ME 562 – Spring 2020
Midterm exam solution

Median score: 50/60

Mean score: 49/60

Problem No. 1

Consider a B-coordinate system xyz is attached to a rigid body. A second A-coordinate system XYZ is fixed is space. Initially, the A- and B-coordinate axes are aligned with a common origin at O. Point P on the rigid body has B-coordinates of $(10,\text{-}6,15)$ ft. The rigid body undergoes three successive rotations about point O:

i) a 90° rotation about the y-axis, followed by

ii) a 36.87° rotation about the x-axis, followed by

iii) a 180° rotation about the z-axis.

For this problem:

a) Determine the <u>*A-coordinates*</u> of P after this sequence of rotations.

b) Determine the <u>*direction angles*</u> for line OP as measured from the A-coordinate axes.

c) Determine the <u>*Euler axis of rotation*</u> and the rotation of the body about this axis as a result of this sequence of rotations.

SOLUTION

$$[R]=[R_z(180°)][R_x(36.87°)][R_y(90°)]$$

$$=\begin{bmatrix} cos(180°) & sin(180°) & 0 \\ -sin(180°) & cos(180°) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(36.87°) & sin(36.87°) \\ 0 & -sin(36.87°) & cos(36.87°) \end{bmatrix}\begin{bmatrix} cos(90°) & 0 & -sin(90°) \\ 0 & 1 & 0 \\ sin(90°) & 0 & cos(90°) \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.8 & 0.6 \\ 0 & -0.6 & 0.8 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ -0.6 & -0.8 & 0 \\ 0.8 & -0.6 & 0 \end{bmatrix}$$

$$[T_{B\to A}]=[R]^T=\begin{bmatrix} 0 & -0.6 & 0.8 \\ 0 & -0.8 & -0.6 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore:

$$\{r_P\}^A = [T_{B\to A}]\{r_P\}^B = \begin{bmatrix} 0 & -0.6 & 0.8 \\ 0 & -0.8 & -0.6 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 10 \\ -6 \\ 15 \end{Bmatrix} = \begin{Bmatrix} 15.6 \\ -4.2 \\ 10 \end{Bmatrix}$$

The direction angles relative to A-axes for OP:

$$\alpha = cos^{-1}\left(\frac{15.6}{\sqrt{(15.6)^2+(-4.2)^2+(10)^2}}\right) = 34.8°$$

$$\beta = cos^{-1}\left(\frac{-4.2}{\sqrt{(15.6)^2+(-4.2)^2+(10)^2}}\right) = 102.8°$$

$$\gamma = cos^{-1}\left(\frac{10}{\sqrt{(15.6)^2+(-4.2)^2+(10)^2}}\right) = 58.4°$$

The Euler axis is found from the solution of:

$$0 = \left[[T_{B\to A}] - [I]\right]\{u\} = \left[\begin{bmatrix} 0 & -0.6 & 0.8 \\ 0 & -0.8 & -0.6 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} -1 & -0.6 & 0.8 \\ 0 & -1.8 & -0.6 \\ 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -u_1 - 0.6u_2 + 0.8u_3 \\ -1.8u_2 - 0.6u_3 \\ u_1 - u_3 \end{Bmatrix}$$

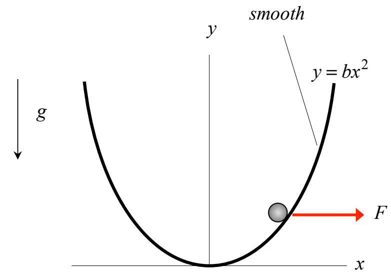
Using $u_1 = 1$:

$$u_3 = u_1 = 1 \quad \& \quad u_2 = -\frac{0.6}{1.8}u_3 = -\frac{1}{3} \quad \Rightarrow \quad \{u\} = \begin{Bmatrix} 1 \\ -1/3 \\ 1 \end{Bmatrix}$$

and:

$$\phi = cos^{-1}\left(\frac{tr[T_{B\to A}]-1}{2}\right) = cos^{-1}\left(\frac{-0.8-1}{2}\right) = cos^{-1}(-0.9) = 154.1°$$

Problem 2



A particle of mass *m* is able to slide along a smooth, parabolic surface defined by the following equation: $y = bx^2$, where $b = 0.25/meter$, and *x* and *y* are given in meters. The particle is acted upon by a constant horizontal force $F = mg/2$ when the particle is motionless at $y = 0$.

a) Determine the speed of the particle as a function of the position variable y.

b) Determine the maximum height, y_{max}, attained by the particle during its subsequent motion.

c) Determine the acceleration vector of the particle at the position of $y_{max}/2$.

SOLUTION

PART a):

$$U_{1\to2} = \int_1^2 \left(\vec{F} \bullet \hat{e}_t ds\right) = \int_1^2 \left(F_x dx + F_y dy\right) = F\int_1^2 dx$$

$$= F\left(x_2 - x_1\right) = F\left(\sqrt{\frac{y_2}{b}} - \sqrt{\frac{y_1}{b}}\right) = \left(\frac{mg}{2}\right)\sqrt{\frac{y}{b}}$$

Using gravitational datum at y = 0:

$$V_1 = 0 \quad \& \quad V_2 = mgy$$

Also:

$$T_1 = 0 \quad \& \quad T_2 = \frac{1}{2}mv^2$$

From this, we have:

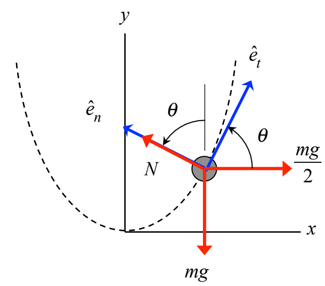
$$T_1 + V_1 + U_{1\to2} = T_2 + V_2 \quad \Rightarrow \quad \left(\frac{mg}{2}\right)\sqrt{\frac{y}{b}} = \frac{1}{2}mv^2 + mgy \quad \Rightarrow \quad \boxed{v = \sqrt{g}\sqrt{\sqrt{\frac{y}{b}} - 2y}}$$

PART b):

The maximum height is reached when $v = 0$:

$$0 = \sqrt{g}\sqrt{\sqrt{\frac{y_{\max}}{b}} - 2y_{\max}} \Rightarrow \sqrt{\frac{y_{\max}}{b}} = 2y_{\max} \Rightarrow \frac{y_{\max}}{b} = 4y_{\max}^2 \Rightarrow \boxed{y_{\max} = \frac{1}{4b} = 1}$$

PART c) – NOT REQUIRED



Kinetics:

$$\sum F_x = -N\sin\theta + \frac{mg}{2} = m\ddot{x} \quad \Rightarrow \quad N\sin\theta = \frac{mg}{2} - m\ddot{x}$$

$$\sum F_y = N\cos\theta - mg = m\ddot{y} \quad \Rightarrow \quad N\cos\theta = mg + m\ddot{y}$$

Therefore:

$$\frac{N\sin\theta}{N\cos\theta} = \frac{mg/2 - m\ddot{x}}{mg + m\ddot{y}} \quad \Rightarrow \quad \tan\theta = \frac{g - 2\ddot{x}}{2\left(g + \ddot{y}\right)} \quad \Rightarrow$$

$$g - 2\ddot{x} = \left(g + \ddot{y}\right)\tan\theta \quad \Rightarrow \quad \ddot{y} = g\left(\cot\theta - 1\right) - 2\cot\theta\,\ddot{x} \qquad (1)$$

Kinematics

$$y = bx^2 \quad \Rightarrow \quad \dot{y} = 2bx\dot{x} \quad \Rightarrow \quad \boxed{\ddot{y} = 2b\left(\dot{x}^2 + x\ddot{x}\right)} \qquad (2)$$

Equate (1) and (2):

$$\ddot{y} = g\left(\cot\theta - 1\right) - 2\cot\theta\,\ddot{x} = 2b\left(\dot{x}^2 + x\ddot{x}\right) \quad \Rightarrow$$

$$\boxed{\ddot{x} = \frac{g\left(\cot\theta - 1\right) - 2b\dot{x}^2}{2\left(bx + \cot\theta\right)}} \qquad (3)$$

with:

$$x = \sqrt{\frac{y_{\max}/2}{b}} = \sqrt{\frac{1/8b}{b}} = \frac{1}{b\sqrt{8}}$$

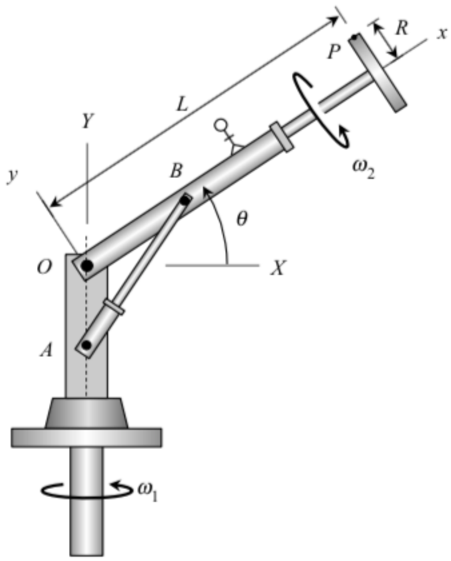
$$\left.\frac{dy}{dx}\right|_{y_{\max/2}} = 2bx = 2b\left(\frac{1}{b\sqrt{8}}\right) = \frac{1}{\sqrt{2}} = \tan\theta \quad \text{AND} \quad \cos\theta = \frac{\sqrt{2}}{\sqrt{1+\left(\sqrt{2}\right)^2}} = \sqrt{\frac{2}{3}}$$

$$v = \sqrt{g}\sqrt{\sqrt{\frac{1}{4b^2}} - 2\left(\frac{1}{8b}\right)} = \frac{1}{\sqrt{2}}\sqrt{\frac{g}{b}}$$

$$\dot{x} = v\cos\theta = \left(\frac{1}{\sqrt{2}}\sqrt{\frac{g}{b}}\right)\sqrt{\frac{2}{3}} = \frac{1}{\sqrt{3}}\sqrt{\frac{g}{b}}$$

Substitute these numbers into equations (1) and (2).

Problem 3



Shaft OA is rotating with a constant rate of ω_1 about a fixed, vertical axis. Sleeve OB is being raised at a constant rate of $\dot{\theta}$ by the telescoping arm AB. A shaft rotates within the sleeve with a constant rate of ω_2, with a disk of radius R being attached to the end of the shaft. The shaft is being extended at a constant rate of $\dot{L}$. A set of *xyz*-coordinate axes are attached to sleeve OB, as shown. At the position shown, point P on the perimeter of the disk is in the y-direction relative to the shaft.

For the position shown:

a) Determine the angular velocity and angular acceleration of the sleeve OB.
b) Determine the velocity and acceleration of P on the disk.

Write your answers as vectors in terms of their xyz-components.

HINT: Consider using an observer attached to OB, as shown.

SOLUTION

$\vec{\omega} = \omega_1 \hat{J} + \dot{\theta}\hat{k}$ = angular velocity of the sleeve (observer)

$\vec{\alpha} = \dot{\theta}\dot{\hat{k}} = \dot{\theta}\left(\vec{\omega} \times \hat{k}\right) = \dot{\theta}\left(\omega_1 \hat{J} + \dot{\theta}\hat{k}\right) \times \hat{k} = \dot{\theta}\omega_1\left(\hat{J} \times \hat{k}\right)$ = angular acceleration of sleeve

where: $\hat{J} = sin\theta\hat{i} + cos\theta\hat{j}$. Therefore:

$\vec{\omega} = \omega_1 sin\theta\hat{i} + \omega_1 cos\theta\hat{j} + \dot{\theta}\hat{k}$

$\vec{\alpha} = \dot{\theta}\omega_1\left(sin\theta\hat{i} + cos\theta\hat{j}\right) \times \hat{k} = \dot{\theta}\omega_1 cos\theta\hat{i} - \dot{\theta}\omega_1 sin\theta\hat{j}$

From the view of the observer we have:

$$\left(\vec{v}_P\right)_{rel} = \dot{L}\hat{i} + R\omega_2\hat{k}$$

$$\left(\vec{a}_P\right)_{rel} = -R\omega_2^2\hat{j}$$

Using the above gives:

$$\vec{v}_P = \vec{v}_O + \left(\vec{v}_P\right)_{rel} + \vec{\omega}\times\vec{r}_{P/O}$$

$$= \vec{0} + \dot{L}\hat{i} + R\omega_2\hat{k} + \left(\omega_1 sin\theta\hat{i} + \omega_1 cos\theta\,\hat{j} + \dot{\theta}\hat{k}\right)\times\left(L\hat{i} + R\hat{j}\right)$$

$$\vec{a}_P = \vec{a}_O + \left(\vec{a}_{P/O}\right)_{rel} + \vec{\alpha}\times\vec{r}_{P/O} + 2\vec{\omega}\times\left(\vec{v}_{P/O}\right)_{rel} + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}_{P/O}\right)$$

$$= \vec{0} - R\omega_2^2\hat{j} + \left(-\dot{\theta}\omega_1 cos\theta\hat{i} + \dot{\theta}\omega_1 sin\theta\,\hat{j}\right)\times\left(L\hat{i} + R\hat{j}\right)$$

$$+ 2\left(\omega_1 sin\theta\hat{i} + \omega_1 cos\theta\,\hat{j} + \dot{\theta}\hat{k}\right)\times\left(R\omega_2\hat{k}\right)$$

$$+ \left(\omega_1 sin\theta\hat{i} + \omega_1 cos\theta\,\hat{j} + \dot{\theta}\hat{k}\right)\times\left[\left(\omega_1 sin\theta\hat{i} + \omega_1 cos\theta\,\hat{j} + \dot{\theta}\hat{k}\right)\times\left(L\hat{i} + R\hat{j}\right)\right]$$