ME 563-Fall 2020

Homework No. 3

Due: September 19, 2020 11:59 pm on Gradescope

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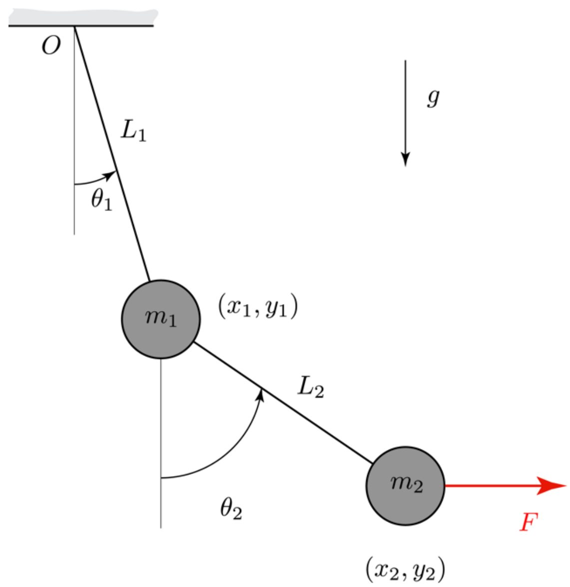
Homework Problem 3.1

A double pendulum consists of two bobs of mass m_1 and m_2, suspended by inextensible, massless strings of length L_1 and L_2.



a) Determine the expression for potential energy U and the generalized forces corresponding to F for the generalized coordinates. Use these results to determine the angles θ_1 and θ_2 corresponding to static equilibrium. Leave these angles in terms of the ratio F/mg.

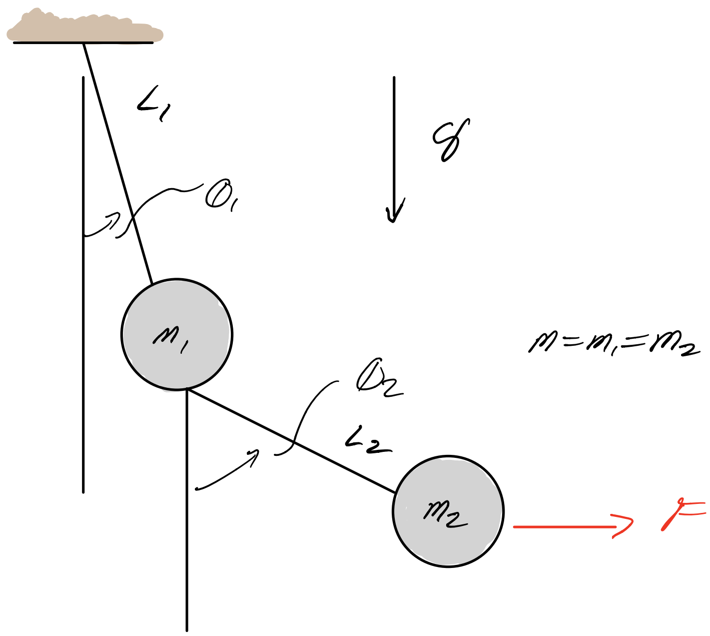
b) Write down an expression for the kinetic energy T in terms of the generalized coordinates θ_1 and θ_2 and their time derivatives. From this expression, identify the elements m_{ij}, where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{\theta}_i\dot{\theta}_j$$

c) Determine the mass matrix [M] and the stiffness matrix [K] corresponding small oscillations about the equilibrium state for $F/mg = 0$.

d) Determine the mass matrix [M] and the stiffness matrix [K] corresponding small oscillations about the equilibrium state for $F/mg = 2$. Compare these with those found in part c).

Problem 1)



a) $U = -mgL_1\cos\theta_1 - mg(L_1\cos\theta_1 + L_2\cos\theta_2)$

$$= -2mgL_1\cos\theta_1 - mgL_2\cos\theta_2$$

$$\vec{r}_2 = (L_1\sin\theta_1 + L_2\sin\theta_2)\hat{i} - (L_1\cos\theta_1 + L_2\cos\theta_2)\hat{j}$$

$$\delta\vec{r}_2 = (L_1\cos\theta_1\,\delta\theta_1 + L_2\cos\theta_2\,\delta\theta_2)\hat{i} + (L_1\sin\theta_1\,\delta\theta_1 + L_2\sin\theta_2\,\delta\theta_2)\hat{j}$$

$$\delta W^{(nc)} = \vec{F}\cdot\delta\vec{r}_2$$

$$= F\hat{i}\cdot\{(L_1\cos\theta_1\,\delta\theta_1 + L_2\cos\theta_2\,\delta\theta_2)\hat{i} + (L_1\sin\theta_1\,\delta\theta_1 + L_2\sin\theta_2\,\delta\theta_2)\hat{j}\}$$

$$= FL_1\cos\theta_1\,\delta\theta_1 + FL_2\cos\theta_2\,\delta\theta_2$$

$Q_1 = FL_1\cos\theta_1$ and $Q_2 = FL_2\cos\theta_2$

Determine equilibrium points

$$\frac{\partial U}{\partial \theta_1} = Q_1 \longrightarrow 2mgL_1\sin\theta_1 = FL_1\cos\theta_1$$

$$2mg\sin\theta_1 = F\cos\theta_1$$

$$2\tan\theta_1 = F/mg \longrightarrow \tan\theta_1 = F/2mg$$

$\theta_{1eq} = \tan^{-1}(F/2mg)$

and

$$\frac{\partial U}{\partial \theta_2} = 0 \longrightarrow mgL_2 \sin\theta_2 = FL_2\cos\theta_2$$

$mg\sin\theta_2 = F\cos\theta_2$
$\tan\theta_2 = F/mg$
$\theta_{2eq} = \tan^{-1}(F/mg)$

b) write Kinetic Energy

$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} = L_1\sin\theta_1\hat{i} - L_1\cos\theta_1\hat{j}$

$\vec{r}_2 = \vec{r}_1 + \vec{r}_{2/1} = (L_1\sin\theta_1 + L_2\sin\theta_2)\hat{i} - (L_1\cos\theta_1 + L_2\cos\theta_2)\hat{j}$

The velocity vectors ...

$\vec{v}_1 = \dot{\theta}_1 L_1\cos\theta_1\hat{i} + \dot{\theta}_1 L_1\sin\theta_1\hat{j}$ and $\vec{v}_2 = (\dot{\theta}_1 L_1\cos\theta_1 + \dot{\theta}_2 L_2\cos\theta_2)\hat{i} + (\dot{\theta} L_1\sin\theta_1 + \dot{\theta}_2 L_2\sin\theta_2)\hat{j}$

The kinetic energy ......

$$T = \frac{1}{2} m\,\vec{v}_1\cdot\vec{v}_1 + \frac{1}{2} m\,\vec{v}_2\cdot\vec{v}_2 = \frac{1}{2}(2mL_1^2\dot{\theta}_1^2 + 2mL_1L_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + mL_2^2\dot{\theta}_2^2)$$

$m_{11} = 2mL_1^2$
$m_{12} = m_{21} = mL_1L_2\cos(\theta_1-\theta_2)$
$m_{22} = mL_2^2$

Stiffness Calculation

$$k_{11} = \left.\frac{\partial^2 U}{\partial\theta_1^2}\right|_{\bar{\theta}_{eq}} = 2mgL_1\cos(\theta_{1eq}), \quad k_{12} = k_{21} = \left.\frac{\partial^2 U}{\partial\theta_1\partial\theta_2}\right|_{\bar{\theta}_{eq}} = 0$$

$$k_{22} = \left.\frac{\partial^2 U_2}{\partial\theta_2^2}\right|_{\bar{\theta}_{eq}} = mgL_2\cos(\theta_{2eq})$$

c) Consider $F/mg = 0$ and $F/2mg = 0$

$\theta_{1eq} = \theta_{2eq} = \tan^{-1}(0) = 0$
$\cos(\theta_{1eq}) = \cos(\theta_{2eq}) = 1$

The linear mass and stiffness matrices

$$[M] = \begin{bmatrix} 2mL_1^2 & mL_1L_2 \\ mL_1L_2 & mL_2^2 \end{bmatrix}, \quad [k] = mg\begin{bmatrix} 2L_1 & 0 \\ 0 & L_1 \end{bmatrix}$$

d) Consider $F/mg = 2$ and $F/gmg = 1$

$$\theta_{1eq} = \tan^{-1}(1) = \pi/4$$

$$\theta_{2eq} = \tan^{-1}(2) = 1.0715$$

$$\cos(\theta_{1eq}) = 1/\sqrt{2}$$

$$\cos(\theta_{2eq}) = 0.447$$

The linear mass and stiffness matrices

$$[M] = \begin{bmatrix} 2\, mL_1^2 & 0.949\, mL_1L_2 \\ 0.949\, mL_1L_2 & mL_2^2 \end{bmatrix}, \quad [k] = mg\begin{bmatrix} 1.414\, L_1 & 0 \\ 0 & 0.447\, L_2 \end{bmatrix}$$

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Homework Problem 3.2

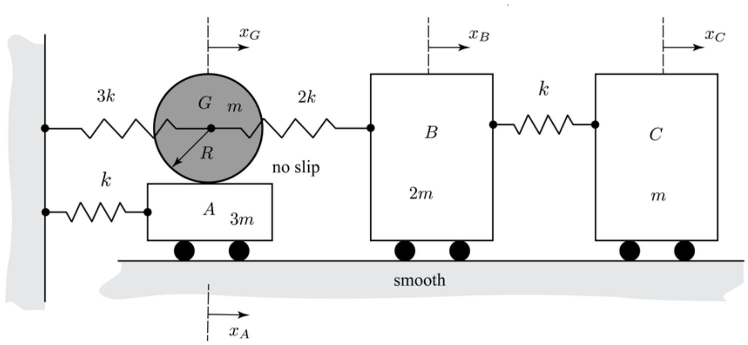
Consider the system below, whose motion is described by the absolute coordinates shown.

a) Write down the potential energy function U for this four-DOF system and use the following results from lecture to develop the stiffness matrix for the system:

$$K_{ij} = \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\mathbf{q}_0}$$

b) Use the method of influence coefficients to develop the flexibility matrix $[A] = [K]^{-1}$.

c) Check your results in a) and b) above by verifying that $[A][K]=[I]$ where $[I]$ is the identity matrix.



a) The potential energy can be written as

$$U = \tfrac{1}{2} k x_A^2 + \tfrac{1}{2}(3k) x_G^2 + \tfrac{1}{2}(2k)(x_B - x_G)^2 + \tfrac{1}{2} k (x_C - x_B)^2$$

Equilibrium position : $x_A = x_G = x_B = x_C = 0$

$$\partial U / \partial x_A = k x_A , \quad \partial U / \partial x_G = 3k x_G + 2k(x_G - x_B) = 5k x_G - 2k x_B$$

$$\partial U / \partial x_B = 2k(x_B - x_G) + k(x_B - x_C) = 3k x_B - 2k x_G - k x_C$$

$$\partial U / \partial x_C = k x_C \qquad x_A = q_1 , \; x_G = q_2 , \; x_B = q_3 , \; x_C = q_4$$

$$K_{11} = \frac{\partial^2 U}{\partial x_A^2} = k , \quad K_{12} = K_{21} = \frac{\partial^2 U}{\partial x_A \partial x_G} = 0 , \quad K_{13} = K_{31} = \frac{\partial^2 U}{\partial x_A \partial x_B} = 0 , \quad K_{14} = \frac{\partial^2 U}{\partial x_A \partial x_C} = 0$$

$$k_{22} = \frac{\partial^2 U}{\partial x_b^2} = 5k, \quad k_{23} = k_{32} = \frac{\partial^2 U}{\partial x_b \partial_b} = -2k, \quad k_{24} = k_{42} = \frac{\partial^2 U}{\partial x_b \partial x_c} = 0$$

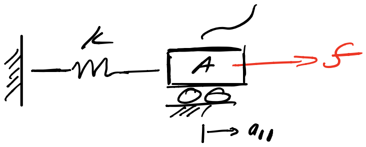
$$k_{33} = \frac{\partial^2 U}{\partial^2 x_B} = 3k \qquad k_{34} = k_{43} = \frac{\partial^2 U}{\partial x_b \partial x_c} = -k$$

$$k_{44} = \frac{\partial^2 U}{\partial^2 x_c} = k$$

$$[K] = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 5k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix}$$

b) Apply a unit load @ A

No force here

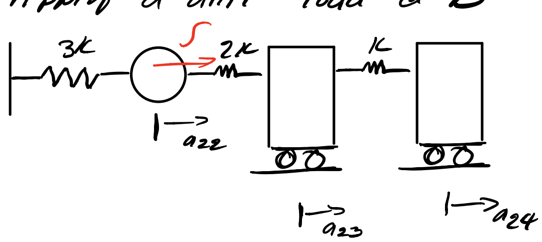


$\Sigma F_x: \ -k a_{11} + \overset{1}{f} = 0, \quad a_{11} = 1/k$

$a_{21} = a_{31} = a_{41} = 0$ since not connected

By reciprocity $a_{12} = a_{21} = 0, \quad a_{13} = a_{31} = 0, \quad a_{41} = a_{14} = 0$

Apply a unit load @ B



$a_{22} = a_{23} = a_{24}$ zero strain to right

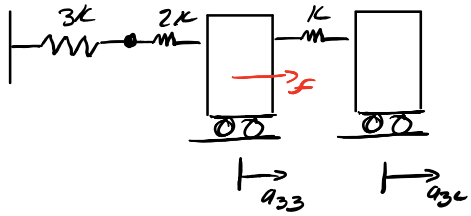
$\Sigma F_x: \ -3k a_{22} + \overset{1}{f} = 0$

$a_{22} = 1/3k, \qquad a_{24} = 1/3k$

$a_{23} = 1/3k,$

By reciprocity $a_{42} = a_{24} = 1/3k, \quad a_{23} = a_{32} = 1/3k$

Apply a unit load @ C



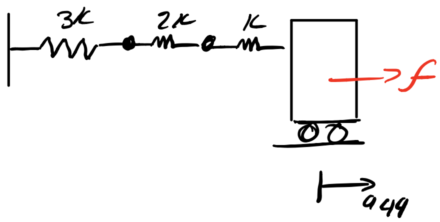
$$\frac{1}{k_{eq}} = \frac{1}{3k} + \frac{1}{2k}$$

$$k_{eq} = \frac{(3k)(2k)}{3k+2k} = \frac{6}{5}k$$

$$\Sigma F_x : -k_{eq}\, a_{33} + f = 0 \qquad a_{33} = \frac{1}{k_{eq}} = \frac{5}{6k}$$

$$a_{34} = a_{43} = \frac{5}{6k}$$

Apply a unit load @ D



$$\frac{1}{k_{eq1}} = \frac{1}{3k} + \frac{1}{2k}, \quad k_{eq1} = \frac{(3k)(2k)}{3k+2k} = \frac{6}{5}k$$

$$\frac{1}{k_{eq2}} = \frac{1}{k_{eq1}} + \frac{1}{k} = \frac{5}{6k} + \frac{1}{k}$$

$$k_{eq2} = \frac{k_{eq1}\,k}{k_{eq1}+k} = \frac{(6/5\,k)(k)}{6/5\,k + k}$$

$$k_{eq2} = \frac{6/5\,k^2}{11/5\,k} = \frac{6}{11}k$$

$$\Sigma F_x : -k_{eq2}\, a_{44} + 1 = 0$$

$$a_{44} = \frac{1}{k_{eq2}} = \frac{11}{6k}$$

$$[a] = \begin{bmatrix} 1/k & 0 & 0 & 0 \\ 0 & 1/3k & 1/3k & 1/3k \\ 0 & 1/3k & 5/6k & 5/6k \\ 0 & 1/3k & 5/6k & 11/6k \end{bmatrix}$$

$$[a][K] = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 5k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{bmatrix} 1/k & 0 & 0 & 0 \\ 0 & 1/3k & 1/3k & 1/3k \\ 0 & 1/3k & 5/6k & 5/6k \\ 0 & 1/3k & 5/6k & 11/6k \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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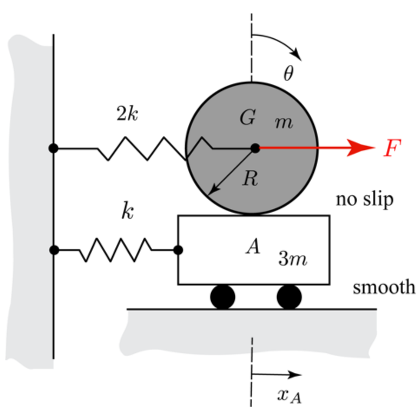
Homework Problem 3.3

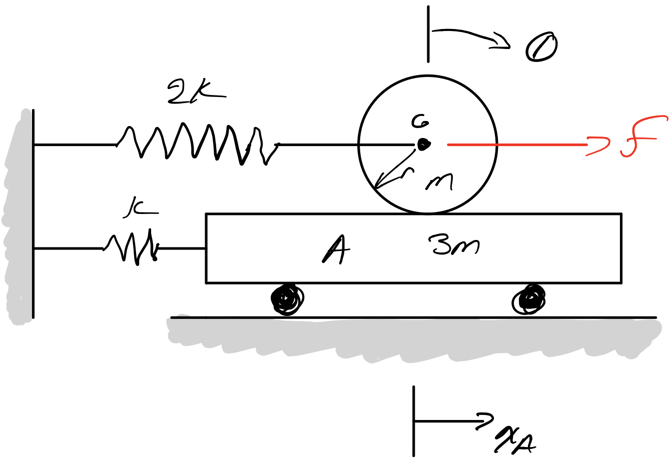
A disk of mass m and radius r rolls without slip on a cart of mass 3m. A force F is applied to the cart. The disk has an angular displacement θ. Let x, describe the absolute motion of the cart. The springs is unstretched when x_A=0. Assume the surface between the cart and ground is smooth.

a) Use a *Newton Euler formulation* to find the equations of motion. Your final equations should not include any forces of reaction. The final differential equation are in terms of x_A and θ and should be in matrix form.

b) Use *Lagrange's Equations* to find the equations of motion. The generalize coordinates are x_A and θ. The final differential equation are in terms of x_A and θ and should be in matrix form.

c) Compare the form of the mass matrix, stiffness matrix and forcing vector of the *Newtonian* formulation with the *Lagrangian* formulation. In particular, are they the same? If not, in which formulation are the matrices symmetric ? Do the forcing vectors differ? Explain any similarities or differences.





$x_G = x_A + r\theta$

$\vec{r}_A = x_A \hat{i}$ $\qquad$ $d\vec{r}_A = dx_A \hat{i}$

$\vec{r}_G = (x_A + r\theta)\hat{i}$ $\qquad$ $d\vec{r}_G = (dx_A + r\,d\theta)\hat{i}$

$\vec{v}_A = \dot{x}_A \hat{i}$ $\qquad$ $\vec{v}_G = (\dot{x}_A + r\dot{\theta})\hat{i}$

$$T = \tfrac{1}{2}(3m)\,\vec{v}_A \cdot \vec{v}_A + \tfrac{1}{2} m\, \vec{v}_G \cdot \vec{v}_G + \tfrac{1}{2} I_G \dot{\theta}^2$$

$$T = \tfrac{1}{2}(3m)\,\dot{x}_A^2 + \tfrac{1}{2} m(\dot{x}_A^2 + r^2\dot{\theta}^2 + 2\dot{x}_A r\dot{\theta}) + \tfrac{1}{2} I_G \dot{\theta}^2$$

$$U = \tfrac{1}{2} k\, x_A^2 + \tfrac{1}{2}(2k)(x_A + r\theta)^2$$

$$dW^{NC} = f\hat{i} \cdot d\vec{r}_G = f\,dx_A + rf\,d\theta$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_A}\right) - \frac{\partial T}{\partial x_A} + \frac{\partial U}{\partial x_A} = Q_{x_A}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} = Q_\theta$$

$$\frac{\partial T}{\partial \dot{x}_A} = 3m\dot{x}_A + m\dot{x}_A + mr\dot{\theta} = 5m\dot{x}_A + mr\dot{\theta},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_A}\right) = 4m\ddot{x}_A + mr\ddot{\theta}$$

$$\frac{\partial T}{\partial x_A} = 0, \quad \frac{\partial U}{\partial x_A} = kx_A + 2k(x_A + r\theta), \text{ and } Q_{x_A} = f$$

$$\boxed{4m\ddot{x}_A + mr\ddot{\theta} + 3kx_A + 2kr\theta_A = f}$$

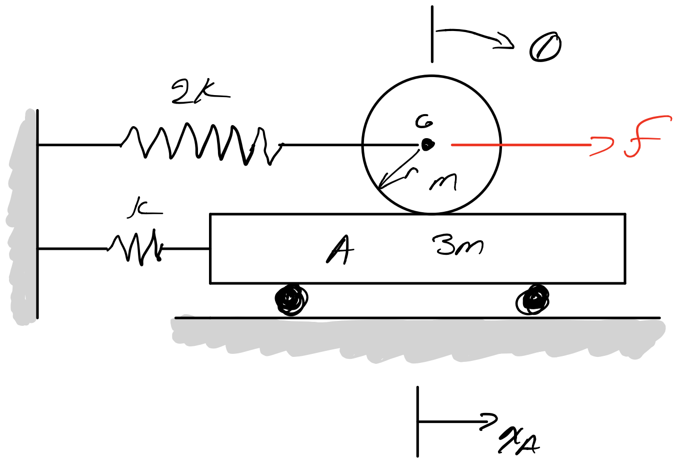
$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta} + I_G\dot{\theta} + mr\dot{x}_A$$

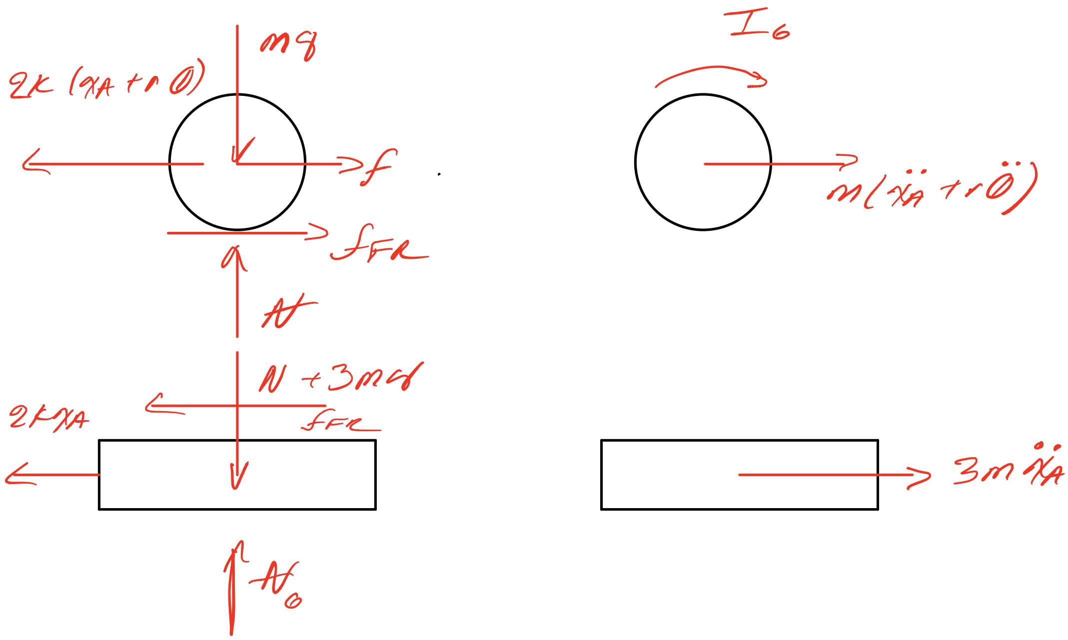
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + I_G\ddot{\theta} + mr\ddot{x}_A$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial U}{\partial \theta} = 2k(x_A + r\theta)r, \quad \text{and} \quad Q_\theta = rf$$

$$\boxed{(I_G + mr^2)\ddot{\theta} + mr\ddot{x}_A + 2krx_A + 2kr^2\theta = rf}$$

$$\begin{pmatrix} 4m & mr \\ mr & I_G + mr^2 \end{pmatrix} \begin{Bmatrix} \ddot{x}_A \\ \ddot{\theta} \end{Bmatrix} + \begin{pmatrix} 3k & 2kr \\ 2kr & 2kr^2 \end{pmatrix} \begin{Bmatrix} x_A \\ \theta \end{Bmatrix} = \begin{Bmatrix} f \\ rf \end{Bmatrix}$$





$\xrightarrow{+}$ ΣF_x: $f_{FR} + f - 2k(x_A + r\theta) = m(\ddot{x}_A + r\ddot{\theta})$ 1)

$+\uparrow$ ΣF_y: $-mg + N = 0$ 2)

ΣM_G: $-f_{FR}\, r = I_G \ddot{\theta}$ 3)

$\xrightarrow{+}$ ΣF_A: $-k x_A - f_{FR} = 3m\ddot{x}_A$ 4)

The wheel EOM can be found by combing 1) and 3)

$$-\frac{I_G}{r}\ddot{\theta} + f = m(\ddot{x}_A + r\ddot{\theta}) + 2k(x_A + r\theta)$$

$$\boxed{m(\ddot{x}_A) + (I_G/r + mr)\ddot{\theta} + 2k(x_A + r\theta) = f}$$

Compine 3) and 4)

$$\boxed{3m\ddot{x}_A + \frac{I_G}{r}\ddot{\theta} + kx_A = 0}$$

$$\begin{pmatrix} m & I_G/r + mr \\ 3m & I_G \end{pmatrix} \begin{Bmatrix} \ddot{x}_A \\ \ddot{\theta} \end{Bmatrix} + \begin{pmatrix} 2k & 2kr \\ k & 0 \end{pmatrix} \begin{Bmatrix} x_A \\ \theta \end{Bmatrix}$$

$$= \begin{Bmatrix} f \\ 0 \end{Bmatrix}$$