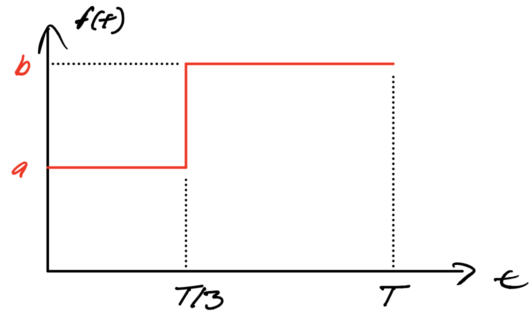
# Problem 1




All calculations done in MATLAB and the expressions may differ from hand calculations due to the simplification

Approximate $f(t)$ as a Fourier Series

$$f(t) \cong f_0 + \sum_{n=1}^{\infty} f_{s,n} \sin n\Omega t + f_{c,n} \cos n\Omega t$$

$$f_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\left\{\int_0^{T/3} a\,dt + \int_{T/3}^{T} b\,dt\right\}$$

$$f_0 = \frac{1}{3}(a+2b)$$

$$f_{s,n} = \frac{2}{T}\int_0^T f(t)\sin n\Omega t\,dt$$

$$f_{s,n} = \frac{2}{T}\left\{\int_0^{T/3} a\sin n\Omega t\,dt + \int_{T/3}^{T} b\sin n\Omega t\,dt\right.$$

$$f_{s,n} = \frac{1}{n\pi}\left\{2a\sin\left(\frac{2\pi}{3}n\right)^2 - 4b\cos\left(\frac{2\pi n}{3}\right)\left(\cos\left(\frac{2\pi n}{3}\right)^2 - 1\right)\right\}$$

$$f_{c,n} = \frac{2}{T}\int_0^T f(t)\cos n\Omega t\,dt$$

$$f_{c,n} = \frac{2}{T}\left\{\int_0^{T/3} a\cos n\Omega t\,dt + \int_{T/3}^{T} b\cos n\Omega t\,dt\right\}$$

$$f_{c,n} = \frac{1}{n\pi}\left\{a\sin\left(\frac{2\pi n}{3}\right) + b\left(2\sin\left(\frac{2\pi n}{3}\right) - 4\sin\left(\frac{2\pi n}{3}\right)^3\right)\right\}$$

Now, solving the equation

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

using the convolution integral

The force can be written as a piecewise function

$$\boxed{f(t) = \begin{cases} a, & 0 \le t \le T/3 \\ b & T/3 \le t \le T \end{cases}}$$

from $0 \le t \le T/3$

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{(v_0 + x_0\omega_n)}{\omega_d}\sin\omega_d t\right) + \int_0^t \cancelto{a}{f(\tau)}\,\frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau)\,d\tau$$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{(v_0 + x_0\omega_n)}{\omega_d}\sin\omega_d t\right) + \frac{a}{m\omega_n^2} - \frac{a e^{-\zeta\omega_n t}(\omega_d \cos\omega_d t + \omega_n\zeta\sin\omega_d t)}{m\omega_d\omega_n^2}}$$

from $0 \le t < T$

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{(v_0 + x_0\omega_n)}{\omega_d}\sin\omega_d t\right) + \int_0^{T/3} \cancelto{a}{f(\tau)}\,\frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau)\,d\tau + \int_0^{T} \cancelto{b}{f(\tau)}\,\frac{e^{-\zeta\omega_n(t-\tau)}}{m\omega_d}\sin\omega_d(t-\tau)\,d\tau$$

$$
\begin{aligned}
x(t) = {} & e^{-\zeta w_n t}\left(x_0 \cos w_d t + \frac{(v_0 + x_0 w_n)}{w_d} \sin w_d t\right) \\
& + \frac{b_0}{w_n^2} + \frac{(a-b)\,\sigma_2 \sigma_4}{m w_d w_n^2} \\
& - \frac{a e^{-\zeta w_n t}}{m w_d w_n^2}\left(w_d \cos w_d t + \zeta w_n \sin w_d t\right)
\end{aligned}
$$

$$\sigma_2 = e^{-\zeta w_n (t + T/3)}$$

$$\sigma_4 = w_d \cos w_d (T/3 - t) - \zeta w_n \sin w_d (T/3 - t)$$

Now, solving the equation

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0$$

using a Fourier series

$$f(t) \cong f_0 + \sum_{n=1}^{\infty} \{ f_{s,n} \sin n\Omega t + f_{c,n} \cos n\Omega t \}$$

The homogenous solution

$$x_h(t) = e^{-\zeta \omega_n t} (A_0 \cos \omega_d t + B_0 \sin \omega_d t)$$

$$x_p = C + \sum_{n=1}^{\infty} A_n \cos n\Omega t + \sum_{n=1}^{\infty} B_n \sin n\Omega t$$

We can take plug $x_p, \dot{x}_p, \ddot{x}_p$ into governing equation

$$\sum_{n=1}^{\infty} \{ -n A_n \Omega^2 \sin(n\Omega t) - n B_n \Omega^2 \cos(n\Omega t)$$

$$+ 2\zeta \omega_n \Omega n A_n \cos(n\Omega t) - 2\zeta \omega_n \Omega n B_n \sin(n\Omega t)$$

$$+ \omega_n^2 C + \omega_n^2 A_n \sin(n\Omega t) + \omega_n^2 B_n \cos n\Omega t$$

$$= f_0 + f_{s,n} \sin n\Omega t + f_{c,n} \cos n\Omega t) \}$$

Seperate out terms

constant: $\omega_n^2 C = f_0$

$\sin n\Omega t$ : $\sum_{n=1}^{\infty} (-n\Omega^2 + \omega_n^2) A_n - 2\zeta \omega_n \Omega n B_n = f_{s,n})$

$\cos n\Omega t$ : $\sum_{n=1}^{\infty} (-n\Omega^2 + \omega_n^2) B_n + 2\zeta \omega_n \Omega n A_n = f_{c,n} \}$

Solving $C = f_0/\omega_n^2$

$$A_n = \frac{(\omega_n^2 - n\Omega^2) f_{sn} + 2\zeta\omega_n n\Omega f_{cn}}{\Delta_n}$$

$$B_n = \frac{(\omega_n^2 - n\Omega^2) f_{cn} - 2\zeta\omega_n n\Omega f_{sn}}{\Delta_n}$$

$$\Delta_n = n\Omega^4 + \omega_n^4 + n^2\Omega^2\omega_n^2(4\zeta^2 - 2)$$

Now the total solution is

$$x(t) = e^{-\zeta\omega_n t}(A_0 \cos\omega_d t + B_0 \sin\omega_d t) +$$
$$C + \sum_{n=1}^{\infty} A_n \cos n\Omega t + \sum_{n=1}^{\infty} B_n \sin n\Omega t$$

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(A_0 \cos\omega_d t + B_0 \sin\omega_d t) +$$
$$e^{-\zeta\omega_n t}(-\omega_d A_0 \sin\omega_d t + \omega_d B_0 \cos\omega_d t)$$
$$+ \sum_{n=1}^{\infty} -n\Omega A_n \sin n\Omega t + \sum_{n=1}^{\infty} n\Omega B_n \cos n\Omega t$$

solving for $A_0$ and $B_0$ usin $x(0) = x_0$ and $\dot{x}(0) = v_0$

$$x_0 = A_0 + \cancel{C}^{\,f_0/\omega_n^2} + \sum_{n=1}^{\infty} A_n$$
$$v_0 = -\zeta\omega_n(A_0) + \omega_d B_0 + \sum_{n=1}^{\infty} n\Omega B_n$$

Solving we can write as a matrix equations of the form $Ax = b$

$$\begin{bmatrix} 1 & 0 \\ -\zeta\omega_n & +\omega_d \end{bmatrix} \begin{Bmatrix} A_0 \\ B_0 \end{Bmatrix} = \begin{Bmatrix} -f_0/\omega_n^2 - \sum_{n=1}^{\infty} A_n + x_0 \\ -\sum_{n=1}^{\infty} n\Omega B_n + v_0 \end{Bmatrix}$$

$$\begin{Bmatrix} A_0 \\ b_0 \end{Bmatrix} = \frac{1}{\omega_d} \begin{bmatrix} \omega_d & +\zeta\omega_n \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_0 - f_0/\omega_n^2 - \sum_{n=1}^{\infty} A_n \\ v_0 - \sum_{n=1}^{\infty} n\Omega b_n \end{Bmatrix}$$

In summary, the response can be written as follows for convolution integral

$0 \le t \le T/3$

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{(v_0 + x_0\omega_n)}{\omega_d}\sin\omega_d t\right) + \frac{a}{m\omega_n^2} - \frac{a e^{-\zeta\omega_n t}(\omega_d\cos\omega_d t + \omega_n\zeta\sin\omega_d t)}{m\omega_d\omega_n^2}$$

$T/3 \le t \le T$

$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t + \frac{(v_0 + x_0\omega_n)}{\omega_d}\sin\omega_d t\right)$$

$$+ \frac{b_0}{\omega_n^2} + \frac{(a-b)\sigma_2\sigma_4}{m\omega_d\omega_n^2}$$

$$- \frac{a e^{-\zeta\omega_n t}}{m\omega_d\omega_n^2}\left(\omega_d\cos\omega_d t + \zeta\omega_n\sin\omega_d t\right)$$

$$\sigma_2 = e^{-\zeta\omega_n(t + T/3)}$$

$$\sigma_4 = \omega_d\cos\omega_d(T/3 - t) - \zeta\omega_n\sin\omega_d(T/3 - t)$$

and for the Fourier Series

$$x_n(t) = e^{-\zeta w_n t}\left(A_0 \cos w_d t + B_0 \sin w_d t\right) + C + \sum_{n=1}^{\infty} A_n \cos n\Omega t + \sum_{n=1}^{\infty} B_n \sin n\Omega t$$

where

$$C = f_0 / w_n^2$$

$$A_n = \frac{(w_n^2 - n\Omega^2) f_{s,n} + 2\zeta w_n n\Omega f_{c,n}}{\Delta_n}$$

$$B_n = \frac{(w_n^2 - n\Omega^2) f_{c,n} - 2\zeta w_n n\Omega f_{s,n}}{\Delta_n}$$

$$\Delta_n = n\Omega^4 + w_n^4 + n^2\Omega^2 w_n^2 (4\zeta^2 - 2)$$

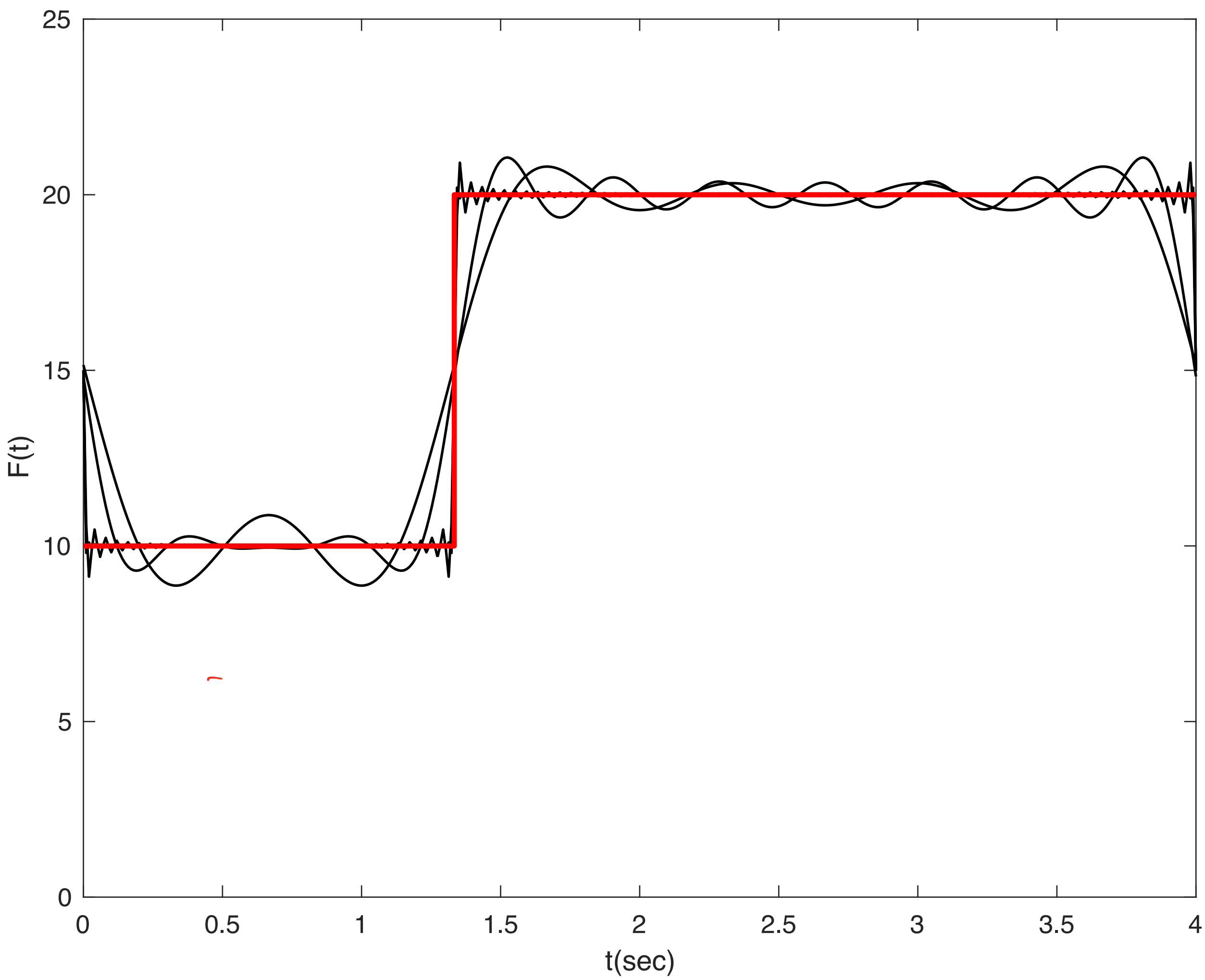
$$f_0 = \tfrac{1}{3}(a + 2b)$$

$$f_{s,n} = \frac{1}{n\pi}\left\{2a \sin\left(\tfrac{2\pi}{3} n\right)^2 - 4b\cos\left(\tfrac{2\pi n}{3}\right)\left(\cos\left(\tfrac{2\pi n}{3}\right)^2 - 1\right)\right\}$$

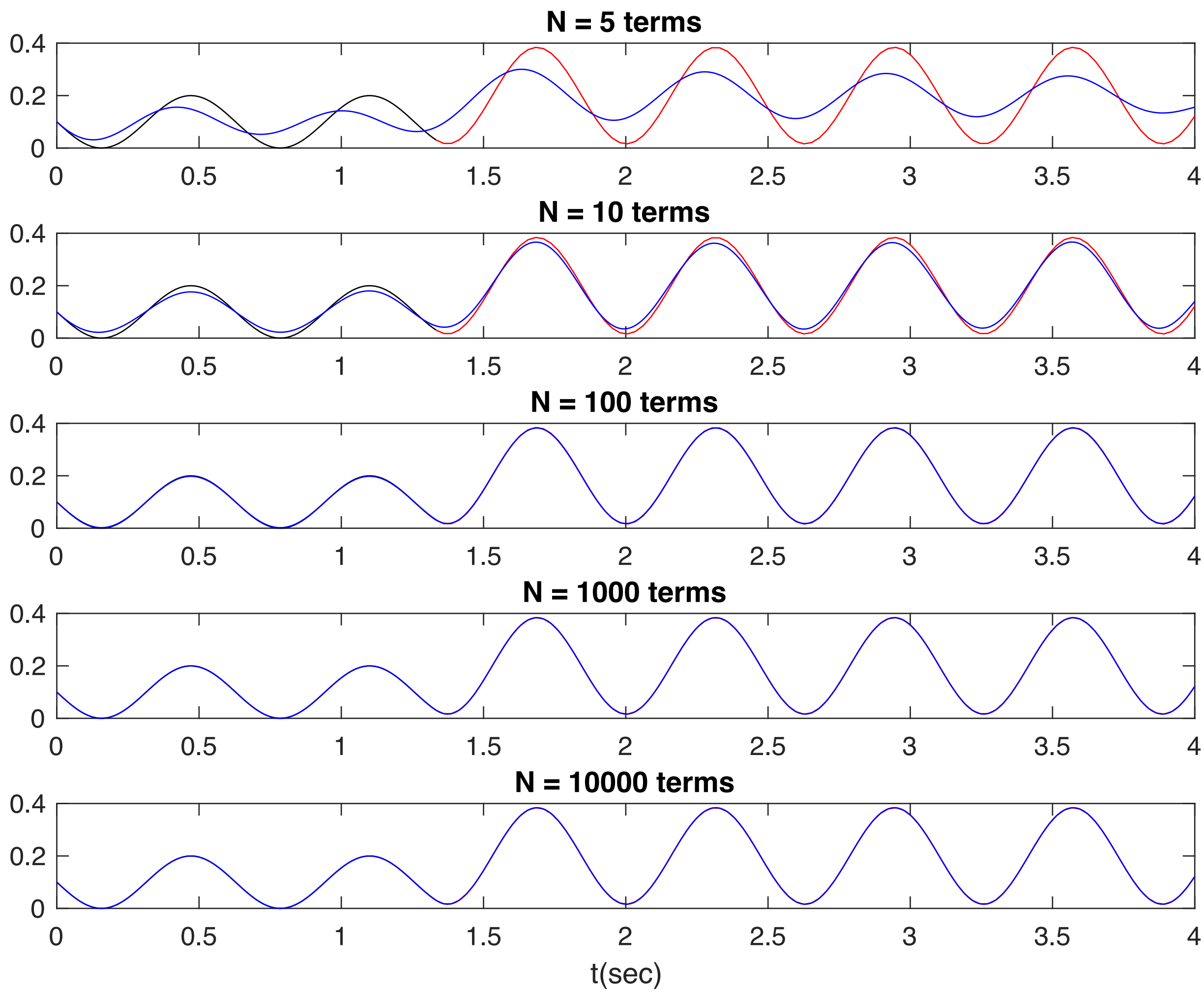
$$f_{c,n} = \frac{1}{n\pi}\left\{a \sin\left(\tfrac{2\pi n}{3}\right) + b\left(2\sin\left(\tfrac{2\pi n}{3}\right) - 4\sin\left(\tfrac{2\pi n}{3}\right)^3\right)\right\}$$

$$\begin{Bmatrix} A_0 \\ B_0 \end{Bmatrix} = \frac{1}{w_d}\begin{bmatrix} w_d & \zeta w_n \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_0 - f_0/w_n^2 - \sum_{n=1}^{\infty} A_n \\ v_0 - \sum_{n=1}^{\infty} n\Omega B_n \end{Bmatrix}$$

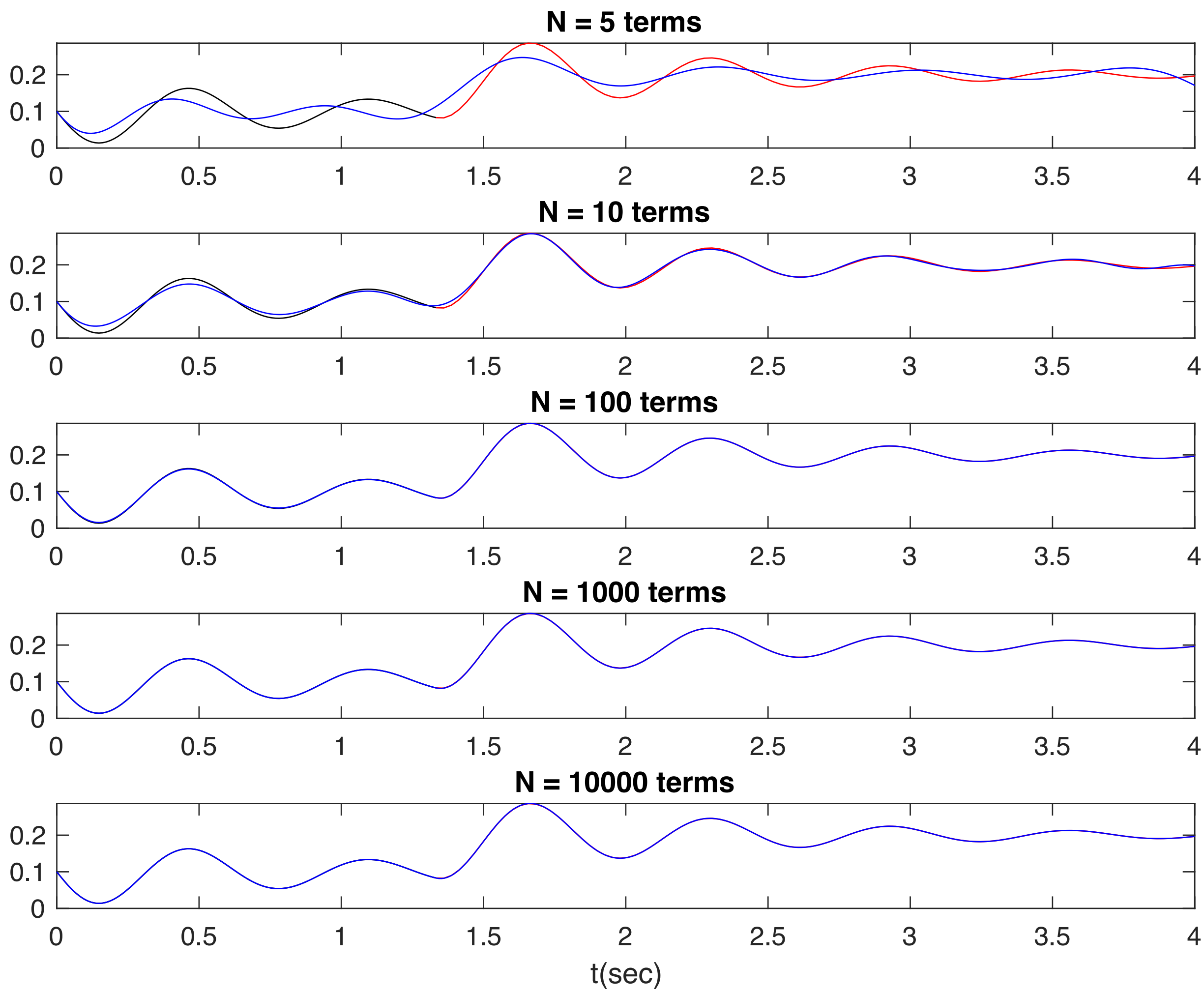
Fourier fit for $N = 5, 10, 100, 1000$, and $10000$



Response when c=0



Response when $c=2$



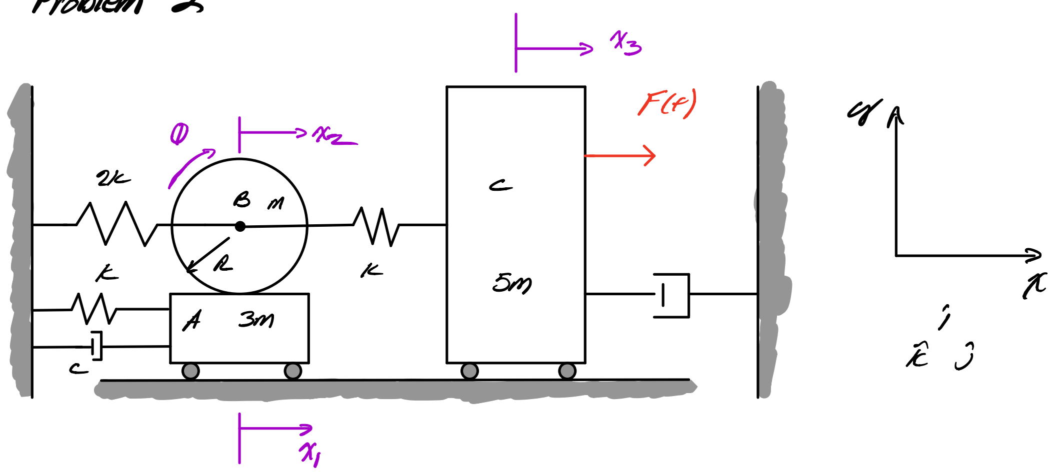
```
% Oscillator Parameters
close all
mass    =  1;
damp    =  2;
stiff   =  100;

% Mass Normalization
Omega_n =  sqrt(stiff/mass);
Zeta    =  damp/2*1/sqrt(mass*stiff);
Omega_d =  Omega_n*sqrt(1-Zeta^2);

% Initial Conditions
X0      =  0.1;
V0      = -1.0;

% Force Parameters
a_actual = 10;
b_actual = 20;
T_actual = 4;
Omega_actual = 2*pi/T_actual;

% Define Force to Plot
F_actual = [a_actual a_actual b_actual b_actual];
t_actual = [0 T_actual/3 T_actual/3 T_actual];

figure(1);
line(t_actual, F_actual, 'color', 'r', 'linewidth',2)
ylabel('F(t)')
xlabel('t(s)')
box on

syms a b T t Omega real
syms n integer


%% Fourier Series %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Constant Term
fo = 1/T*(int(a,t,[0,T/3])+ int(b, t,[T/3,T]));
fo = simplify(fo)
fof = matlabFunction(fo); % Change to a function
fo;
% Cosine Term
fc1 = 2/T*int(a*cos(n*2*pi/T*t), t,[0,T/3]);
fc2 = 2/T*int(b*cos(n*2*pi/T*t), t,[T/3,T]);
assume(n,'integer')
fc1 = simplify(fc1);
fc2 = simplify(fc2);
fc  = fc1 + fc2;
fcf = matlabFunction(fc);
fc
% Sine Term
fxp1 = 2/T*int(a*sin(n*2*pi/T*t),t,[0,T/3]);
fxp2 = 2/T*int(b*sin(n*2*pi/T*t), t,[T/3,T]);
fxp1 = simplify(fxp1);
fxp2 = simplify(fxp2);
fs   = fxp1 + fxp2;
fsf  = matlabFunction(fs); % convert to inline function
fs
```

```
% Number of Fourier Terms
Nv = [5 10 100 1000 10000];
tv = linspace(0, T_actual, 400);

for i  = 1:length(Nv)
    N  = Nv(i);
    Fo = fof(a_actual, b_actual);
    Fs = 0;
    Fc = 0;
    for nv = 1:N
        Fs = Fs + fsf(a_actual, b_actual, nv)*sin(nv*2*pi*tv/T_actual);
        Fc = Fc + fcf(a_actual, b_actual, nv)*cos(nv*2*pi*tv/T_actual);
    end

    F_fourier = Fo+Fs+Fc;

    figure(2)
    line(tv, F_fourier, 'color', 'k', 'linewidth', 1)
    box on
    axis([0 4 0 25])
end
line(t_actual, F_actual, 'color', 'r', 'linewidth',2)
xlabel('t(sec)')
ylabel('F(t)')


%% Convolution Integral
syms h m zeta omega_n omega_d tau a b  x0 v0 reals
assume(zeta>0 & zeta<1)

h   = exp(-zeta*omega_n*(t-tau))*1/(m*omega_d)*sin(omega_d*(t-tau));

xh  = exp(-zeta*omega_n*t)*((zeta*omega_n*x0+v0)/(omega_d)*sin(omega_d*t)+x0*cos↙
(omega_d*t))

x1  = int(a*h,tau,[0,t]);
x1  = simplify(x1+xh)       %Add in homogenous solution
x1c = matlabFunction(x1) %Convert to Matlab Function

x2a  = int(a*h,tau,[0,T/3]);
x2b  = int(b*h,tau,[T/3,t]);
x2   = simplify(x2a+x2b+xh)
x2c  = matlabFunction(x2) %Convert to Matlab Function


%% Harmonic Balance with Fourier Series
syms x(t) FFo FFsn FFcn C1 C2 An Bn C A0 B0 xo vo

% Homogenous Solution
xhf     = exp(-zeta*omega_n*t)*(A0*cos(omega_d*t) + B0*sin(omega_d*t));
Dxhf    = diff(xhf,t)

% Particular Soution
xpf    = C+ An*sin(n*Omega*t) + Bn*cos(n*Omega*t)
Dxpf   = diff(xpf,t)
Dx2xpf = diff(Dxpf,t)

Eqn1 = Dx2xpf + 2*zeta*omega_n*Dxpf +omega_n^2*xpf -(FFo + FFsn*sin(n*Omega*t) + FFcn*cos↙
```

```
(n*Omega*t))

Sin_terms = coeffs(Eqn1, sin(n*Omega*t));
Sin_terms = Sin_terms(2)

Cos_terms = coeffs(Eqn1, cos(n*Omega*t));
Cos_terms = Cos_terms(2)

Constant_terms = subs(Eqn1, [sin(n*Omega*t) cos(n*Omega*t)],[0, 0])

Csol = solve(Constant_terms ==0,C)
Sol=solve(Sin_terms==0, Cos_terms==0,[An Bn])

ASol=Sol.An;
BSol=Sol.Bn;

%IC homogeneous
xho=subs(xhf,t,0)
vho=subs(Dxhf,t,0)

AAn = matlabFunction(ASol)
BBn = matlabFunction(BSol)

for ii = 1:length(Nv)

    N    = Nv(ii);
    Xpf  = 0; % The particular solution
    Xpfo = 0; % Particular solution
    Vpfo = 0;

    for nv = 1:N
        fsn  = fsf(a_actual, b_actual, nv);
        fcn  = fcf(a_actual, b_actual, nv);
        an   = AAn(fcn, fsn, Omega_actual, nv, Omega_n, Zeta);
        bn   = BBn(fcn, fsn, Omega_actual, nv, Omega_n, Zeta);

        Xpf  = Xpf + an*sin(nv*Omega_actual*tv) + bn*cos(nv*Omega_actual*tv);

        Xpfo = Xpfo + bn;
        Vpfo = Vpfo + an*nv*Omega_actual;
    end
    fon  = fof(a_actual, b_actual);
    Xpfo = Xpfo + fon/Omega_n^2;
    Xpf  = Xpf  + fon/Omega_n^2;

    F   = [X0-Xpfo;V0-Vpfo];
    IcM = [1 0; -Omega_n*Zeta Omega_d];
    Icv = IcM\F;

    Ao=Icv(1);
    Bo=Icv(2);

    Xf = exp(-Zeta*Omega_n*tv).*(Ao*cos(Omega_d*tv) + Bo*sin(Omega_d*tv));
    Xf = Xf + Xpf;

    %Time Vectors for Convolutions- notreally efficient to put in the loop
    %but it makes the code easier to read
    tv1  = linspace(0,T_actual/3,100);
```

```
    tv2  = linspace(T_actual/3,T_actual,100);
    X1c  = x1c(a_actual, mass, Omega_d, Omega_n, tv1, V0, X0, Zeta);
    X2c  = x2c(T_actual, a_actual, b_actual, mass, Omega_d, Omega_n, tv2, V0, X0, Zeta);

    figure(3)
    subplot(length(Nv),1,ii)
    title(['N = ',num2str(N),' terms'])
    line(tv1,X1c,'color','k') % Plot 0<t<T/3 Convolution
    line(tv2,X2c,'color','r') % Plot T/3<t<T Convolution
    line(tv,Xf,'color','b')   % Plot Fourier Series
    box on
end
xlabel('t(sec)')

box on
```

Problem 2




Begin by deriving the equations of motion using the linearized Lagrange's Methods

$$T = \tfrac{1}{2}(3m)\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}I_B\omega^2 + \tfrac{1}{2}(5m)\dot{x}_3^2$$

$$I_B = \tfrac{1}{2}mR^2$$

Next by kinematics

$$\vec{v}_2 = \vec{v}_1 + \vec{\omega} \times \vec{r}_{2/1}$$

where $\vec{v}_2 = \dot{x}_2\hat{i}$, $\vec{\omega} = -\dot{\theta}\hat{k}$, $\vec{r}_{2/1} = R\hat{j}$, and $\vec{v}_1 = \dot{x}_1\hat{i}$

$$\dot{x}_2\hat{i} = \dot{x}_1\hat{i} + (-\dot{\theta}\hat{k}) \times R\hat{j} = (\dot{x}_1 + \dot{\theta}R)\hat{i}$$

Now, $\dot{\theta} = (\dot{x}_2 - \dot{x}_1)/R$

$$T = \tfrac{1}{2}(3m)\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mR^2\right)\frac{(\dot{x}_2 - \dot{x}_1)^2}{R^2} + \tfrac{1}{2}(5m)\dot{x}_3^2$$

$$T = \tfrac{1}{2}(3m)\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m\right)(\dot{x}_2 + \dot{x}_1 - 2\dot{x}_1\dot{x}_2) + \tfrac{1}{2}(5m)\dot{x}_3^2$$

$$T = \tfrac{1}{2}\left(\overset{7/2\,m}{\cancel{3m + \tfrac{1}{2}m}}\right)\dot{x}_1^2 + \tfrac{1}{2}\left(\overset{3/2\,m}{\cancel{m + \tfrac{1}{2}m}}\right)\dot{x}_2^2 + \tfrac{1}{2}(-m)\dot{x}_1\dot{x}_2 + \tfrac{1}{2}(5m)\dot{x}_3^2$$

$$T = \tfrac{1}{2}\underset{m_{11}}{\left(\tfrac{7}{2}m\right)}\dot{x}_1^2 + \tfrac{1}{2}\underset{m_{12}}{\left(\tfrac{3}{2}m\right)}\dot{x}_2 + \tfrac{1}{2}\underset{m_{12}+m_{21}}{(-m)\,\dot{x}_1 x_2} + \tfrac{1}{2}\underset{m_{33}}{(5m)}\dot{x}_3^2$$

The mass matrix can be written as

$$[m] = m\underset{\sim \bar{m}}{\begin{bmatrix} 7/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}} = m\,[\bar{m}]$$

Now, the potential energy can be written as

$$U = \tfrac{1}{2}k x_1^2 + \tfrac{1}{2}(2k)x_2^2 + \tfrac{1}{2}k(x_3 - x_2)^2$$

Using linearized Lagrange Method

$$k_{11} = \frac{\partial^2 U}{\partial x_1^2} = k$$

$$k_{12} = k_{21} = \frac{\partial^2 U}{\partial x_2 \partial x_1} = 0$$

$$k_{13} = k_{31} = \frac{\partial^2 U}{\partial x_3 \partial x_1} = 0$$

$$k_{22} = \frac{\partial^2 U}{\partial x_2^2} = 2k + k = 3k$$

$$k_{23} = k_{32} = \frac{\partial}{\partial x_2}\left(2k x_2 - k(x_3 - x_2)\right) = -k$$

$$k_{33} = \frac{\partial^2 U}{\partial x_3^2} = k x_3$$

$$[k] = k\overset{\sim \bar{k}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & +3 & -1 \\ 0 & -1 & 1 \end{bmatrix}} = k\,[\bar{k}]$$

The Rayleigh damping

$$R = \tfrac{1}{2} c \dot{x}_1^2 + \tfrac{1}{2} c \dot{x}_3^2$$

$c_{11} = c, \quad c_{12} = c_{21} = 0, \quad c_{13} = c_{31} = 0$

$c_{22} = 0, \quad c_{23} = 0$

$c_{33} = c$

$$[c] = c \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = c\left[\bar{c}\right]$$

Virtual work

$$dW = \bar{F}\hat{\imath} \cdot dx_3\hat{\imath} = \underbrace{F(t)}_{Q_3} \cdot dx_3$$

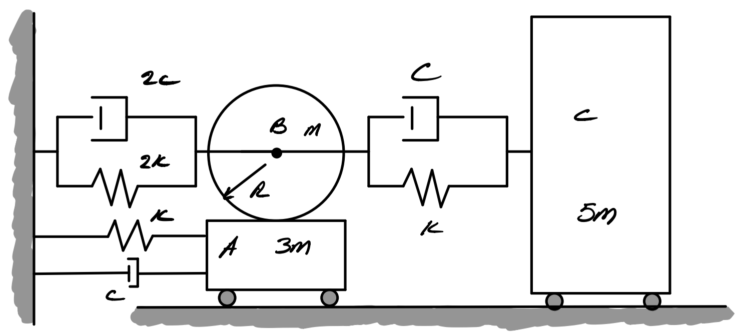
$$\bar{F} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix}$$

$$[m]\ddot{\bar{x}} + [c]\dot{\bar{x}} + [K]\bar{x} = \bar{F}$$

The system is not proportionally damped

Note the system cannot be made to be proportional to the mass matrix due to the rotational inertia which causes inertial coupling between $x_1$ and $x_2$

However, the damping can be made proportional to the stiffness




In determining the mode shapes set $[c]=0$

assume $\vec{x} = \vec{X}e^{j\omega t}$

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0} \longrightarrow \underbrace{\left[-\omega^2[m] + [k]\right]}_{D}\vec{X} = \vec{0}$$

$$Det(D) = -25m^3\omega^6 + 65km^2\omega^4 - 47/2\,k^2m\omega^2 + 2k^3 = 0$$

In MATLAB

$\omega_1 = 0.359\sqrt{k/m}$

$\omega_2 = 0.533\sqrt{k/m}$

$\omega_3 = 1.479\sqrt{k/m}$

$$\vec{X}_1 = \frac{1}{\sqrt{m}}\begin{Bmatrix} -0.018 \\ 0.156 \\ 0.438 \end{Bmatrix} \qquad \vec{X}_2 = \frac{1}{\sqrt{m}}\begin{Bmatrix} 0.531 \\ -0.015 \\ 0.036 \end{Bmatrix}$$

$$\vec{X}_3 = \frac{1}{\sqrt{m}}\begin{Bmatrix} -0.135 \\ -0.821 \\ 0.083 \end{Bmatrix}$$

Solve forced vibration problem $\bar{x} = \bar{\bar{X}} e^{i\Omega t}$

$$\left(-\Omega^2 [m] + i\Omega [c] + [k]\right) \cancel{e^{i\Omega t}} = F_0 \cancel{e^{i\Omega t}}$$

Recast as

$$-\Omega^2 m [\bar{m}] + i\Omega c [\bar{c}] + k [\bar{k}] = \bar{F}_0 \text{, where}$$

$$[\bar{m}] = \begin{bmatrix} 7/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad [\bar{c}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[\bar{k}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

divide the matrix equation by

$$\left(-\Omega^2 m/k [\bar{m}] + i\Omega c/k [\bar{c}] + [\bar{k}]\right) \bar{\bar{X}} = \bar{F}$$

$$\left(-\underbrace{\Omega^2 m/k}_{\mu^2} [\bar{m}] + \underbrace{i\Omega \frac{c}{\sqrt{m}} \frac{\sqrt{m}}{\sqrt{k}}}_{3\mu} \frac{1}{\sqrt{k}} [\bar{c}] + [\bar{k}]\right) \bar{\bar{X}} = \bar{F}$$

$$\left(-\mu^2 [\bar{m}] + i\, 3\mu [\bar{c}] + [\bar{k}]\right) \bar{\bar{X}} = \bar{F}$$

$$\begin{bmatrix} -\frac{7}{2}\mu^2 + i\xi\mu + 1 & \frac{1}{2}\mu^2 & 0 \\ \frac{1}{2}\mu^2 & -\frac{3}{2}\mu^2 + 3 & -3\mu \\ 0 & -3\mu & 5\mu^2 + i\xi\mu + 1 \end{bmatrix} \begin{Bmatrix} X_{1p} \\ X_{2p} \\ X_{3p} \end{Bmatrix} =$$

$$\begin{Bmatrix} 0 \\ 0 \\ F_0 \end{Bmatrix}$$

$$X_{1p} = \frac{\begin{vmatrix} 0 & \frac{1}{2}\mu^2 & 0 \\ 0 & -\frac{3}{2}\mu^2+3 & -3\mu \\ F_0 & -3\mu & 5\mu^2 + i\xi\mu + 1 \end{vmatrix}}{\Delta} = \frac{-F_0\mu^2}{\Delta}$$

$$X_{2p} = \frac{\begin{vmatrix} -\frac{7}{2}\mu^2 + i\xi\mu + 1 & 0 & 0 \\ \frac{1}{2}\mu^2 & 0 & -3\mu \\ 0 & F_0 & 5\mu^2 + i\xi\mu + 1 \end{vmatrix}}{\Delta} = \frac{F_0(-7\mu^2 + 2\xi\mu\hat{i} + 2)}{2\Delta}$$

$$X_{3p} = \frac{\begin{vmatrix} -\frac{7}{2}\mu^2 + i\xi\mu + 1 & \frac{1}{2}\mu^2 & 0 \\ \frac{1}{2}\mu^2 & -\frac{3}{2}\mu^2+3 & 0 \\ 0 & -3\mu & F_0 \end{vmatrix}}{\Delta} = \frac{F_0((10\mu^4 - 24\mu^2 + 6) + (6\xi\mu - 3\xi\mu^3)i)}{2\Delta}$$

$$\Delta = (-25\mu^6 + \tfrac{3}{2}\xi^2\mu^4 + 65\mu^4 - 3\mu^2\xi^2 - \tfrac{47}{2}\mu^2 + 2)$$
$$+ i\,(\tfrac{25}{2}\xi\mu^5 + \tfrac{57}{2}\xi\mu^3 + 5\xi\mu)$$

Resonance occurs at $\Omega = \omega_1, \omega_2$ and $\omega_3$. In nondimensional terms . . .

$$\mu_1 = \omega_1/\sqrt{k/m} = 0,354$$

$$\mu_2 = \omega_2/\sqrt{k/m} = 0,533$$

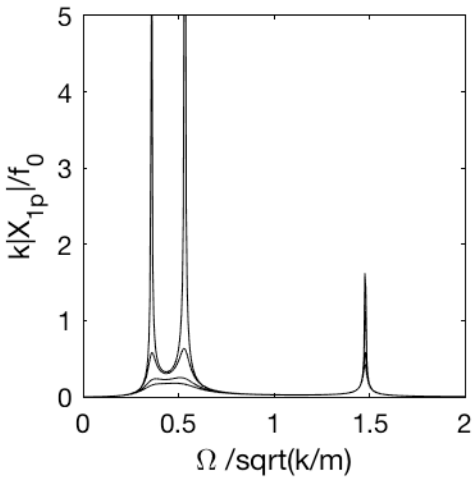
$$\mu_3 = \omega_3/\sqrt{k/m} = 1,479$$

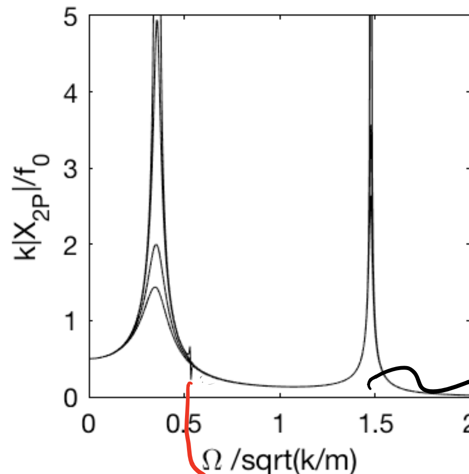
Antiresonance occurs at $\xi = 0$, when the response goes to zero

$$X_{1P} = \frac{-F_0\mu^2}{2\Delta}$$

$$X_{2P} = \frac{F_0(-7\mu^2 + 2)}{2\Delta} \Rightarrow \mu = \sqrt{2/7} = 0,534$$

$$X_{3P} = \frac{F_0(10\mu^4 + 6 - 24\mu^2)}{2\Delta} \Rightarrow \mu = 0,532,\ 1,45$$

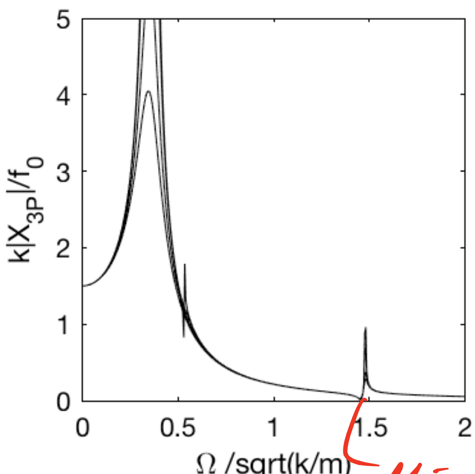



$\mu = 1.45$

sharp notch

$\mu = 0.534$

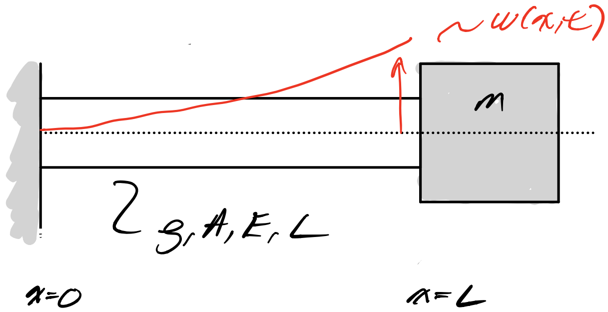



$\mu = 1.45$

```
close all
clear all
clc

fprintf(['\n\n\nStarting file >>' mfilename '<< at ' datestr(now,0) '\n\n']);
format long
strp = [0 0 1; 0.5 0.5 0.8; 0 0.5 0.6; 0 0.5 0.4; 0 0.5 0.2];
syms m k omega mu zeta Fo

FS =12;

% Free Vibration Analysis
MM = [ 7/2 -1/2 0 ;
      -1/2  3/2 0 ;
       0    0    5 ];

CC = [ 1  0  0;
       0  0  0;
       0  0  1];

 KK = [1   0   0 ;
       0   3  -1 ;
       0  -1   1 ];

[X,d] =eig(KK, MM);
[omegan,id] = sort(sqrt(diag(d)))
X= X(:,id)

% Symbolic analysis  for CE
MMs = m*MM
KKs = k*KK

Mass  =-mu^2*MM
Damp  = zeta*i*mu*CC
Stiff = KK

DD    = Mass+Damp+Stiff
Delta = simplify(det(DD))

%Cramer's Rule
DD1      = DD;
DD1(1,1) = 0; DD1(2,1) = 0; DD1(3,1)= Fo
Delta1   = simplify(det(DD1))

DD2      = DD;
DD2(1,2) = 0; DD2(2,2) = 0; DD2(3,2)= Fo
Delta2   = simplify(det(DD2))

DD3      = DD;
DD3(1,3) = 0; DD3(2,3) = 0; DD3(3,3)= Fo
Delta3   = simplify(det(DD3))


% Steady State Amplitudes
zeta_v = [0,0.2,0.5,0.7];
mu_v   = linspace(0, 5, 1000)';

for ii = 1:length(zeta_v)
```

```
    zeta = zeta_v(ii);
    for jj = 1:length(mu_v)
        H       = -mu_v(jj)^2*MM + i*zeta*mu_v(jj)*CC + KK;
        Y(:,jj) = inv(H)*[0;0;1];
    end

    figure(1)
    subplot(1,3,1)
    line(mu_v, abs(Y(1,:)), 'color' , strp(ii,:)),hold on
    axis([0,2,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_1|/f_0')
    box on
    axis square

    subplot(1,3,2)
    line(mu_v, abs(Y(2,:)), 'color' , strp(ii,:)),hold on
    axis([0,2,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_2|/f_0')
    box on
    axis square

    subplot(1,3,3)
    line(mu_v, abs(Y(3,:)), 'color' , strp(ii,:)),hold on
    axis([0,2,0,5]),xlabel('\Omega /sqrt(k/m)'),ylabel('k|Y_3|/f_0')
    box on
    axis square
end
```

Problem ③




Note the governing equation

$$EI\frac{\partial^4 w}{\partial x^4} = -\rho A\frac{\partial^2 w}{\partial t^2}$$

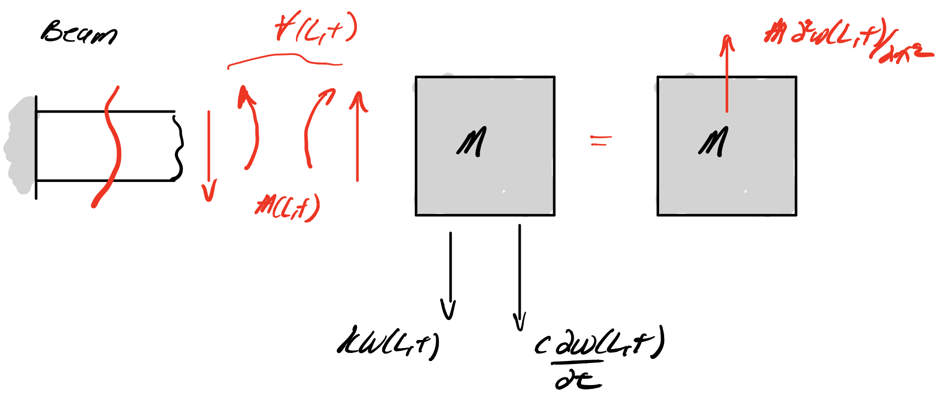
The boundary conditions can be written as

@ $x=0$

① $w(0,t) = W(0)\,T(t) \longrightarrow W(0)=0$

② $\frac{\partial w}{\partial x}(0,t) = W'(0)\,T(t) \longrightarrow W'(0)=0$

@ $x=L$, use $\Sigma F$ and $\Sigma M$




$\circlearrowleft \Sigma M$: $\quad M(L,t) = EI\frac{\partial^2 w(L,t)}{\partial x^2} = 0$

$+\uparrow \Sigma F_y$: $\quad V(L,t) - k\,w(L,t) - c\frac{\partial w(L,t)}{\partial t} = M\frac{\partial^2 w(L,t)}{\partial t^2}$

③ $EI\dfrac{\partial^2 w(L,t)}{\partial x^2} = EI\,W''(L)T(t) = 0 \longrightarrow W''(L) = 0$

④ $EI\dfrac{\partial^3 w(L,t)}{\partial x^3} - k w(L,t) - c\dfrac{\partial w}{\partial t}(L,t) = M\dfrac{\partial^2 w(L,t)}{\partial t^2}$

assuming

$T(t) = Ce^{i\omega t}$

$\dot{T}(t) = +i\omega Ce^{i\omega t}$

$\ddot{T}(t) = -\omega^2 Ce^{i\omega t}$

Then

④ $EI\,W'''(L)T(t) - kW(L)T(t) - cW(L)T'(t) = MW(L)T''(t)$

$EI\,W'''(L)\cancel{T(t)} - kW(L)\cancel{T(t)} - ci\omega W(L)\cancel{T(t)} = -\omega^2 MW(L)\cancel{T(t)}$

$EI\,W'''(L) - kW(L) - ic\omega W(L) = -\omega^2 MW(L)$

The boundary conditions are...

$W(0) = 0$

$W'(0) = 0$

$W''(L) = 0$

$EI\,W'''(L) - k\,W(L) - ic\omega W(L) = -\omega^2 MW(L)$

Note the last equation is problematic, we can split between real and imaginary parts

$R:\ EI\,W'''(L) - kW(L) + \omega^2 MW(L) = 0$

$I:\quad ic\omega W(L) = 0$

The imaginary equation implies that $W(L) = 0$, and from the real equation we have

Now, the undamped condition is equivalent to looking at real part of the 4th boundary condition

$$EI\,W'''(L) - k\,W(L) + \omega^2 M\,W(L) = 0$$

Ok,

$$W(x) = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$$

where $\beta^2 = \sqrt{\frac{\rho A}{EI}}\,\omega$

Thus the previous boundary conditions can be written as

① $W(0) = 0$

② $W'(0) = 0$

③ $W''(L) = 0$

④ $EI\,W'''(L) - k\,W(L) + \omega^2 M\,W(L) = 0$

$\hookrightarrow W'''(L) = \frac{(k - \omega^2 M)}{(EI)}\,W(L)$

Now let's apply

$$W(0) = a\cancelto{1}{\cosh 0} + b\cancelto{0}{\sinh 0} + c\cancelto{1}{\cos 0} + d\cancelto{0}{\sin 0} = 0$$

$$W'(0) = \beta\left(a\cancelto{0}{\sinh 0} + b\cancelto{1}{\cosh 0} - c\cancelto{0}{\sin 0} + d\cancelto{1}{\cos 0}\right) = 0$$

$0 = a + c \longrightarrow c = -a$

$0 = \beta(b + d) \longrightarrow d = -b \qquad \beta \neq 0$

$$W(x) = a(\cosh\beta x - \cos\beta x) + b(\sinh\beta x - \sin\beta x)$$

$$W'(x) = \beta a(\sinh\beta x + \sin\beta x) + \beta b(\cosh\beta x - \cos\beta x)$$

$$W''(x) = \beta^2 a(\cosh\beta x + \cos\beta x) + \beta^2 b(\sinh\beta x + \sin\beta x)$$

$$W'''(x) = \beta^3 a(\sinh\beta x - \sin\beta x) + \beta^3 b(\cosh\beta x + \cos\beta x)$$

③ $0 = \beta^2 a(\cosh\beta L + \cos\beta L) + \beta^2 b(\sinh\beta L + \sin\beta L)$

④ $0 = \beta^3 a(\sinh\beta L - \sin\beta L) + \beta^3 b(\cosh\beta L + \cos\beta L)$
$+ \left(\frac{\omega^2 M - k}{EI}\right)\left(a(\cosh\beta L - \cos\beta L) + b(\sinh\beta L - \sin\beta L)\right)$

look at this term

Recall $\omega = \sqrt{\frac{EI}{\rho A}}\,\beta^2 \longrightarrow \omega^2 = \frac{EI}{\rho A}\beta^4$

then

$$\frac{\omega^2 M - k}{EI} = \frac{\omega^2 M}{EI} - \frac{k}{EI} = \frac{\cancel{EI}}{\rho A}\frac{M}{\cancel{EI}}\beta^4 - \frac{k}{EI}$$

$$= \frac{M}{\rho A}\beta^4 - \frac{k}{EI}$$

④ $0 = \beta^3 a(\sinh\beta L - \sin\beta L) + \beta^3 b(\cosh\beta L + \cos\beta L)$
$+ \left(\frac{M\beta^4}{\rho A} - \frac{k}{EI}\right)\left(a(\cosh\beta L - \cos\beta L) + b(\sinh\beta L - \sin\beta L)\right)$

$$0 = (\beta L)^3 a(\sinh\beta L - \sin\beta L) + (\beta L)^3 b(\cosh\beta L + \cos\beta L) +$$
$$\left(\frac{M(\beta L)^4}{\rho A L} - \frac{k L^3}{EI}\right)\left(a(\cosh\beta L - \cos\beta L) + b(\sinh\beta L - \sin\beta L)\right)$$

Recall $M = \rho A L$ and $\alpha = \frac{kL^3}{EI}$

$$0 = (\beta L)^3 a(\sinh\beta L - \sin\beta L) + (\beta L)^3 b(\cosh\beta L + \cos\beta L) +$$
$$((\beta L)^4 - \alpha)(a(\cosh\beta L - \cos\beta L) + b(\sinh\beta L - \sin\beta L))$$

Write as a matrix ...

$$\underbrace{\begin{bmatrix} \cosh\sqrt{B}L + \cos\sqrt{B}L & \sinh\sqrt{B}L + \sin\sqrt{B}L \\ (\sqrt{B}L)^3(\sinh\sqrt{B}L - \sin\sqrt{B}L) + & (\sqrt{B}L)^3(\cosh\sqrt{B}L + \cos\sqrt{B}L) + \\ ((\sqrt{B}L)^4 - \alpha)(\cosh\sqrt{B}L - \cos\sqrt{B}L) & ((\sqrt{B}L)^4 - \alpha)(\sinh\sqrt{B}L - \sin\sqrt{B}L) \end{bmatrix}}_{D} \begin{Bmatrix} a \\ b \end{Bmatrix} = \vec{0}$$

$$\Delta = \det(D) = CE$$

Now, solve for modeshapes using 1st row of matrix equation

$$(\cosh \beta L + \cos \beta L)a + (\sinh \beta L + \sin \beta L)b = 0$$

$$b = \frac{(\cosh \beta L + \cos \beta L)}{(\sinh \beta L + \sin \beta L)} a$$

$$\bar{W}(x) = a(\cosh \beta x - \cos \beta x) + b(\sinh \beta x - \sin \beta x)$$

$$\boxed{\bar{W}(x) = a(\cosh \beta x - \cos \beta x) + \frac{(\cosh \beta L + \cos \beta L)a}{(\sinh \beta L + \sin \beta L)}(\sinh \beta x - \sin \beta x)}$$

If we consider damping the $c\omega W(L) = 0$

this implies $\bar{W}(L) = 0$

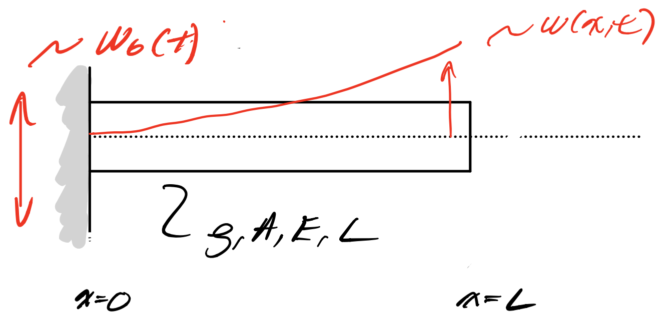
$$\bar{W}(L) = a(\cosh \beta L - \cos \beta L) + \frac{(\cosh \beta L + \cos \beta L)a}{\sinh \beta L + \sin \beta L}(\sinh \beta L - \sin \beta L) = 0$$

This implies $a = 0$

$\therefore$ damping at boundaries doesn't affect boundary conditions since the result would be no motion

See code

Consider,




Note the governing equation $\quad EI\dfrac{\partial^4 w}{\partial x^4} = -\rho A \dfrac{\partial^2 w}{\partial t^2}$

Now, $\quad w(x,t) = w_R(x,t) + w_b(t)$

$$\partial^4 w/\partial x^4 = \frac{\partial^4 w_R}{\partial x^4}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w_R}{\partial t^2} + \ddot{w}_b(t)$$

$$EI\frac{\partial^4 w_R}{\partial x^4} = -\rho A\frac{\partial^2 w_R}{\partial t^2} - \rho A \ddot{w}_b(t)$$

$$\boxed{EI\frac{\partial^4 w_R}{\partial x^4} + \rho A \ddot{w}_b(t) = \rho A \frac{\partial^2 w_R}{\partial t^2}}$$

The term $A\ddot{w}_b(t)$ acts like distributed forcing and does not change the boundary conditions.

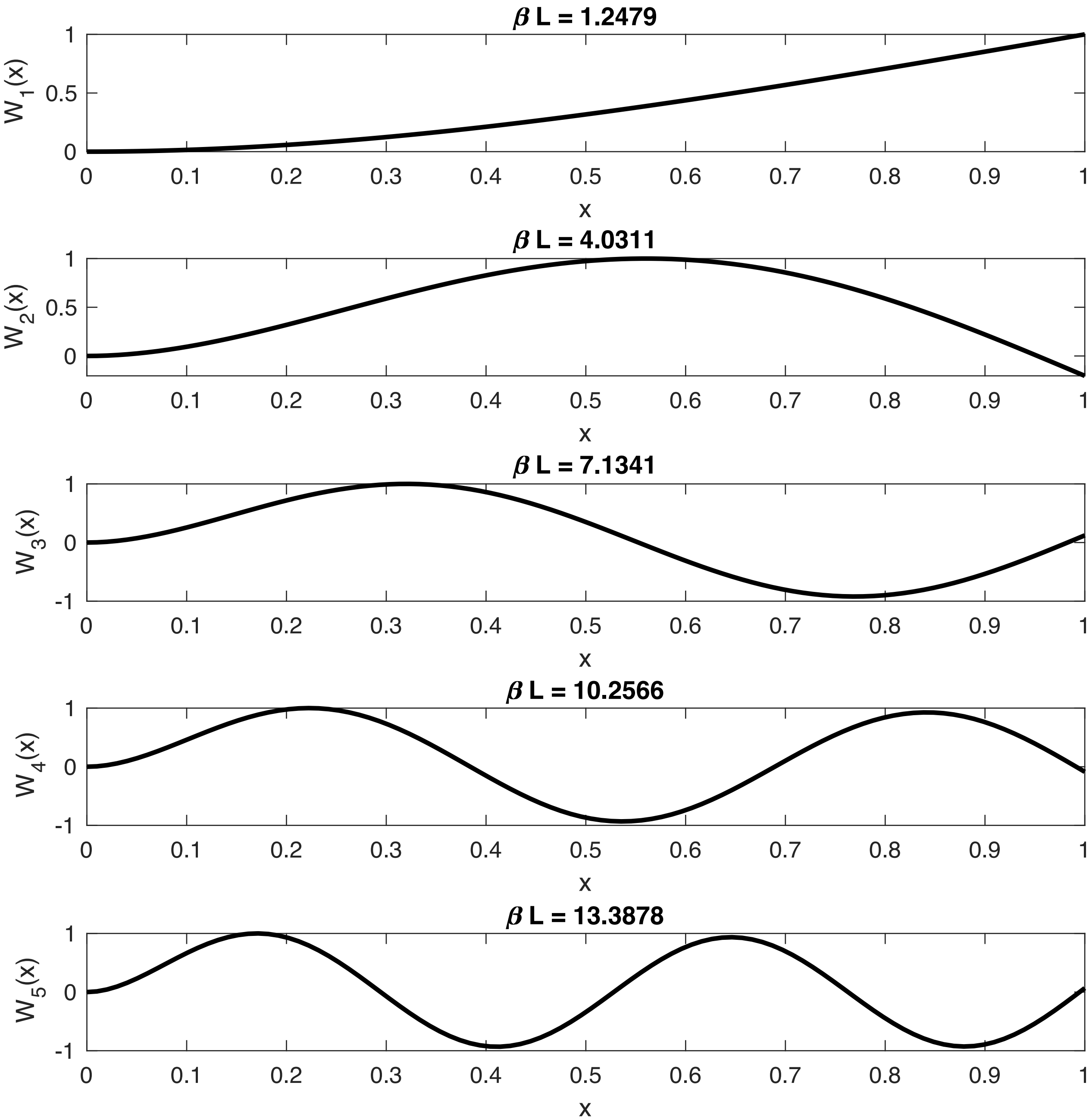
The boundary conditions can be written as

$w_R(0,t) = 0, \qquad EI\,\partial^2 w_R/\partial x^2 = 0$

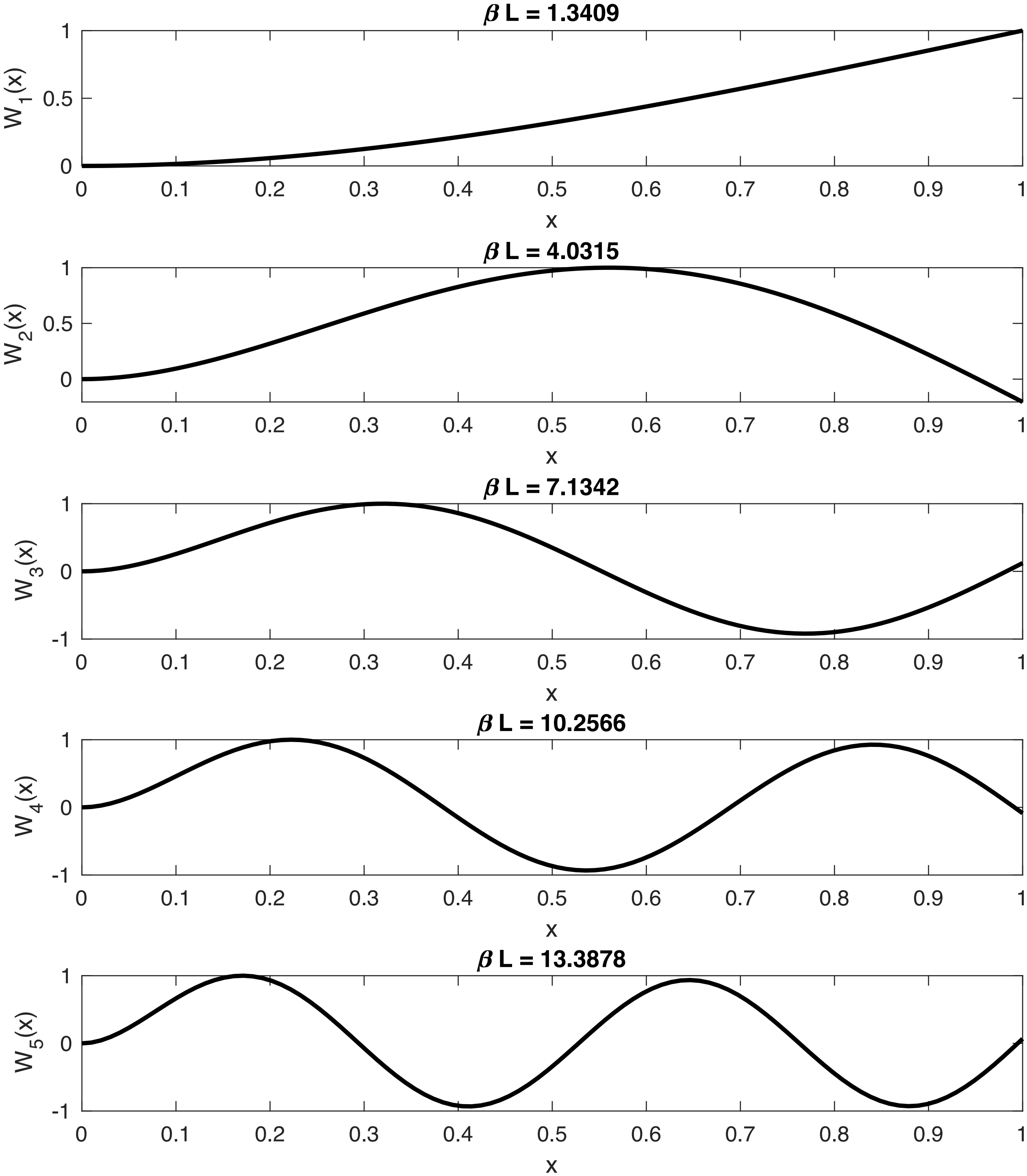
$w_R'(0,t) = 0, \qquad EI\,\partial^3 w_R/\partial x^3 = 0$

same as a fixed free beam.

$a = 0$ same as HW 5.3

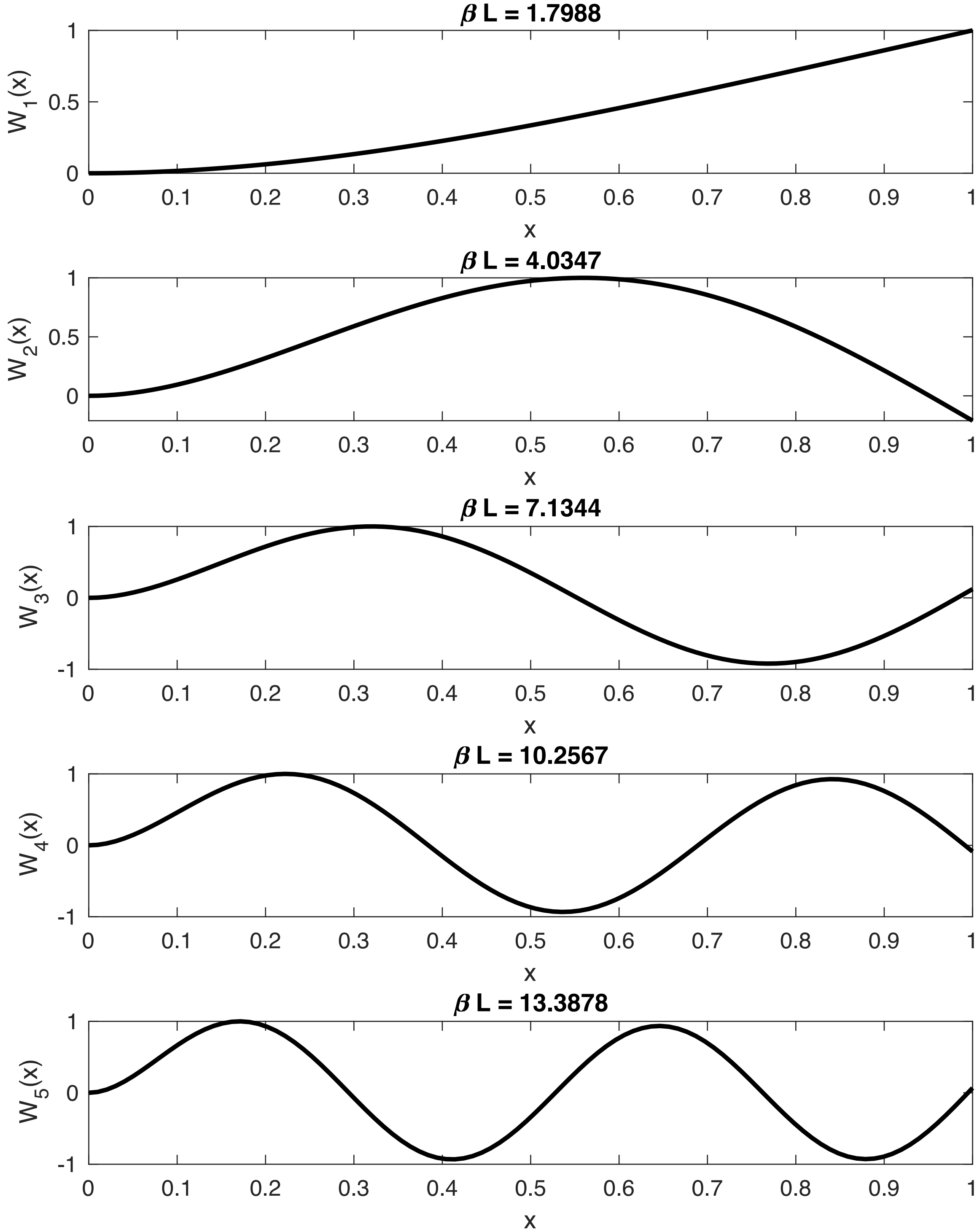


$\alpha=1$



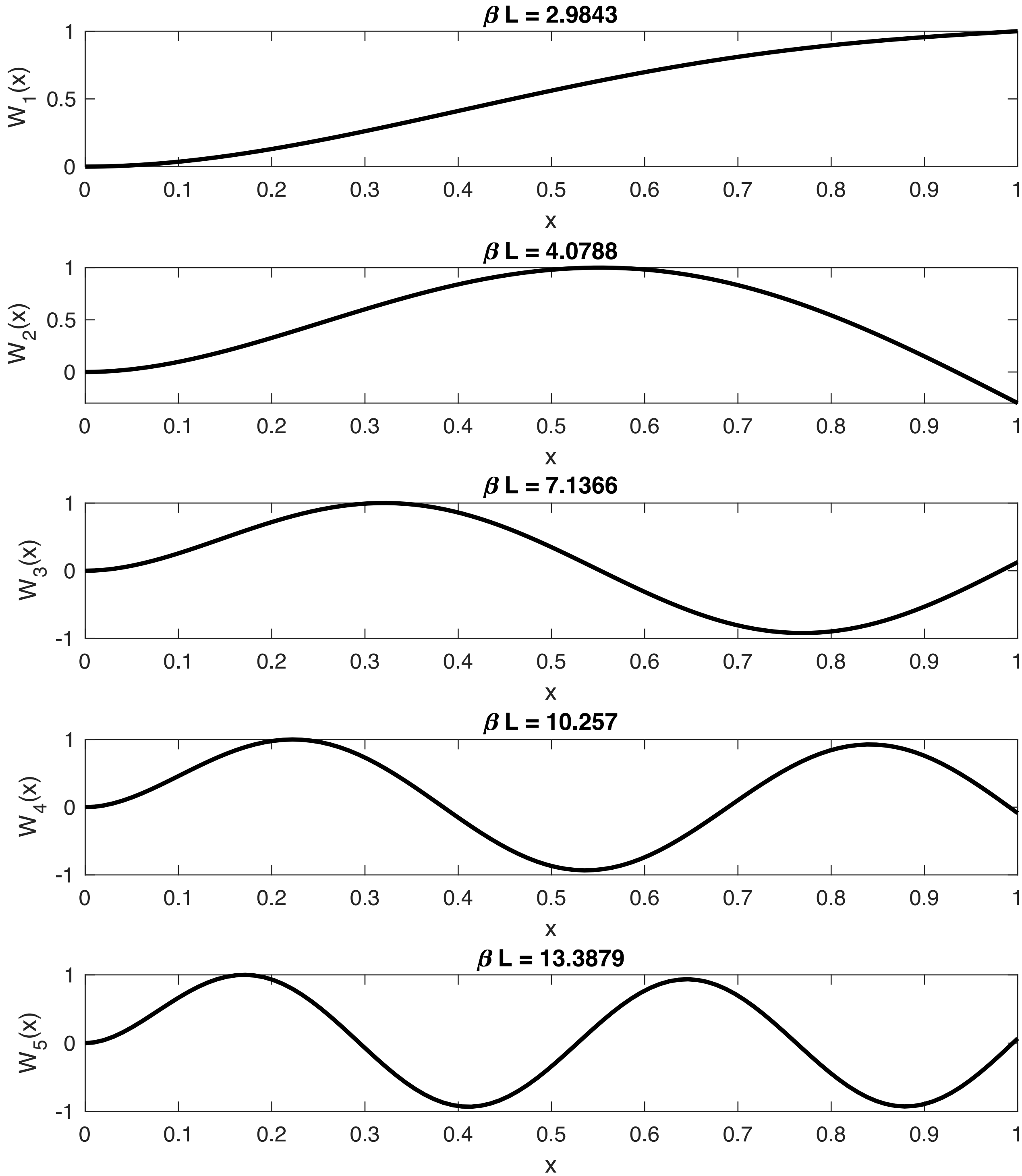
β L = 1.3409
W1(x)
x
β L = 4.0315
W2(x)
x
β L = 7.1342
W3(x)
x
β L = 10.2566
W4(x)
x
β L = 13.3878
W5(x)
x

$a = 10$



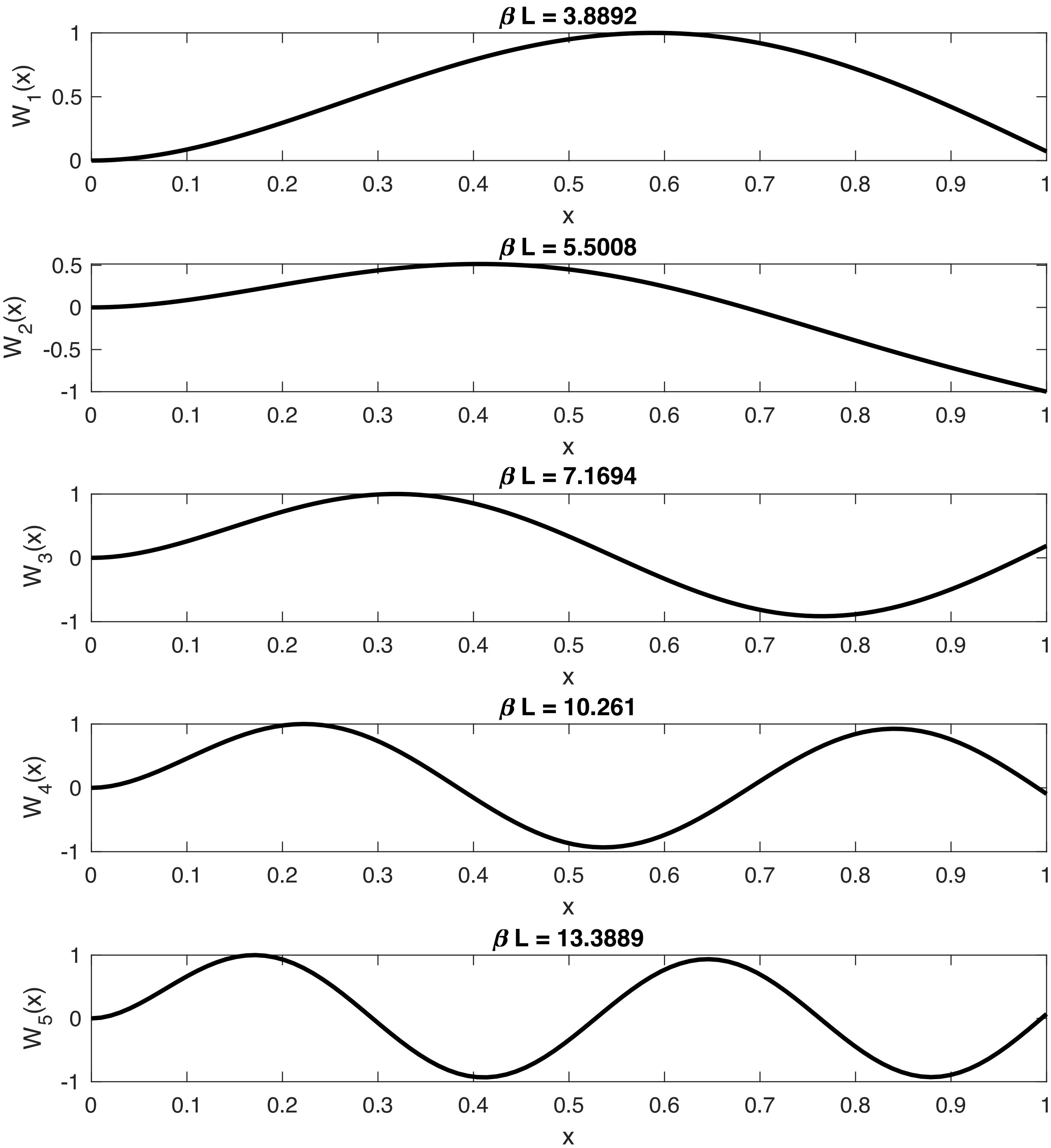
β L = 1.7988
W1(x)
β L = 4.0347
W2(x)
β L = 7.1344
W3(x)
β L = 10.2567
W4(x)
β L = 13.3878
W5(x)
x
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

$\alpha = 100$



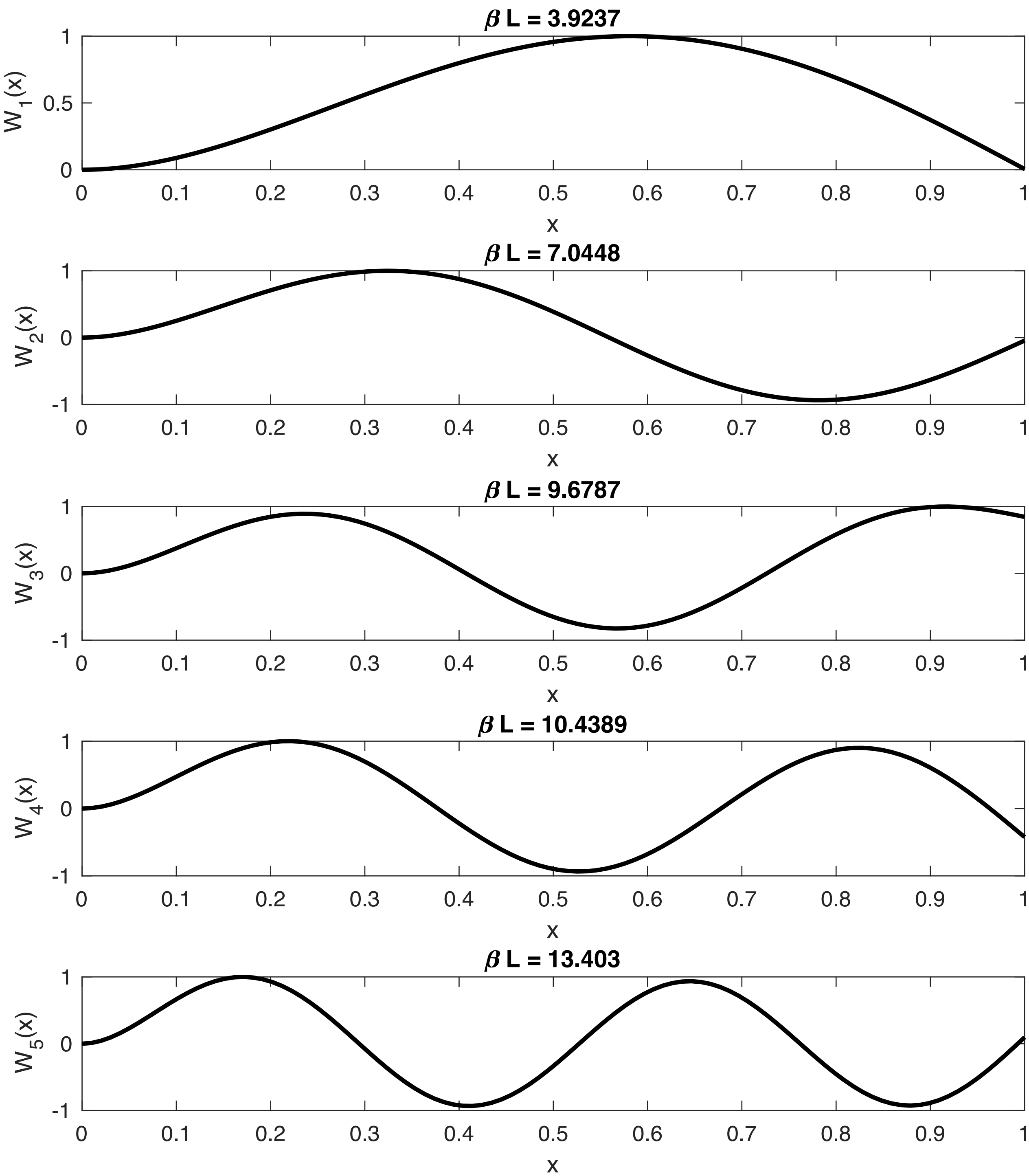
β L = 2.9843
W1(x)
β L = 4.0788
W2(x)
β L = 7.1366
W3(x)
β L = 10.257
W4(x)
β L = 13.3879
W5(x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
-1

$\alpha = 1000$



β L = 3.8892
W_1(x)
β L = 5.5008
W_2(x)
β L = 7.1694
W_3(x)
β L = 10.261
W_4(x)
β L = 13.3889
W_5(x)
x

$\alpha = 10000$



β L = 3.9237
W1(x)
β L = 7.0448
W2(x)
β L = 9.6787
W3(x)
β L = 10.4389
W4(x)
β L = 13.403
W5(x)
x

```
clc
clear
close all
fprintf(['\n\n\nStarting file >>' mfilename '<< at ' datestr(now,0) '\n\n']);
format long
% Exam 2 Problem 3

% x= Beta*L
syms B x l a b c d M E I w rho L A BL alpha

W = a*(cosh(B*x)-cos(B*x))+b*(sinh(B*x)-sin(B*x))

% Write as matrix pull out coefficents
BC11 = cosh(BL)+cos(BL);
BC12 = sinh(BL)+sin(BL);
BC21 = (BL).^3.*(sinh(BL)-sin(BL))+((BL).^4 - alpha).*(cosh(BL)-cos(BL));
BC22 = (BL).^3.*(cosh(BL)+cos(BL))+((BL).^4 - alpha).*(sinh(BL)-sin(BL));

BC   = [BC11 BC12; BC21 BC22];

CE    = simplify(det(BC));
CE    = matlabFunction(CE);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
alpha = 0;
CEfun = @(BL) CE(BL,alpha); % This is for optimization
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
BL    = linspace(0,5*pi,10^4);
CEv   = CE(BL,alpha);

% This CE does not have  plot that we can easily identify our iniitial
% guess
% In order to get initial guess we look at places where the is a sign
% change in the CE
[Val1, loc1] = find(abs(diff(sign(CEv)))==2); % This time loc1 is indices
%
figure
plot(BL(1:end),sign(CEv),'r')
axis([0 inf -1.2 1.2])

% Initial Guesses
x0v = BL(loc1);
options_all=[];
% set your length L
L=1;
for i =1:5
    x0 = x0v(i);
    fprintf(['\n\n\n >> Mode' num2str(i) '<<  \n\n']);
    % Fzero
    [xval, fval, exitflag] = fzero(CEfun, x0, options_all);
    betaL(i) = xval;%This is Beta*L
    beta(i)  = betaL(i)/L; % THis is Beta

end
%Solve for Modeshapes
%solve fofr a constant
bsol = -BC11/BC12;
%plug into W(x)
Wmode= simplify(subs(W,b,bsol));
```

```
pretty(Wmode)
%Plot mode shapes
x  =linspace(0,L,100);
figure
for i =1:5
    B  = beta(i);
    BL = betaL(i);
    a  = 1;
    WmodeP = eval(Wmode);
    Wmax   = max(abs(WmodeP));
    WmodeP = WmodeP/Wmax;
    subplot(5,1,i)
    line(x,WmodeP,'linewidth',2,'color','k')
    xlabel('x')
    ylabel(['W_',num2str(i),'(x)'])
    title(['\beta L = ', num2str(BL)])
    box on
end
```