Prob 6.1 (a) 3 DOFs

(b) Virtual Work

$$\delta U = (F\hat{I})\cdot(\delta x_1 \hat{I}) + (-F\hat{J})\cdot[\delta(x_1 + x_2\cos\theta)\hat{I} - \delta(x_2\sin\theta)\hat{J}]$$
$$+(F\hat{I})\cdot[\delta(x_1 + x_2\cos\theta + x_3\cos\theta)\hat{I} - \delta(x_2\sin\theta + x_3\sin\theta)\hat{J}]$$

$$= \underbrace{(2F)}_{Q_1}\delta x_1 + \underbrace{F(\sin\theta + \cos\theta)}_{Q_2}\delta x_2 + \underbrace{F\cos\theta}_{Q_3}\delta x_3$$

$$T = \frac{1}{2}(3m)v_A^2 + \frac{1}{2}m v_B^2 + \frac{1}{2}m v_C^2$$
$$= \frac{1}{2}3m\dot{x}_1^2 + \frac{1}{2}m[(\dot{x}_1 + \dot{x}_2\cos\theta)^2 + (\dot{x}_2\sin\theta)^2]$$
$$+\frac{1}{2}m[(\dot{x}_1 + \dot{x}_2\cos\theta + \dot{x}_3\cos\theta)^2 + (\dot{x}_2\sin\theta + \dot{x}_3\sin\theta)^2]$$
$$= \frac{5}{2}m\dot{x}_1^2 + m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + 2m\dot{x}_1\dot{x}_2\cos\theta + m\dot{x}_2\dot{x}_3$$
$$+ m\dot{x}_1\dot{x}_3\cos\theta$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}kx_3^2 + \frac{1}{2}k\overbrace{(x_2^2 + x_3^2 + 2x_2x_3)}^{(x_2+x_3)^2}$$
$$= kx_1^2 + kx_2^2 + kx_3^2 + kx_2x_3$$

$$R = \frac{1}{2}c\dot{x}_1^2$$

(c) Lagrangian Equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial \dot{x}_i} + \frac{\partial V}{\partial q_i} = Q_i$$

$\Rightarrow i=1$: $5m\ddot{x}_1 + 2m\ddot{x}_2\cos\theta + m\ddot{x}_3\cos\theta + c\dot{x}_1 + 2kx_1 = 2F$

$i=2$: $2m\ddot{x}_2 + 2m\ddot{x}_1\cos\theta + m\ddot{x}_3 + 2kx_2 + kx_3 = F(\sin\theta + \cos\theta)$

$i=3$: $m\ddot{x}_3 + m\ddot{x}_2 + m\ddot{x}_1\cos\theta + 2kx_3 + kx_2 = F\cos\theta.$

Problem 6.2

(a) 2 DOFs $\quad q_1 = x \quad q_2 = \theta$

(b) $T = \frac{1}{2} m v_0^2 + \frac{1}{2} m v_A^2$

$\vec{v}_A = \vec{v}_0 + \vec{\omega} \times r_{A/O}$

$= (\dot{x}\hat{I}) + (\dot{\theta}\hat{k}) \times (L\cos\theta\,\hat{I} + L\sin\theta\,\hat{J})$

$= (\dot{x} - L\dot{\theta}\sin\theta)\hat{I} + L\dot{\theta}\cos\theta\,\hat{J}$

$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\left[(\dot{x} - L\dot{\theta}\sin\theta)^2 + (L\dot{\theta}\cos\theta)^2\right]$

$= m\dot{x}^2 + \frac{1}{2} m L^2\dot{\theta}^2 - mL\sin\theta\,\dot{x}\dot{\theta}$

$V = \frac{1}{2} k x^2 + mgL\sin\theta$

$\delta U = \vec{F} \cdot \delta\vec{r}_A$

$\delta\vec{r}_A = \delta(\vec{r}_0 + \vec{r}_{A/O})$

$= \delta\left[(x + L\cos\theta)\hat{I} + L\sin\theta\,\hat{J}\right]$

$= \underbrace{(\delta x - L\sin\theta\,\delta\theta)\hat{I} + L\cos\theta\,\delta\theta\,\hat{J}}$

$\delta U = (-F\cos\theta\,\hat{I} - F\sin\theta\,\hat{J}) \cdot (\quad)$

$= \underbrace{-F\cos\theta}_{Q_1}\,\delta x + \underbrace{0}_{Q_2}\,\delta\theta$

(c) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$

$i=1 \quad 2m\ddot{x} - mL\cos\theta\,\dot{\theta}^2 - mL\sin\theta\,\ddot{\theta} + kx = -F\cos\theta$

$i=2 \quad mL\ddot{\theta} - m\sin\theta\,\ddot{x} + mg\cos\theta = 0.$

Problem 6.3

(a) 2 DOFs $\quad q_1 = X_A \quad q_2 = x_p$

(b) $T = \frac{1}{2}(2m)\, v_A^2 + \frac{1}{2} m v_p^2$

$\vec{v}_p = \vec{v}_A + (\vec{v}_{P/A})_{rel} \; \vec{\omega} \times \vec{r}_{P/A}$

$= (\dot{x}_A \hat{i}) + (\dot{x}_p \hat{e}) + (\Omega \hat{k}) \times (x_p \hat{e})$

$\hat{e} = \cos\theta \hat{I} + \sin\theta \hat{J}$

$= (\dot{X}_A + \dot{x}_p \cos\theta - x_p \Omega \sin\theta) I + (\dot{x}_p \sin\theta + x_p \Omega \cos\theta) \hat{J}$

$T = \frac{1}{2}(2m) \dot{X}_A^2 + \frac{1}{2} m \left[(\;)^2 + (\;)^2 \right]$

$= \frac{3}{2} m \dot{X}_A^2 + \frac{1}{2} m \dot{x}_p^2 + m \dot{X}_A \dot{x}_p \cos\theta - m \dot{X}_A x_p \Omega \sin\theta$

$+ \frac{1}{2} m x_p^2 \Omega^2$

$V = \frac{1}{2} k X_A^2 + \frac{1}{2} k x_p^2 \times 2$

$= \frac{1}{2} k X_A^2 + k x_p^2$

$Q_1 = Q_2 = 0$

(c) By inspection

$m_{11} = 3m \quad m_{22} = m \quad m_{12} = m_{21} = m\cos\theta$

(d) $T_1 = -m \dot{X}_A x_p \Omega \sin\theta$

$h_1 = -m x_p \Omega \sin\theta$

$h_2 = 0$

$H_{12} = \frac{\partial h_1}{\partial x_p} - \frac{\partial h_2}{\partial X_A} = -m\Omega \sin\theta$

$H_{21} = \frac{\partial h_2}{\partial X_A} - \frac{\partial h_1}{\partial x_p} = m \Omega \sin\theta$

$H_{11} = H_{22} = 0$

(e) $T_0 = \frac{1}{2} m x_p^2 \Omega^2$

$P = V - T_0 = \frac{1}{2} k X_A^2 + k x_p^2 - \frac{1}{2} m x_p^2 \Omega^2$

(f) $\sum_{k=1}^{2} m_{pk}\ddot{q}_k + \sum_{j=1}^{2} H_{pj}\dot{q}_j + \frac{\partial P}{\partial q_p} + \frac{\partial h_p}{\partial t}$

$+ \sum_{k=1}^{2} \frac{\partial m_{pk}}{\partial t}\dot{q}_k + \sum_{k=1}^{2}\sum_{j=1}^{2} \underbrace{\gamma_{pjk}}_{0}\,\dot{q}_j\dot{q}_k = \underbrace{Q_p^{(nc)}}_{0}$

$p=1 \quad m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + H_{11}\dot{q}_1 + H_{12}\dot{q}_2 + \underbrace{\frac{\partial P}{\partial q_1}}_{kx_A} + \underbrace{\frac{\partial h_1}{\partial t}}_{-m\dot{x}_p\Omega\sin\theta - m x_p\Omega^2\cos\theta} + \frac{\partial m_{11}}{\partial t}\dot{q}_1 + \underbrace{\frac{\partial m_{12}}{\partial t}}_{-m\Omega\sin\theta}\dot{q}_2 = 0$

$\Rightarrow \boxed{3m\ddot{x}_A + m\cos\theta\,\ddot{x}_p - 3m\Omega\sin\theta\,\dot{x}_p - m x_p\Omega^2\cos\theta + kx_A = 0}$

$p=2 \quad m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + H_{21}\dot{q}_1 + \underbrace{H_{22}}_{0}\dot{q}_2 + \underbrace{\frac{\partial P}{\partial q_2}}_{2kx_p - m\Omega^2 x_p} + \frac{\partial h_2}{\partial t}^{0} + \underbrace{\frac{\partial m_{21}}{\partial t}}_{-m\Omega\sin\theta}\dot{q}_1 + \underbrace{\cancel{\frac{\partial m_{22}}{\partial t}}}_{0}\dot{q}_2 = 0$

$\Rightarrow \boxed{m\cos\theta\,\ddot{x}_A + m\ddot{x}_p + (2k - m\Omega^2)x_p = 0}$

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Problem 6.4

(a) 1 independent DOF.

(b) $T = \frac{1}{2} m v_p^2$

$= \frac{1}{2} m \left(\dot{z} \hat{e}_k + \dot{R} \hat{e}_R + R\dot{\phi} \hat{e}_\phi \right)^2$

Derive the relation between ϕ & h.

$\vec{a}_p = \ddot{z} \hat{e}_k + (\ddot{R} - R\dot{\phi}^2) \hat{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi}) \hat{e}_\phi$

$\Sigma F_k = m\ddot{z} \Rightarrow N\sin\theta - mg = m\ddot{z}$ ①

$\Sigma F_R = m(\ddot{R} - R\dot{\phi}^2) \quad -N\cos\theta = \frac{m}{\cos\theta}(R\dot{\phi}^2 - \ddot{R})$ ②

Geometry: $\tan\theta = R/h \Rightarrow \ddot{R} = \ddot{h}\tan\theta$ ③

①, ②, ③ give: $\ddot{h} = h\dot{\phi}\sin^2\theta - g\cos^2\theta$

$\vec{H}_0 = m \vec{r}_{P/O} \times \vec{v}_p$

$\vec{H}_{01} = m R_0 \hat{e}_R \times [-v_0 \hat{e}_\phi] = -mH\tan\theta\, v_0 \hat{e}_k$

$\vec{H}_{02} = mR\hat{e}_R \times [\dot{h}\hat{e}_k + \dot{h}\tan\theta\, \hat{e}_R + h\tan\theta\, \dot{\phi} \hat{e}_\phi]$

$= -mR\dot{h} \hat{e}_\phi + mR\tan\theta\, \dot{\phi}\, \hat{e}_k$

$-mH\tan\theta = mRh\tan\theta \Rightarrow \dot{\phi} = -\frac{Hv_0}{h^2\tan\theta}$

$\Rightarrow v_p = \dot{h}\hat{e}_k + \dot{h}\tan\theta\, \hat{e}_R + h\tan\theta\, \dot{\phi}\, \hat{e}_\phi$

$T = \frac{1}{2} m \left[\dot{h}^2 + \dot{h}^2 \tan^2\theta + h^2 \tan^2\theta\, \dot{\phi}^2 \right]$

$= \frac{1}{2} \frac{m\dot{h}^2}{\cos^2\theta} + \frac{1}{2} m h^2 \tan^2\theta\, \dot{\phi}^2$

$V = mgh$

$Q = 0$

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(c) $\mathcal{L} = T - V = \frac{1}{2} m \dot{h}^2 / \cos^2\theta + \frac{1}{2} m h^2 \tan^2\theta \, \dot{\phi}^2 - mgh.$

$\frac{\partial \mathcal{L}}{\partial h} = m h \tan^2\theta \, \dot{\phi}^2 - mg$

$\frac{\partial \mathcal{L}}{\partial \phi} = 0$ ignorable.

$p = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m h^2 \tan^2\theta \, \dot{\phi} = \text{constant} \equiv \beta$

$h_0 = H \quad \dot{\phi}_0 = -\frac{v_0}{\rho_0} = -\frac{v_0}{H \tan\theta}$

$\boxed{p = -mH \tan\theta \, v_0}$

(d) $p = \beta \Rightarrow -mH \tan\theta \, v_0 = m h^2 \tan^2\theta \, \dot{\phi}$

$\dot{\phi} = -\frac{H v_0}{h^2 \tan\theta}$

$R = L - \beta\dot{\phi} = \frac{1}{2} m \dot{h}^2 / \cos^2\theta + \frac{1}{2} m h^2 \tan^2\theta \, \dot{\phi}^2 - mgh$

$- (-mH \tan\theta \, v_0) \dot{\phi}$

$\left(\dot{\phi}^2 = \frac{H^2 v_0^2}{h^4 \tan^2\theta}\right)$

$R = \frac{1}{2} m \dot{h}^2 / \cos^2\theta - \frac{1}{2} m \frac{H^2}{h^2} v_0^2 - mgh.$

(e) $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{h}}\right) - \frac{\partial R}{\partial h} = 0$

$\Rightarrow m \ddot{h} - m \frac{H^2}{h^3} v_0^2 \cos^2\theta + mg \cos^2\theta = 0$

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