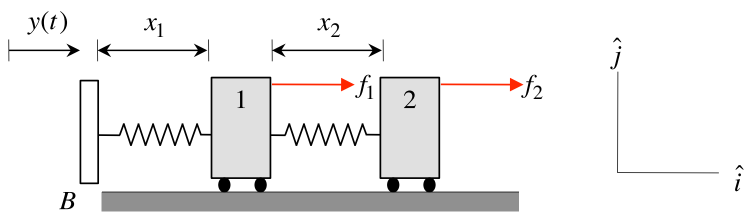
Example of a rheonomic system

One of the most common instances of rheonomic systems are those on which "base excitation" is applied. Consider the two-DOF system shown below where the base "B" of the system is given a prescribed motion of $y(t) = y_0 \sin\omega t$. The motion of particles 1 and 2 is described by the relative coordinates x_1 and x_2. With that, the position vectors for these two particles are given by:

$$\vec{r}_1 = \left[y_0 sin\omega t + x_1\right]\hat{i} = \vec{r}_1\left(x_1, t\right)$$

$$\vec{r}_2 = [y_0 sin\omega t + x_1 + x_2]\,\hat{i} = \vec{r}_2\,(x_1, x_2, t)$$

where we see here that due to the prescribed motion of base B, the position vectors for the two particles are explicit functions of time.



Using the chain rule of differentials:

$$d\vec{r}_1 = [y_0\omega cos\omega t dt + dx_1]\,\hat{i}$$

$$d\vec{r}_2 = [y_0\omega cos\omega t dt + dx_1 + dx_2]\,\hat{i}$$

and the definition of a virtual displacement:

$$\delta\vec{r}_1 = [\delta x_1]\,\hat{i}$$

$$\delta\vec{r}_2 = [\delta x_1 + \delta x_2]\,\hat{i}$$

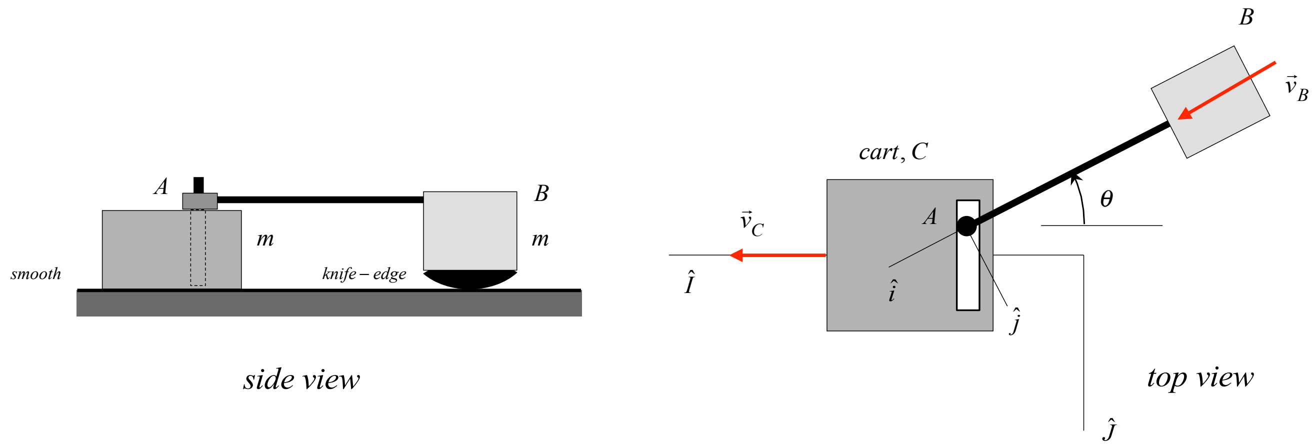
we can write the work and virtual work done by the applied forces:

$$dU = \vec{f}_1 \cdot d\vec{r}_1 + \vec{f}_2 \cdot d\vec{r}_2 = (y_0\omega cos\omega t)\,(f_1 + f_2)\,dt + (f_1 + f_2)\,dx_1 + f_2 dx_2$$

$$\delta U = \vec{f}_1 \cdot \delta\vec{r}_1 + \vec{f}_2 \cdot \delta\vec{r}_2 = (f_1 + f_2)\,\delta x_1 + f_2\delta x_2$$

The difference between the work and virtual work expressions above is that virtual work does not include the influence of the prescribed displacement of the base within the work expression, whereas it must be included in the work expression. This is how using virtual displacements in the Lagrangian formulation simplifies the EOM's and is the recommended formulation for deriving EOM's for rheonomic systems.

Nonholonomic systems. Consider the following system made up of particles A and B. Particle



A moves within a slot cut into cart C (of mass M), as shown, where C is moving with a prescribed speed (v_C) and direction (X). The slot is perpendicular to the prescribed direction of motion for C. Particle B is attached to A with a massless, rigid rod of length L. B rides along on the same horizontal surface as C on a knife-edge support. This support allows for smooth sliding of B along the direction of the rod but does not allow slip perpendicular to the rod. Rotation about the contact point of the knife-edge with the ground is possible. Enforcing these constraints on a rigid body kinematics equation relating the velocities of A and B gives:

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \\
v_B \hat{i} &= \dot{X}_A \hat{I} + \dot{Y}_A \hat{I} + \left(\dot{\theta}\hat{k}\right) \times \left(-L\hat{i}\right) \\
&= \dot{X}_A \hat{I} + \dot{Y}_A \hat{I} - L\dot{\theta}\hat{j}
\end{aligned}$$

From the figure we see that:

$$\begin{aligned}
\hat{I} &= cos\theta\hat{i} - sin\theta\hat{j} \\
\hat{J} &= sin\theta\hat{i} + cos\theta\hat{j}
\end{aligned}$$

Therefore:

$$v_B \hat{i} = \left(\dot{X}_A cos\theta + \dot{Y}_A sin\theta\right)\hat{i} + \left(-\dot{X}_A sin\theta + \dot{Y}_A cos\theta - L\dot{\theta}\right)\hat{j}$$

Balancing coefficients gives the following two constraints:

$$v_B = \dot{X}_A cos\theta + \dot{Y}_A sin\theta = v_C cos\theta + \dot{Y}_A sin\theta$$

$$0 = -\dot{X}_A sin\theta + \dot{Y}_A cos\theta - L\dot{\theta} = -v_C sin\theta + \dot{Y}_A cos\theta - L\dot{\theta}$$

Using the first constraint equation above in the kinetic energy expression for the system gives:

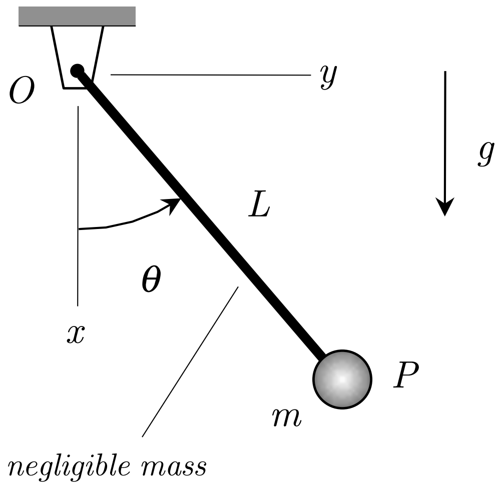
$$\begin{aligned}
T &= \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}Mv_C^2 \\
&= \frac{1}{2}m\left(v_C^2 + \dot{Y}_A^2\right) + \frac{1}{2}m\left(v_C^2 cos^2\theta + \dot{Y}_A^2 sin^2\theta\right) + \frac{1}{2}Mv_C^2
\end{aligned}$$

From this we see that the kinetic energy is a function of both coordinates Y_A and θ. The second constraint above shows that these coordinates are NOT independent; however, due to the nature of this constraint, we are not able to enforce it. (Do you see the problem? The constraint is non-integrable; that is, we cannot relate θ to Y_A without actually solving the problem first.) The Lagrangian formulation developed here relies on independent coordinates, and therefore we cannot use this form of Lagrange's equations to develop the EOM for this single DOF system.

Systems having non-integrable constraints (such as the example above) are known as "nonholonomic". Since we cannot enforce these constraints *apriori*, Lagrange's equations cannot be applied to such systems. There are a number of ways to modify Lagrange's equations to handle nonholonomic systems. We will not pursue that here. You are encouraged to read advanced dynamics textbooks if you are interested in learning more. Should we encounter such systems in this course, we will avoid this complication and develop the EOM's through the Newton-Euler formulation.

Example I.4.1

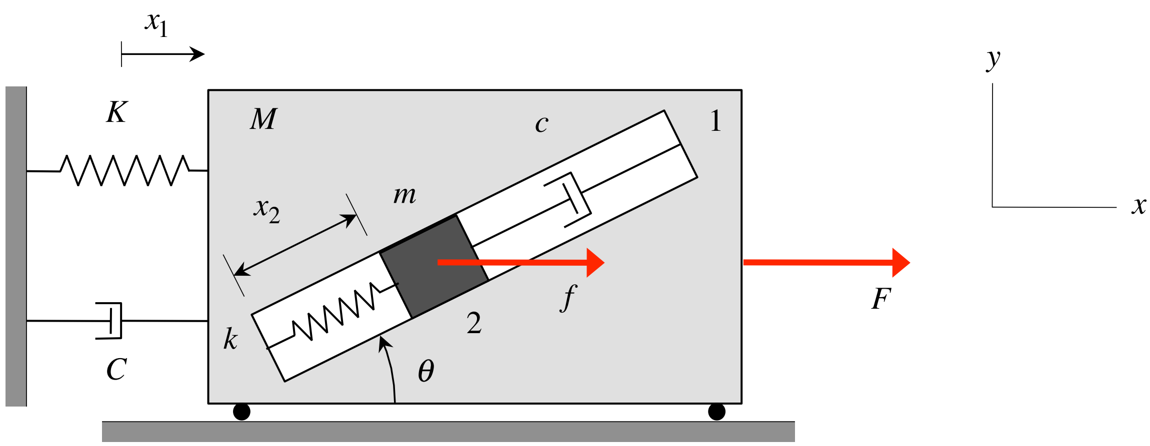
Find the EOM for the simple pendulum using θ as the generalized coordinate.



Chapter I

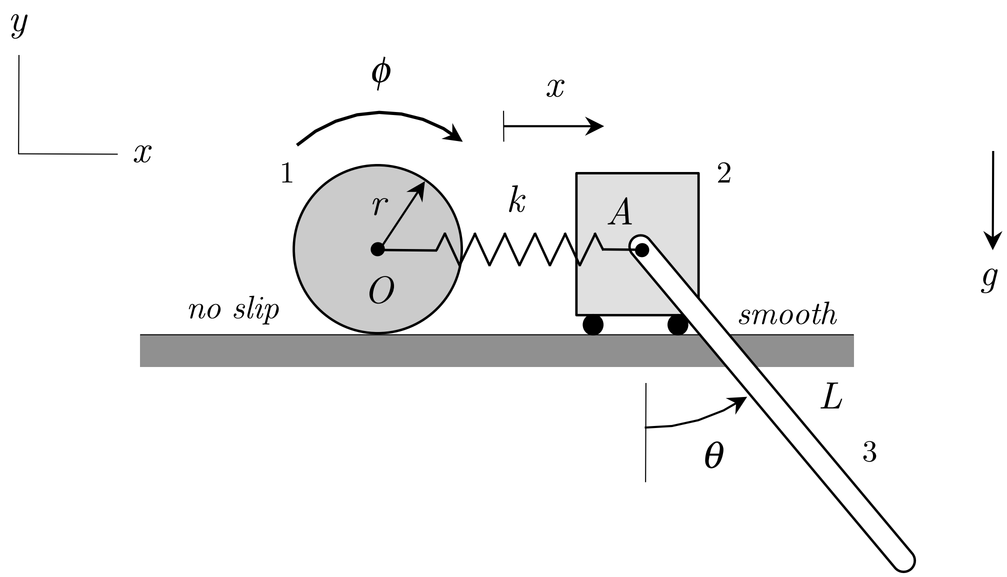
Example I.4.2

Particles 1 and 2 (having masses of m and M, respectively) move in a HORIZONTAL plane. Generalized coordinates x_1 and x_2 describe the position of the system where x_2 is measured relative to particle 1. Forces f and F act in the x-direction on particles 1 and 2, respectively. The springs are unstretched when $x_1 = x_2 = 0$. Find the EOM's of the system using generalized coordinates x_1 and x_2. Ignore gravity. Consider all of the surfaces to be smooth.



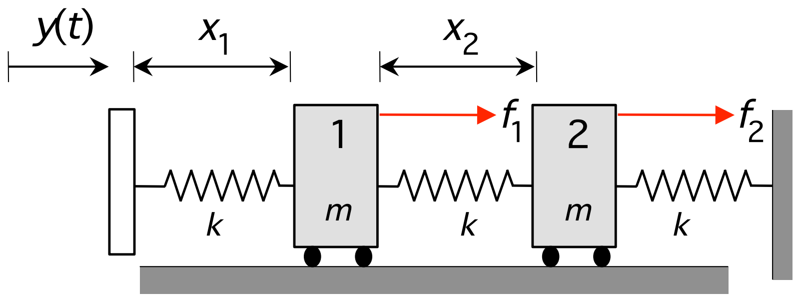
Example I.4.3

Find the EOM's of system shown below using x, ϕ and θ as generalized coordinates. The spring is unstretched when $x = \phi = 0$. All three bodies are to be considered to be homogeneous in their mass distribution, with each body have a mass of m.



Example I.4.4

If particle 0 is given a prescribed motion of $y(t) = y_0 sin\Omega t$, find the EOM's for the system. Note that x_1 and x_2 are relative coordinates. The springs are unstretched when $y = x_1 = x_2 = 0$.



I.5 EOM's: Linearization

Objectives

The EOM's for a system are typically nonlinear, with the nonlinearities arising from geometric and material effects. In this course, we will study small oscillations of systems. For this, we will use a linearized set of free response EOM's arising from Lagrange's equations. In this section of the notes, we will develop a systematic approach for this linearization process for free motion of the system.

Background

- The kinetic energy for discrete models is of the form:

$$T = \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} m_{ik}\dot{q}_i\dot{q}_k$$

  where the symmetric mass coefficients m_{ik} are given by:

$$m_{ik} = \sum_{j=1}^{n}\left[m_j \frac{\partial \vec{r}_j}{\partial q_i}\cdot\frac{\partial \vec{r}_j}{\partial q_k} + I_j \frac{\partial \theta_j}{\partial q_i}\frac{\partial \theta_j}{\partial q_k}\right]$$

- For free response, the system is at an equilibrium state $\vec{q}_0$ when $(i = 1, 2, ..., N)$:

$$\left(\frac{\partial U}{\partial q_i}\right)_{\vec{q}_0} = 0$$

- The Taylor series expansion of a function $g(\vec{q})$ about the point $\vec{q}_0$ is given by:

$$g(\vec{q}) = g(\vec{q}_0) + \sum_{i=1}^{N}\left[\frac{\partial g}{\partial q_i}\right]_{\vec{q}_0}(q_i - q_{0i}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\left[\frac{\partial^2 g}{\partial q_i q_k}\right]_{\vec{q}_0}(q_i - q_{0i})(q_k - q_{0k}) + ...$$

Results

Small motion about an equilibrium state of a free system is governed by the following linearized EOM's:

$$[M]\,\ddot{\vec{z}} + [C]\,\dot{\vec{z}} + [K]\,\vec{z} = \vec{0}$$

where

$$\vec{z}(t) = \vec{q}(t) - \vec{q}_0$$

$$M_{ik} = (m_{ik})_{\vec{q}_0} = M_{ki}$$

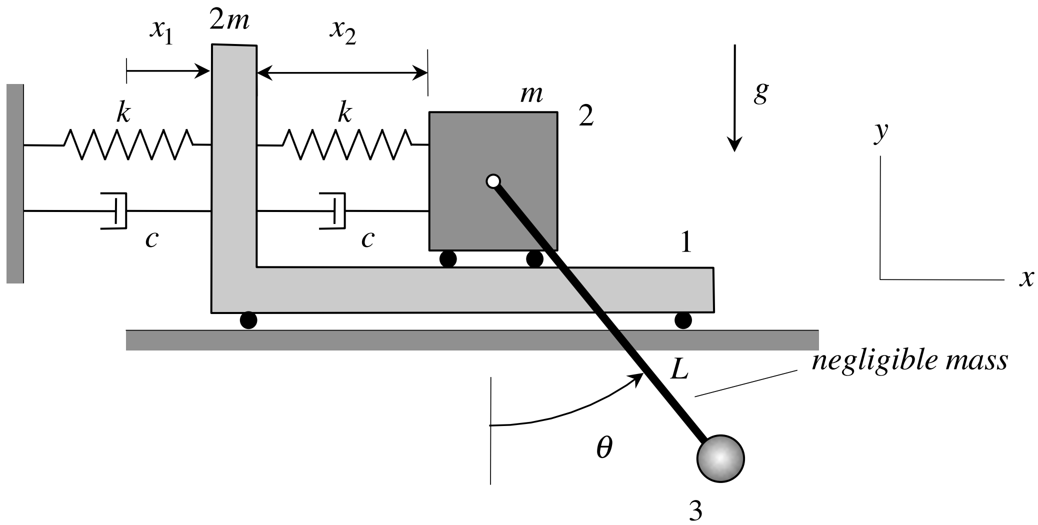
$$C_{ik} = \left(\frac{\partial^2 R}{\partial q_i \partial q_k}\right)_{\vec{q}_0} = C_{ki}$$

$$K_{ik} = \left(\frac{\partial^2 U}{\partial q_i \partial q_k}\right)_{\vec{q}_0} = K_{ki}$$

Derivation and remarks

Example I.5.1

In the system shown, the springs are unstretched when $x_1 = x_2 = 0$. Find the mass, damping and stiffness matrices for small motion about the equilibrium state corresponding to the generalized coordinates shown. Note that x_2 is a relative coordinate.



$$\vec{r}_1 = x_1\hat{i}$$

$$\vec{r}_2 = (x_1 + x_2)\hat{i}$$

$$\vec{r}_3 = (x_1 + x_2 + Lsin\theta)\hat{i} + (-Lcos\theta)\hat{j}$$

$$\vec{v}_1 = \dot{x}_1\hat{i}$$

$$\vec{v}_2 = (\dot{x}_1 + \dot{x}_2)\hat{i}$$

$$\vec{v}_3 = (\dot{x}_1 + \dot{x}_2 + L\dot{\theta}cos\theta)\hat{i} + (L\dot{\theta}sin\theta)\hat{j}$$

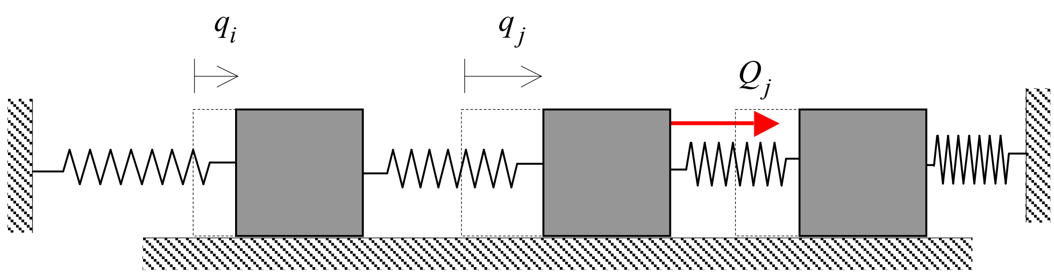
I.6 EOM's: Influence coefficients and the flexibility matrix

Objectives

Later on in our investigation of approximate methods for finding natural frequencies and modal vectors, we will find it necessary to obtain the inverse of the stiffness matrix $[K]$. This inverse, $[K]^{-1}$, is known as the "flexibility matrix". In this section, we will explore a method for obtaining the flexibility matrix directly without the need for performing the inversion of an $N \times N$ matrix.

Background

- Consider a generalized force $Q_j = 1$ (unit magnitude) corresponding to generalized coordinate q_j. Let q_i be the displacement of the ith generalized coordinate as a result of the applied force Q_j. The "influence coefficient" a_{ij} is defined as displacement of the generalized coordinate q_i



  due to the application of a unit magnitude generalized force Q_j.

- We have seen that the stiffness matrix for a conservative system obtained from the Lagrangian formulation is symmetric; that is, $[K]^T = [K]$. Since the order of the transpose and inverse operations on a non-singular matrix is reversible, it can be seen that $[K]^{-1}$ is also symmetric:

$$\left([K]^{-1}\right)^T = \left([K]^T\right)^{-1} = [K]^{-1}$$

Results

- The influence coefficients a_{ij} form the elements of the flexibility matrix; that is, $\left([K]^{-1}\right)_{ij} = a_{ij}$.

- For a linearly elastic system, the influence coefficients a_{ij} defined above are symmetric: a_{ji}. In words, this says that a unit force applied at j will produce the same displacement at i as the displacement at j due to a unit force at i. This is known as "Maxwell's reciprocity theorem".

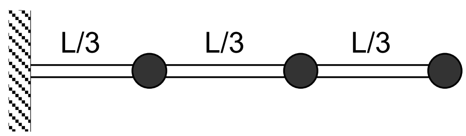
- Method for determing the influence coefficients:
  - Apply a unit generalized force Q_j with $Q_k = 0$ for $k \neq j$.
  - Calculate/measure the change in the displacements of the generalized coordinates $q_i (i = 1, 2, ...N)$. These displacements are the influence coefficients $a_{ij} (i = 1, 2, ...N)$.
  - Use the symmetry of a_{ij} to simplify your calculations; that is, although a symmetric $N \times N$ matrix has N^2 terms, it has only $N(N+1)/2$ unique terms. Furthermore, a_{ij} may be easier to compute than a_{ji} in some instances.

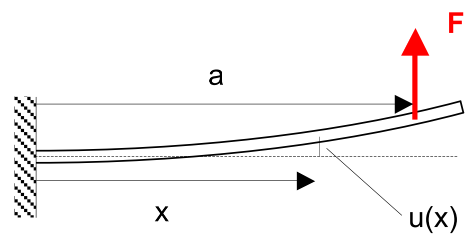
Derivation and remarks

Example I.6.1

Consider a homogeneous, clamped beam having a length of L and flexural rigidity of EI. Find the flexibility matrix for the beam using the transverse deflections at the three particles as generalized coordinates. Note that from mechanics of materials, we know that the deflection at x, $u(x)$, due to a force F applied at location $a \geq x$ is given by:

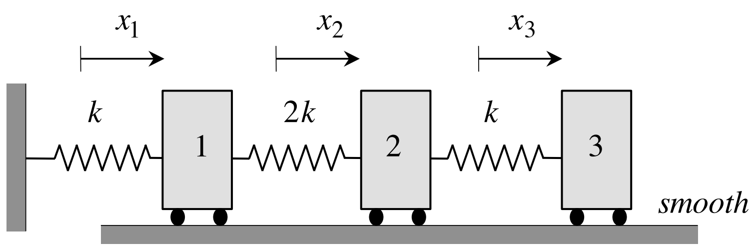
$$u(x) = \frac{F}{6EI}\left[2a^3 - 3a^2\left(a-x\right) + \left(a-x\right)^3\right]$$





Example I.6.2

Find the flexibility matrix for the three-DOF system using the absolute coordinates shown. The springs are unstretched when $x_1 = x_2 = x_3 = 0$.



I.7 EOM's: Summary

Newton-Euler formulation

- Vector approach dealing with each body individually.
- Must deal with ALL forces acting on each body.
- Often times the reduction of the equations to a number equal to the number of DOF's can be tedious.
- Dealing with the signs and direction of spring forces can be tedious.

Power equations

- Scalar approach dealing with the system as a whole.
- Need not deal with forces which do no work on the system as a whole.
- Dealing with the signs and direction of spring forces is more straight-forward.
- Useful for single-DOF systems only.

Lagrange's equations

- Scalar approach dealing with the system as a whole.
- Need not deal with forces which do no virtual work on the system as a whole.
- Dealing with the signs and direction of spring forces is more straight-forward.
- Valid for systems with any number of DOF's provided that one uses independent generalized coordinates.
- Development of generalized forces is a straight-forward procedure when using expressions for virtual work:
  - write down a vector expression for each force, $\vec{F}_j$.
  - write down a vector expression for the differential (or virtual) displacement for the point of application of each force, $d\vec{\rho}_j$
  - then from

    $$dW = \sum_{j=1}^{M} \vec{F}_j \cdot d\vec{\rho}_j = \sum_{j=1}^{M} Q_j dq_j$$

    we can identify the corresponding generalized forces Q_j.

Linearized EOM's

- General form of the linearized EOM's for free response: $[M]\,\ddot{\vec{z}} + [C]\,\dot{\vec{z}} + [K]\,\vec{z} = \vec{0}$
- The mass and stiffness matrices are always symmetric (unlike those found from Newton-Euler).
- These EOM's describe small motion about an equilibrium state: $\vec{z}(t) = \vec{q}(t) - \vec{q}_0$.