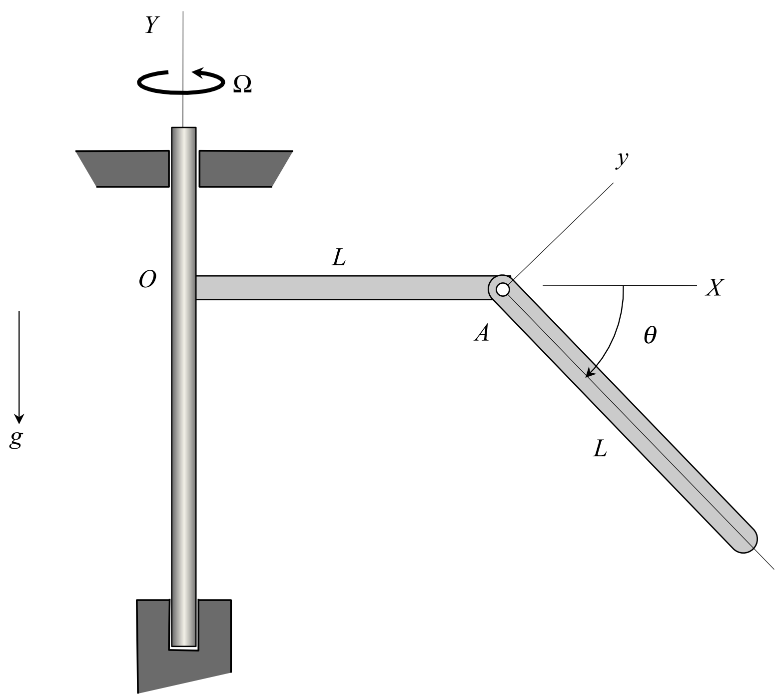
Problem 7.1



A thin homogeneous bar, AB, of mass m, is pinned to arm OA. Arm OA is welded to the vertical shaft, as shown. The vertical shaft is constrained to be rotating with a constant speed of Ω. Derive the differential equation of motion of arm AB in terms of the angle θ.

SOLUTION

Kinematics

$$\vec{\omega} = \Omega\hat{J} - \dot{\theta}\hat{k} = \text{angular velocity of bar AB}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = -\ddot{\theta}\hat{k} - \dot{\theta}(\vec{\omega}\times\hat{k}) = \text{angular acceleration of bar AB}$$

From figure: $\hat{J} = -sin\theta\hat{i} + cos\theta\hat{j}$. Therefore,

$$\vec{\omega} = -\Omega sin\theta\hat{i} + \Omega cos\theta\hat{j} - \dot{\theta}\hat{k}$$

$$\vec{\alpha} = -\ddot{\theta}\hat{k} - \dot{\theta}(-\Omega sin\theta\hat{i} + \Omega cos\theta\hat{j} - \dot{\theta}\hat{k})\times\hat{k}$$

$$= -\dot{\theta}\Omega cos\theta\hat{i} - \dot{\theta}\Omega sin\theta\hat{j} - \ddot{\theta}\hat{k}$$

$$\vec{a}_G = \vec{a}_A + \vec{\alpha}\times\vec{r}_{G/A} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{G/A})$$

$$= -L\Omega^2 I + (-\dot{\theta}\Omega cos\theta\hat{i} - \dot{\theta}\Omega sin\theta\hat{j} - \ddot{\theta}\hat{k})\times\left(\frac{L}{2}i\right)$$

$$+(-\Omega sin\theta\hat{i} + \Omega cos\theta\hat{j} - \dot{\theta}\hat{k})\times\left[(-\Omega sin\theta\hat{i} + \Omega cos\theta\hat{j} - \dot{\theta}\hat{k})\times\left(\frac{L}{2}i\right)\right]$$

where from the figure: $\hat{I} = cos\theta\hat{i} + sin\theta\hat{j}$.

Newton-Euler

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \vec{M}_G = \frac{d\vec{H}_G}{dt}$$

where:

$$\sum \vec{F} = F_{Ax}\hat{i} + F_{Ay}\hat{j} + F_{Az}\hat{k} - mg\hat{J}$$
$$= (F_{Ax} - mgsin\theta)\hat{i} + (F_{Ay} + mgcos\theta)\hat{j} + F_{Az}\hat{k}$$

$$\sum \vec{M}_G = \vec{r}_{A/G} \times \vec{F}_A + M_{Ax}\hat{i} + M_{Ay}\hat{j}$$
$$= \left(-\frac{L}{2}\hat{i}\right) \times (F_{Ax}\hat{i} + F_{Ay}\hat{j} + F_{Az}\hat{k}) + M_{Ax}\hat{i} + M_{Ay}\hat{j}$$
$$= M_{Ax}\hat{i} + \left(M_{Ay} + F_{Az}\frac{L}{2}\right)\hat{j} - F_{Ay}\frac{L}{2}\hat{k}$$

$$\vec{H}_G = I_{G,yy}\omega_y\hat{j} + I_{G,zz}\omega_z\hat{k}$$
$$= I_{G,yy}\Omega cos\theta\hat{j} - I_{G,zz}\dot{\theta}\hat{k}$$

$$\frac{d\vec{H}_G}{dt} = I_{G,yy}\alpha_y\hat{j} + I_{G,zz}\alpha_z\hat{k} + \hat{\omega} \times \vec{H}_G$$
$$= I_{G,yy}(-\dot{\theta}\Omega sin\theta)\hat{j} + I_{G,zz}(-\ddot{\theta})\hat{k}$$
$$+ (-\Omega sin\theta\hat{i} + \Omega cos\theta\hat{j} - \dot{\theta}\hat{k}) \times (I_{G,yy}\Omega cos\theta\hat{j} - I_{G,zz}\dot{\theta}\hat{k})$$

$$I_{G,yy} = I_{G,zz} = \frac{1}{12}mL^2 \tag{1}$$

Writing down the z-component of the Euler equation:

$$\left(\sum \vec{M}_G\right)_z = \left(\frac{d\vec{H}_G}{dt}\right)_z \quad \Rightarrow \quad -F_{Ay}\frac{L}{2} = -I_{G,zz}\ddot{\theta} - I_{G,yy}\Omega^2 cos^2\theta \tag{2}$$

Writing down the y-component of the Euler equation:

$$\left(\sum \vec{F}\right)_y = ma_{Gy} \quad \Rightarrow$$

$$F_{Ay} + mgcos\theta = -mL\Omega^2 sin\theta - m\frac{L}{2}\ddot{\theta} - m\frac{L}{2}\Omega^2 sin\theta cos\theta \quad \Rightarrow$$

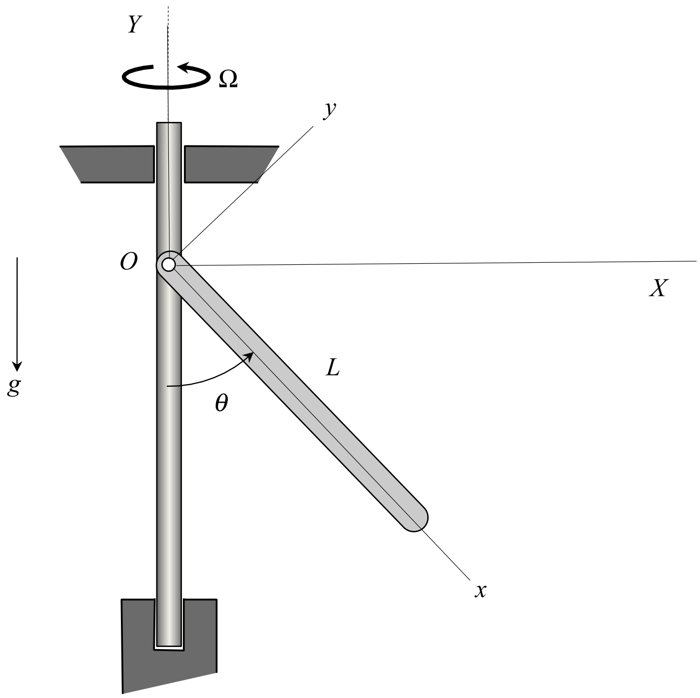
$$F_{Ay} = -m\left[gcos\theta + L\Omega^2 sin\theta + \frac{L}{2}\ddot{\theta} + \frac{L}{2}\Omega^2 sin\theta cos\theta\right] \tag{3}$$

Combining equations (1)-(3):

$$\left[gcos\theta + L\Omega^2 sin\theta + \frac{L}{2}\ddot{\theta} + \frac{L}{2}\Omega^2 sin\theta cos\theta\right]\frac{L}{2} = -\frac{1}{12}mL^2\ddot{\theta} - \frac{1}{12}mL^2\Omega^2 cos^2\theta \quad \Rightarrow$$

$$4\ddot{\theta} + 6\frac{g}{L}cos\theta + \Omega^2(cos^2\theta + 6sin\theta + 3sin\theta cos\theta) = 0$$

Problem 7.2



A thin, homogeneous bar of mass m is pinned to a vertical shaft at end O. The vertical shaft is allowed to rotate freely. The bar is released with initial conditions of: $\theta(0)=10^\circ$, $\dot{\theta}(0)=4\,rad/sec$ and $\Omega(0)=10\,rad/sec$. Determine the total angular velocity of the bar when $\theta=90^\circ$.

SOLUTION

<u>*Kinematics*</u>

$$\vec{\omega}=\Omega\hat{J}+\dot{\theta}\hat{k}=\text{angular velocity of bar}$$

From figure: $\hat{J}=-cos\theta\hat{i}+sin\theta\hat{j}$. Therefore,

$$\vec{\omega}=-\Omega cos\theta\hat{i}+\Omega sin\theta\hat{j}+\dot{\theta}\hat{k}$$

<u>*Conservation of energy*</u>

$$T=\frac{1}{2}I_{O,xx}\omega_x^2+\frac{1}{2}I_{O,yy}\omega_y^2+\frac{1}{2}I_{O,zz}\omega_z^2$$

$$=\frac{1}{2}I_{O,yy}\Omega^2 sin^2\theta+\frac{1}{2}I_{O,zz}\dot{\theta}^2$$

$$V=-mg\frac{L}{2}cos\theta$$

where:

$$I_{O,yy}=I_{O,zz}=\frac{1}{3}mL^2$$

Using:

$$T_2 + V_2 = T_1 + V_1$$

gives:

$$\frac{1}{2} I_{O,yy} \Omega^2 sin^2 90° + \frac{1}{2} I_{O,zz} \dot{\theta}^2 - mg\frac{L}{2} cos 90°$$

$$= \frac{1}{2} I_{O,yy} \Omega_0^2 sin^2\theta_0 + \frac{1}{2} I_{O,zz} \dot{\theta}_0^2 - mg\frac{L}{2} cos\theta_0$$

$$I_{O,yy}\Omega^2 + I_{O,zz}\dot{\theta}^2 = I_{O,yy}\Omega_0^2 sin^2\theta_0 + I_{O,zz}\dot{\theta}_0^2 - mgLcos\theta_0 \tag{1}$$

<u>*Conservation of angular momentum about Y-axis*</u>

$$\vec{H}_O = I_{O,yy}\omega_y \hat{j} + I_{O,zz}\omega_z \hat{k}$$

$$= I_{G,yy}\Omega sin\theta \hat{j} + I_{G,zz}\dot{\theta}\hat{k}$$

Since no moments act on the system about the Y-axis:

$$\left(\vec{H}_{O2}\right) \bullet \hat{J} = \left(\vec{H}_{O1}\right) \bullet \hat{J} \quad \Rightarrow$$

$$\left(I_{G,yy}\Omega sin 90° \hat{j} + I_{G,zz}\dot{\theta}\hat{k}\right) \bullet (-cos 90° \hat{i} + sin 90° \hat{j})$$

$$= \left(I_{G,yy}\Omega_0 sin\theta_0 \hat{j} + I_{G,zz}\dot{\theta}_0 \hat{k}\right) \bullet (-cos\theta_0 \hat{i} + sin\theta_0 \hat{j}) \quad \Rightarrow$$

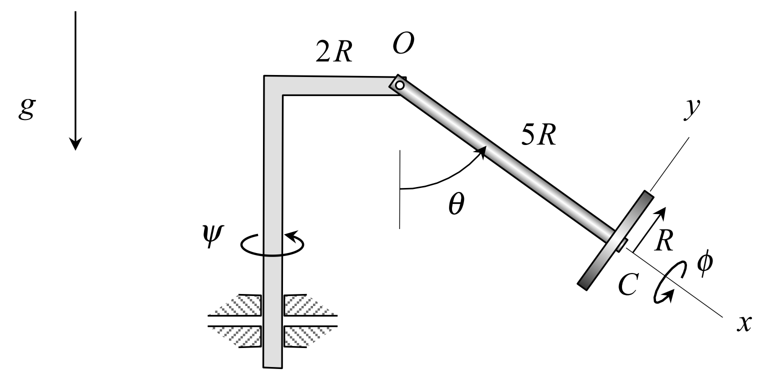
$$I_{G,yy}\Omega = I_{G,yy}\Omega_0 sin^2\theta_0 \;\; \Rightarrow \;\; \Omega = \Omega_0 sin^2\theta_0 \tag{2}$$

Combining (1) and (2):

$$\dot{\theta} = \sqrt{\dot{\theta}_0^2 + \frac{I_{O,yy}}{I_{O,zz}}\Omega_0^2 (sin^2\theta_0 - sin^4\theta_0) - \frac{mgLcos\theta_0}{I_{O,zz}}}$$

$$= \sqrt{\dot{\theta}_0^2 + \Omega_0^2 (sin^2\theta_0 - sin^4\theta_0) - 3cos\theta_0 \frac{g}{L}}$$

Problem 7.3



A thin disk of mass m and radius R is allowed to rotate about end C of shaft OC with *constant* angular speed of $\dot{\phi}$, with the orientation of shaft relative to the vertical being θ. Shaft OC has a mass of m. The rotation of the L-shaped shaft is rotating is represented by the angle ψ. Assume that OC is aligned with the symmetry axis of the disk. Using Lagrange's equations, develop the equations of motion for the system.

SOLUTION

For the ease of analysis, first define the coordinates associated with each rotation:

- Space fixed: X and Y parallel to x and y at the instant shown in the figure, respectively.
- Body fixed associated with ϕ rotation: xyz as shown, so this rotation is in $\hat{\imath}$.
- Body fixed associated with θ rotation: x_1 and y_1 parallel to x and y at the instant shown in the figure, respectively, so this rotation is in $\hat{k}_1$.
- Body fixed associated with ψ rotation: x_2 is along the vertical shaft and y_2 is along the 2R long horizontal bar, so this rotation is in $-\hat{\imath}_2$.

Therefore, the total angular velocity:

$\omega = -\dot{\psi}\hat{\imath}_2 + \dot{\theta}\hat{k}_1 + \dot{\phi}\hat{\imath}$

Recast each term so that they are in the same frame:

$$-\dot{\psi}\hat{\imath}_2 = R_x(\phi)R_z(\theta)R_x(-\psi)\begin{bmatrix}-\dot{\psi}\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\0 & \cos\phi & \sin\phi\\0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\-\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & \cos\psi & -\sin\psi\\0 & \sin\psi & \cos\psi\end{bmatrix}\begin{bmatrix}-\dot{\psi}\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}-\dot{\psi}\cos\theta\\\dot{\psi}\sin\theta\cos\phi\\-\dot{\psi}\sin\theta\sin\phi\end{bmatrix}$$

$$\dot{\theta}\hat{k}_2 = R_x(\phi)R_z(\theta)\begin{bmatrix}0\\0\\\dot{\theta}\end{bmatrix} = \begin{bmatrix}0\\\dot{\theta}\sin\phi\\\dot{\theta}\cos\phi\end{bmatrix}$$

$$\dot{\phi}\hat{\imath} = \begin{bmatrix}\dot{\phi}\\0\\0\end{bmatrix}$$

$$\omega = \left(-\dot{\psi}\cos\theta + \dot{\phi}\right)\hat{I} + (\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi)\,\hat{J} + \left(-\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)\hat{K}$$

For easily calculating the kinetic energy, choose the intersection of the vertical shaft and the horizontal bar as the reference point to calculate the moment of inertia.

$I_{xx} = \frac{1}{2}mR^2$, only contribution from the disk

$I_{yy} = \text{Rod contribution} + \text{Disc contribution}$

$$= \left[\frac{1}{3}m(5R)^2 + m\left(\frac{2R}{\sin\theta}\right)^2\right] + \left[\frac{1}{4}mR^2 + m\left(5R + \frac{2R}{\sin\theta}\right)^2\right]$$

$$= (\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta})mR^2$$

The rod and the disc are symmetrical around x in geometry, so $I_{zz} = I_{yy}$ and, the products of inertia are all zero.

Note that the reference point we choose is a fixed point, so

$$T = T_{rot} = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2$$

$$= \frac{1}{2}mR^2\left[\left(\frac{1}{2}\dot{\psi}^2\cos^2\theta - \dot{\psi}\cos\theta\,\dot{\phi} + \frac{1}{2}\dot{\phi}^2\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left[\left(\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi\right)^2 + \left(-\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)^2\right]\right]$$

$$= \frac{1}{2}mR^2\left[\left(\frac{1}{2}\dot{\psi}^2\cos^2\theta - \dot{\psi}\cos\theta\,\dot{\phi} + \frac{1}{2}\dot{\phi}^2\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)\right]$$

$$V = mg(-5R - 2.5R)\cos\theta = -\frac{15}{2}mg\cos\theta$$

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = \frac{1}{2}mR^2\left[\left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)(2\ddot{\theta}) - \left(\frac{8\cos\theta}{\sin^2\theta} + \frac{20\cos\theta}{\sin^3\theta}\right)\dot{\theta}^2\right]$$

$$\frac{\partial T}{\partial\theta} = \frac{1}{2}mR^2\left[\left(-\dot{\psi}^2\sin\theta\cos\theta + \dot{\psi}\sin\theta\,\dot{\phi}\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left(2\dot{\psi}^2\sin\theta\cos\theta\right) - \left(\frac{8\cos\theta}{\sin^2\theta} + \frac{20\cos\theta}{\sin^3\theta}\right)\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)\right]$$

$$\frac{\partial V}{\partial \theta} = \frac{15}{2} mgR \sin\theta$$

Therefore, the EOM for θ is written as:

$$2\left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\ddot{\theta} - \left[\left(-\dot{\psi}^2 \sin\theta\cos\theta + \dot{\psi}\sin\theta\,\dot{\phi}\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left(2\dot{\psi}^2 \sin\theta\cos\theta\right) - \left(\frac{8\cos\theta}{\sin^2\theta} + \frac{20\cos\theta}{\sin^3\theta}\right)\dot{\psi}^2 \sin^2\theta\right] + 15\frac{g}{R}\sin\theta = 0$$

Lagrangian equation for ψ

$$\left(\frac{\partial T}{\partial \dot{\psi}}\right) = \frac{1}{2} mR^2 \left[\left(\dot{\psi}\cos^2\theta - \cos\theta\,\dot{\phi}\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left(2\dot{\psi}\sin^2\theta\right)\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = \frac{1}{2} mR^2 \left[\left(\ddot{\psi}\cos^2\theta - 2\dot{\psi}\cos\theta\sin\theta\,\dot{\theta} + \sin\theta\,\dot{\theta}\dot{\phi}\right) + \left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right)\left(2\ddot{\psi}\sin^2\theta + 4\dot{\psi}\sin\theta\cos\theta\,\dot{\theta}\right) - \left(\frac{8\cos\theta}{\sin^2\theta} + \frac{20\cos\theta}{\sin^3\theta}\right)\dot{\theta}\left(2\dot{\psi}\sin^2\theta\right)\right]$$

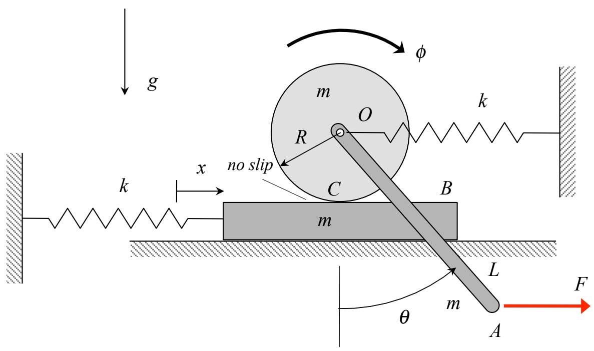
$$\frac{\partial T}{\partial \psi} = \frac{\partial V}{\partial \psi} = 0$$

Therefore, EOM becomes:

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = 0$, i.e.

$$\left[\cos^2\theta + 2\left(\frac{403}{12}\sin^2\theta + 20\sin\theta + 8\right)\right]\ddot{\psi} + \left\{\left[2\left(\frac{403}{12} + \frac{8}{\sin^2\theta} + \frac{20}{\sin\theta}\right) - 1\right]\sin 2\theta - (16\cos\theta + 40\cot\theta)\right\}\dot{\theta}\dot{\psi} + \sin\theta\,\dot{\theta}\dot{\phi} = 0$$

Problem 7.4 – 10 points



Consider the system shown above made up of a homogeneous disk, a thin homogeneous bar OA and a block. The surface on which the block slides is smooth; however, the disk rolls without slipping on the block. The springs are unstretched when $x = \phi = 0$. Using Lagrange's equation, develop the equations of motion for the system.

SOLUTION

<u>*Kinematics*</u>

$$\vec{r}_A = \vec{r}_B + \vec{r}_{O/B} + \vec{r}_{A/O}$$

$$= x\hat{i} + \left(R\phi\hat{i} + R\hat{j}\right) + L\left(\sin\theta\hat{i} - \cos\theta\hat{j}\right) = \left(x + R\phi + L\sin\theta\right)\hat{i} + \left(R - L\cos\theta\right)\hat{j}$$

$$\vec{v}_G = \vec{v}_O + \vec{\omega}_{arm} \times \vec{r}_{G/O}$$

$$= \left(\dot{x} + R\dot{\phi}\right)\hat{i} + \left(\dot{\theta}\hat{k}\right) \times \frac{L}{2}\left(\sin\theta\hat{i} - \cos\theta\hat{j}\right)$$

$$= \left(\dot{x} + R\dot{\phi} + \frac{L}{2}\dot{\theta}\cos\theta\right)\hat{i} + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)\hat{j}$$

Therefore,

$$\delta\vec{r}_A = \left(\delta x + R\delta\phi + L\delta\theta\cos\theta\right)\hat{i} + \left(L\delta\theta\sin\theta\right)\hat{j}$$

<u>*Generalized forces*</u>

$$\delta U = \vec{F} \bullet \delta\vec{r}_A = \left(F\hat{i}\right) \bullet \left[\left(\delta x + R\delta\phi + L\delta\theta\cos\theta\right)\hat{i} + \left(L\delta\theta\sin\theta\right)\hat{j}\right]$$

$$= F\delta x + FR\delta\phi + FL\cos\theta\delta\theta = Q_x\delta x + Q_\phi\delta\phi + Q_\theta\delta\theta$$

<u>*Energies*</u>

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mv_O^2 + \frac{1}{2}I_O\omega_{disk}^2 + \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega_{disk}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\dot{x} + R\dot{\phi}\right)^2 + \frac{1}{2}I_O\dot{\phi}^2$$

$$+ \frac{1}{2}m\left[\left(\dot{x} + R\dot{\phi} + \frac{L}{2}\dot{\theta}\cos\theta\right)^2 + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)^2\right] + \frac{1}{2}I_G\dot{\theta}^2$$

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k\left(x + R\phi\right)^2 - mg\frac{L}{2}cos\theta$$

<u>Lagrange's equations</u>

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = Q_x$$

where:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \frac{d}{dt}\left[m\dot{x} + m\left(\dot{x} + R\dot{\phi}\right) + m\left(\dot{x} + R\dot{\phi} + \frac{L}{2}\dot{\theta}\cos\theta\right)\right]$$

$$= 2m\ddot{x} + 2mR\ddot{\phi} + m\frac{L}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2 sin\theta\right)$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial V}{\partial x} = 2kx + k\left(x + R\phi\right)$$

and:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = Q_\phi$$

where:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left[mR\left(\dot{x} + R\dot{\phi}\right) + I_O\dot{\phi}\right]$$

$$= mR\ddot{x} + \left(mR^2 + I_O\right)\ddot{\phi}$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial V}{\partial \phi} = kR\left(x + R\phi\right) = rKx + kR^2\phi$$

and:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

where:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= \frac{d}{dt}\left\{ m\left[\left(\dot{x} + R\dot{\phi} + \frac{L}{2}\dot{\theta}\cos\theta\right)\frac{L}{2}\cos\theta + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)\frac{L}{2}\dot{\theta}\sin\theta\right] + I_G\dot{\theta}\right\} \\
&= \frac{d}{dt}\left\{ m\left[\frac{L}{2}\dot{x}\cos\theta + \frac{L}{2}R\dot{\phi}\cos\theta + \frac{L^2}{4}\dot{\theta}\right] + I_G\ddot{\theta}\right\} \\
&= m\left[\frac{L}{2}\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}sin\theta\right) + \frac{LR}{2}\left(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}sin\theta\right)\right] + \left(m\frac{L^2}{4} + I_G\right)\ddot{\theta}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial T}{\partial \theta} &= m\left[\left(\dot{x} + R\dot{\phi} + \frac{L}{2}\dot{\theta}\cos\theta\right)\left(-\frac{L}{2}\dot{\theta}sin\theta\right) + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)\left(\frac{L}{2}\dot{\theta}cos\theta\right)\right] \\
&= m\left[\left(\dot{x} + R\dot{\phi}\right)\left(-\frac{L}{2}\dot{\theta}sin\theta\right)\right]
\end{aligned}$$

$$\frac{\partial V}{\partial \theta} = mg\frac{L}{2}sin\theta$$