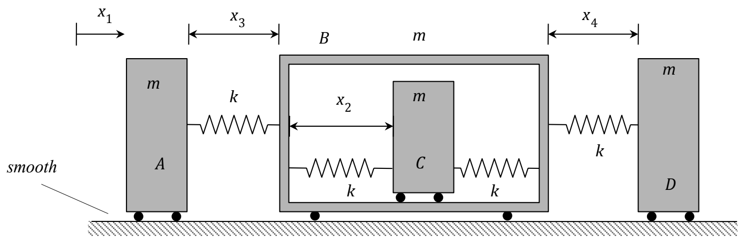
ME562 – Spring 2020
Purdue University
West Lafayette, IN

Final Exam – SOLUTION

Mean score: 65.3/80 (81.6%)
Median score: 66/80 (82.5%)

Problem No. 1 **SOLUTION**



A system is made up of particles A, B, C and D, all of mass *m*. The particles are interconnectd by springs, as shown, with each spring having a stiffness of *k*. Let x_1 represent the absolute motion of particle A, whereas x_2 is the motion of C relative to B, x_3 is the motion of B relative to A, and x_4 is the motion of D with respect to B. The springs are all unstretched when $x_1 = x_2 = x_3 = x_4 = 0$. At $t = 0$, the system is released with the springs unstretched, with $\dot{x}_2(0) = \dot{x}_3(0) = \dot{x}_4(0) = 0$ and $\dot{x}_1(0) = v_0$.

For this problem:

a) Identify the ignorable coordinate(s) in the system.
b) Use the Routhian formulation to derive the equations of motion in terms of the non-ignorable coordinates.

SOLUTION

Energy expressions

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1 + \dot{x}_3\right)^2 + \frac{1}{2}m\left(\dot{x}_1 + \dot{x}_2 + \dot{x}_3\right)^2 + \frac{1}{2}m\left(\dot{x}_1 + \dot{x}_3 + \dot{x}_4\right)^2$$

$$V = \frac{1}{2}kx_3^2 + 2\left(\frac{1}{2}kx_2^2\right) + \frac{1}{2}kx_4^2$$

Ignorable coordinates

The Lagrangian is given by: $L = T - V$. Since the Lagrangian is independent of the generalized coordinate x_1, that coordinate is ignorable. The corresponding generalized momentum for this coordinate is:

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 + m\left(\dot{x}_1 + \dot{x}_3\right) + m\left(\dot{x}_1 + \dot{x}_2 + \dot{x}_3\right) + m\left(\dot{x}_1 + \dot{x}_3 + \dot{x}_4\right)$$

$$= 4m\dot{x}_1 + m\dot{x}_2 + 3m\dot{x}_3 + m\dot{x}_4 = constant \triangleq \beta_1$$

where $\beta_1 = p_1(0) = 4mv_0$. From this, we have:

$$\dot{x}_1 = \frac{\beta_1}{4m} - \frac{1}{4}\left(\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4\right)$$

Routhian formulation

$$
\begin{aligned}
R &= L-\beta_1\dot{x}_1\\
&=\frac{1}{2}m\left[\frac{\beta_1}{4m}-\frac{1}{4}\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)\right]^2+\frac{1}{2}m\left[\frac{\beta_1}{4m}-\frac{1}{4}\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+\dot{x}_3\right]^2\\
&\quad+\frac{1}{2}m\left[\frac{\beta_1}{4m}-\frac{1}{4}\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+\dot{x}_2+\dot{x}_3\right]^2+\frac{1}{2}m\left[\frac{\beta_1}{4m}-\frac{1}{4}\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+\dot{x}_3+\dot{x}_4\right]^2\\
&\quad-\beta_1\left[\frac{\beta_1}{4m}-\frac{1}{4}\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+\dot{x}_3+\dot{x}_4\right]-\frac{1}{2}kx_3^2-kx_2^2-\frac{1}{2}kx_4^2\\
&=\frac{1}{2}\left(\frac{m}{16}\right)\left[4v_0-\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)\right]^2+\frac{1}{2}\left(\frac{m}{16}\right)\left[4v_0-\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+4\dot{x}_3\right]^2\\
&\quad+\frac{1}{2}\left(\frac{m}{16}\right)\left[4v_0-\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+4\dot{x}_2+4\dot{x}_3\right]^2+\frac{1}{2}\left(\frac{m}{16}\right)\left[4v_0-\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+4\dot{x}_3+4\dot{x}_4\right]^2\\
&\quad-\frac{v_0}{4}\left[4v_0-\left(\dot{x}_2+3\dot{x}_3+\dot{x}_4\right)+4\dot{x}_3+4\dot{x}_4\right]-\frac{1}{2}kx_3^2-kx_2^2-\frac{1}{2}kx_4^2\\
&=\frac{1}{2}\left(\frac{m}{16}\right)\left[\dot{x}_2+3\dot{x}_3+\dot{x}_4-4v_0\right]^2+\frac{1}{2}\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]^2\\
&\quad+\frac{1}{2}\left(\frac{m}{16}\right)\left[-3\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]^2+\frac{1}{2}\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3-3\dot{x}_4-4v_0\right]^2\\
&\quad-\frac{v_0}{4}\left[4v_0-\dot{x}_2+\dot{x}_3+3\dot{x}_4\right]-\frac{1}{2}kx_3^2-kx_2^2-\frac{1}{2}kx_4^2
\end{aligned}
$$

From this, we have:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}_2}\right)&=\frac{d}{dt}\left\{\left(\frac{m}{16}\right)\left[\dot{x}_2+3\dot{x}_3+\dot{x}_4-4v_0\right]+\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]\right\}\\
&\quad+\frac{d}{dt}\left\{-3\left(\frac{m}{16}\right)\left[-3\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]+\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3-3\dot{x}_4-4v_0\right]+\frac{v_0}{4}\right\}\\
&=\left(\frac{m}{16}\right)\left[12\ddot{x}_2+4\ddot{x}_3-4\ddot{x}_4\right]
\end{aligned}
$$

$$
\frac{\partial R}{\partial x_2}=-2kx_2
$$

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}_3}\right)&=\frac{d}{dt}\left\{3\left(\frac{m}{16}\right)\left[\dot{x}_2+3\dot{x}_3+\dot{x}_4-4v_0\right]-\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]\right\}\\
&\quad+\frac{d}{dt}\left\{-\left(\frac{m}{16}\right)\left[-3\dot{x}_2-\dot{x}_3+\dot{x}_4-4v_0\right]-\left(\frac{m}{16}\right)\left[\dot{x}_2-\dot{x}_3-3\dot{x}_4-4v_0\right]+\frac{v_0}{4}\right\}\\
&=\left(\frac{m}{16}\right)\left[4\ddot{x}_2+12\ddot{x}_3+4\ddot{x}_4\right]
\end{aligned}
$$

$$\frac{\partial R}{\partial x_3} = -kx_3$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}_4}\right) = \frac{d}{dt}\left\{\left(\frac{m}{16}\right)\left[\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4 - 4v_0\right] + \left(\frac{m}{16}\right)\left[\dot{x}_2 - \dot{x}_3 + \dot{x}_4 - 4v_0\right]\right\}$$

$$+\frac{d}{dt}\left\{\left(\frac{m}{16}\right)\left[-3\dot{x}_2 - \dot{x}_3 + \dot{x}_4 - 4v_0\right] - 3\left(\frac{m}{16}\right)\left[\dot{x}_2 - \dot{x}_3 - 3\dot{x}_4 - 4v_0\right] + \frac{v_0}{4}\right\}$$

$$= \left(\frac{m}{16}\right)\left[-4\ddot{x}_2 + 4\ddot{x}_3 + 12\ddot{x}_4\right]$$

$$\frac{\partial R}{\partial x_4} = -kx_4$$

The Routhian formulation $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}_j}\right) - \frac{\partial R}{\partial x_j} = 0 \; ; \quad j = 2,3,4$, gives the following three EOMs in terms of the non-ignorable coordinates:

$$\left(\frac{m}{16}\right)\left[12\ddot{x}_2 + 4\ddot{x}_3 - 4\ddot{x}_4\right] + 2kx_2 = 0$$
$$\left(\frac{m}{16}\right)\left[4\ddot{x}_2 + 12\ddot{x}_3 + 4\ddot{x}_4\right] + kx_3 = 0 \qquad (1)$$
$$\left(\frac{m}{16}\right)\left[-4\ddot{x}_2 + 4\ddot{x}_3 + 12\ddot{x}_4\right] + kx_4 = 0$$

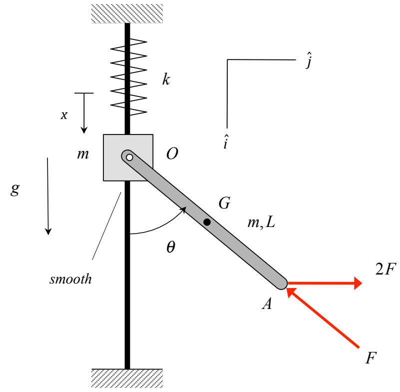
along with: $\dot{x}_1 = 4v_0 - \frac{1}{4}\left(\dot{x}_2 + 3\dot{x}_3 + \dot{x}_4\right)$ (2)

NOT REQUIRED: In matrix form, the EOMs for the non-ignorable coordinates are written as:

$$\left(\frac{m}{16}\right)\begin{bmatrix} 12 & 4 & -4 \\ 4 & 12 & 4 \\ -4 & 4 & 12 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The mass and stiffness matrices are symmetrical, as expected. Note that the Routhian formulation gives a non-singular stiffness matrix, whereas the standard Lagrangian formulation has a singular stiffness matrix (rigid body mode). The rigid motion is seen in the Routhian formulation through the integration of equation (2).

Problem 2



A block of mass m is able to slide along a smooth, vertical guide. A thin, homogeneous bar (of mass m and length L) is pinned to the block at end O. A pair of forces act on the free end of the bar. The force $2F$ acts in the horizontal direction, whereas the force F acts toward pin O for all motion. A spring of stiffness k connects end O of the bar to ground. Use Lagrange's equations to develop the differential equations of motion for the system in terms of the generalized coordinates of x and θ.

SOLUTION

Kinematics

$$\vec{r}_A = x\hat{i} + L\left(cos\theta\hat{i} + sin\theta\hat{j}\right) \Rightarrow \delta\vec{r}_A = \left(\delta x - Lsin\theta\delta\theta\right)\hat{i} + L\delta\theta cos\theta\hat{j}$$

$$\vec{v}_G = \vec{v}_O + \vec{\omega}\times\vec{v}_{G/O} = \dot{x}\hat{i} + \left(\dot{\theta}\hat{k}\right)\times\left(\frac{L}{2}cos\theta\hat{i} + \frac{L}{2}sin\theta\hat{j}\right) = \left(\dot{x} - \frac{L}{2}\dot{\theta}sin\theta\right)\hat{i} + \frac{L}{2}\dot{\theta}cos\theta\hat{j}$$

Energy and work expressions

$$T = \frac{1}{2}mv_O^2 + \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2 \quad ; \quad I_G = \frac{1}{12}mL^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\left(\dot{x} - \frac{L}{2}\dot{\theta}sin\theta\right)^2 + \left(\frac{L}{2}\dot{\theta}cos\theta\right)^2\right] + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 + \frac{L^2\dot{\theta}^2}{4}\left(cos^2\theta + sin^2\theta\right) - L\dot{x}\dot{\theta}sin\theta\right] + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}^2$$

$$= \frac{1}{2}m\left[2\dot{x}^2 + \frac{1}{3}L^2\dot{\theta}^2 - L\dot{x}\dot{\theta}sin\theta\right]$$

$$V = \frac{1}{2}kx^2 - mgx - mg\left(x + \frac{L}{2}cos\theta\right) = \frac{1}{2}kx^2 - mg\left(2x + \frac{L}{2}cos\theta\right)$$

$$\delta U = \left[2F\hat{j} + F\left(-cos\theta\hat{i} - sin\theta\hat{j}\right)\right]\bullet\delta\vec{r}_A$$

$$= \left[\left(-Fcos\theta\right)\hat{i} + \left(2F - Fsin\theta\right)\hat{j}\right]\bullet\left[\left(\delta x - Lsin\theta\delta\theta\right)\hat{i} + \left(L\delta\theta cos\theta\right)\hat{j}\right]$$

$$= \left(-Fcos\theta\right)\left(\delta x - Lsin\theta\delta\theta\right) + \left(2F - Fsin\theta\right)\left(L\delta\theta cos\theta\right)$$

$$= \left(-Fcos\theta\right)\delta x + \left(2FLcos\theta\right)\delta\theta = Q_x\delta x + Q_\theta\delta\theta$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)=\frac{d}{dt}\left[2m\dot{x}-\frac{1}{2}mL\dot{\theta}sin\theta\right]=2m\ddot{x}-\frac{1}{2}mL\ddot{\theta}sin\theta-\frac{1}{2}mL\dot{\theta}^2cos\theta$$

$$\frac{\partial T}{\partial x}=0$$

$$\frac{\partial V}{\partial x}=kx-2mg$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)=\frac{d}{dt}\left[\frac{1}{3}mL^2\dot{\theta}-\frac{1}{2}mL\dot{x}sin\theta\right]=\frac{1}{3}mL^2\ddot{\theta}-\frac{1}{2}mL\ddot{x}sin\theta-\frac{1}{2}mL\dot{x}\dot{\theta}cos\theta$$

$$\frac{\partial T}{\partial \theta}=-\frac{1}{2}mL\dot{x}\dot{\theta}cos\theta$$

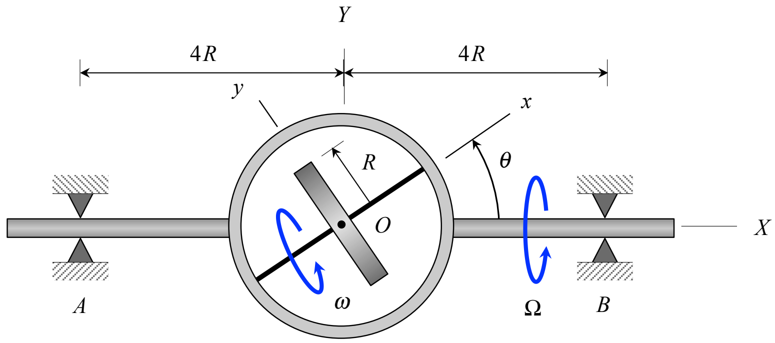
$$\frac{\partial V}{\partial \theta}=\frac{1}{2}mgLsin\theta$$

The above give the two EOMs for the system:

$$2m\ddot{x}-\frac{1}{2}mL\ddot{\theta}sin\theta-\frac{1}{2}mL\dot{\theta}^2cos\theta+kx-2mg=-Fcos\theta$$

$$\frac{1}{3}mL^2\ddot{\theta}-\frac{1}{2}mL\ddot{x}sin\theta+\frac{1}{2}mgLsin\theta=2FLcos\theta$$

Problem 3



A shaft is supported by bearings A and B, with the shaft rigidly supporting a ring. The shaft is rotating with a constant rate of Ω. A thin disk of radius R and mass m is attached to a second shaft that is rotating with a constant rate of ω on bearings in the ring, as shown. Ignore all mass in the system except for that of the disk. For the position shown, determine the reactions on the shaft due to the bearings at A and B.

SOLUTION

$$\vec{\omega} = \Omega\hat{I} + \omega\hat{i} = \Omega\left(cos\theta\hat{i} - sin\theta\hat{j}\right) + \omega\hat{i} = \left(\Omega cos\theta + \omega\right)\hat{i} + \left(-\Omega sin\theta\right)\hat{j} \text{ = ang. vel. of the disk}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \omega\frac{d\hat{i}}{dt} = \omega\left(\vec{\omega}\times\hat{i}\right) = \omega\left(\Omega\hat{I} + \omega\hat{i}\right)\times\hat{i} = \omega\Omega\left(\hat{I}\times\hat{i}\right)$$
$$= \omega\Omega\left(cos\theta\hat{i} - sin\theta\hat{j}\right)\times\hat{i} = \omega\Omega sin\theta\hat{k}$$

From this, we have:

$$\vec{H}_O = I_{O,xx}\omega_x\hat{i} + I_{O,yy}\omega_y\hat{j} + I_{O,zz}\omega_z\hat{k} = \left(\frac{1}{2}mR^2\right)\left(\Omega cos\theta + \omega\right)\hat{i} + \left(\frac{1}{4}mR^2\right)\left(-\Omega sin\theta\right)\hat{j}$$

For use in Euler's equations: $\sum\vec{M}_O = \frac{d\vec{H}_O}{dt}$, we have:

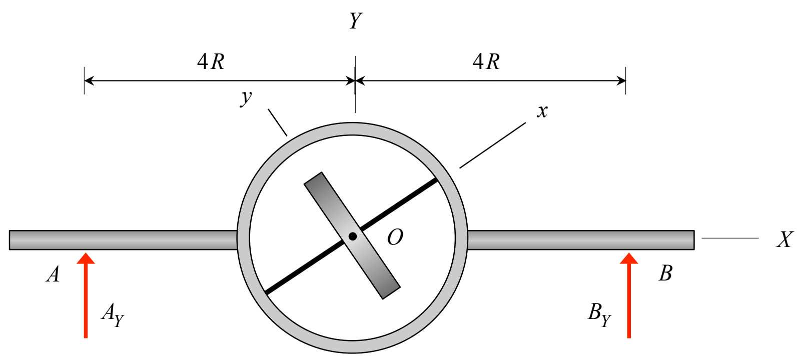
$$\frac{d\vec{H}_O}{dt} = I_{O,xx}\alpha_x\hat{i} + I_{O,yy}\alpha_y\hat{j} + I_{O,zz}\alpha_z\hat{k} + \vec{\omega}\times\vec{H}_O$$
$$= I_{O,zz}\alpha_z\hat{k} + \left(\omega_x\hat{i} + \omega_y\hat{j}\right)\times\left(I_{O,xx}\omega_x\hat{i} + I_{O,yy}\omega_y\hat{j}\right)$$
$$= \left[I_{O,zz}\alpha_z + \omega_x\omega_y\left(I_{O,yy} - I_{O,x}\right)\right]\hat{k}$$
$$= \left[\left(\frac{1}{4}mR^2\right)\omega\Omega sin\theta + \left(\Omega cos\theta + \omega\right)\left(-\Omega sin\theta\right)\left(\frac{1}{4}mR^2 - \frac{1}{2}mR^2\right)\right]\hat{k}$$
$$= \left(\frac{1}{4}mR^2\right)\left[\omega\Omega sin\theta - \left(\Omega cos\theta + \omega\right)\left(-\Omega sin\theta\right)\right]\hat{k}$$
$$= \left(\frac{1}{4}mR^2\right)\left[2\omega\Omega sin\theta + \Omega^2 sin\theta cos\theta\right]\hat{k}$$

From the FBD below, we have:

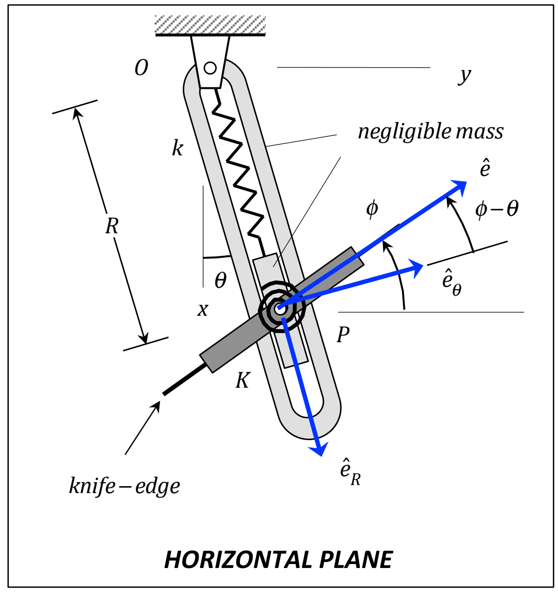
$$\sum F_Y = A_Y + B_Y = 0 \;\Rightarrow\; A_Y = -B_Y$$

$$\sum M_{Oz} = (B_Y - A_Y)(4R) = \left(\frac{d\vec{H}_O}{dt}\right)_z = \left(\frac{1}{4}mR^2\right)\left[2\omega\Omega sin\theta + \Omega^2 sin\theta cos\theta\right] \;\Rightarrow$$

$$\boxed{B_Y = \left(\frac{1}{32}mR\Omega\right)\left[2\omega + \Omega cos\theta\right]sin\theta = -A_Y}$$



Problem 4



A knife-edge is attached to rigid link having a mass m and a mass moment of inertia I about its centroid P. The link is pinned to a slider this is able to move within a smooth slot cut into an arm, with the arm pinned to ground at end O. Let R represent the radial distance between O and P, θ be the rotation angle of the arm and ϕ be the rotation angle of the knife-edge. A rotational spring of stiffness K is connected between the link and the slider. As the knife-edge moves on a horizontal plane, no side-slip between the knife-edge and the surface is allowed. A spring of stiffness k is connected between O and P, as shown. The rotational spring is unstretched when $\phi = 0$, and the spring between O and P is unstretched when $R = R_0$.

a) Determine the constraint that exists among R, θ and ϕ due to the no slide-slip condition imposed by the knife-edge.
b) Use Lagrange's equations with Lagrange multipliers to derive the EOMs for the generalized coordinates of R, θ and ϕ.

SOLUTION

<u>*Kinematics*</u>

$$\vec{v}_P = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

For unit vector running along the knife-edge:

$$\hat{e} = -sin(\phi-\theta)\hat{e}_R + cos(\phi-\theta)\hat{e}_\theta$$

For no side-slip at the knife-edge:

$$\begin{aligned}\vec{0} &= \hat{e}\times\vec{v}_P \\ &= \left[-sin(\phi-\theta)\hat{e}_R + cos(\phi-\theta)\hat{e}_\theta\right]\times\left[\dot{R}(\hat{e}_R) + R\dot{\theta}\hat{e}_\theta\right] \\ &= \left[-sin(\phi-\theta)(R\dot{\theta}) - cos(\phi-\theta)(\dot{R})\right]\hat{k}\end{aligned}$$

Therefore, the following constraint exists among the three generalized coordinates of the problem:

$$\boxed{0 = cos(\phi-\theta)\dot{R} + Rsin(\phi-\theta)\dot{\theta}} = a_{1R}\dot{R} + a_{1\theta}\dot{\theta} + a_{1\phi}\dot{\phi} \qquad (1)$$

<u>*Energy expressions*</u>

$$T = \frac{1}{2}mv_P^2 + \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}m\left[(\dot{R})^2 + (R\dot{\theta})^2\right] + \frac{1}{2}I\dot{\phi}^2$$

$$V = \frac{1}{2}k(R-R_0)^2 + \frac{1}{2}K(\phi-\theta)^2$$

Lagrange's equations

- For the R coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}}\right)=\frac{d}{dt}\left[m\dot{R}\right]=m\ddot{R}$$

$$\frac{\partial T}{\partial R}=mR\dot{\theta}^2$$

$$\frac{\partial V}{\partial x}=k\left(R-R_0\right)$$

Using $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}}\right)-\frac{\partial T}{\partial R}+\frac{\partial V}{\partial x}=a_{1R}\lambda$ gives:

$$\boxed{m\ddot{R}-mR\dot{\theta}^2+k\left(R-R_0\right)=cos\left(\phi-\theta\right)\lambda} \tag{2}$$

- For the θ coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)=\frac{d}{dt}\left[mR^2\dot{\theta}\right]=2mR\dot{R}\dot{\theta}+mR^2\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta}=0$$

$$\frac{\partial V}{\partial \theta}=-K\left(\phi-\theta\right)$$

Using $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right)-\frac{\partial T}{\partial \theta}+\frac{\partial V}{\partial \theta}=a_{1\theta}\lambda$ gives:

$$\boxed{mR^2\ddot{\theta}+2mR\dot{R}\dot{\theta}+K\theta-K\phi=Rsin\left(\phi-\theta\right)\lambda} \tag{3}$$

- For the ϕ coordinate:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right)=\frac{d}{dt}\left[I\dot{\phi}\right]=I\ddot{\phi}$$

$$\frac{\partial T}{\partial \phi}=0$$

$$\frac{\partial V}{\partial \phi}=K\left(\phi-\theta\right)$$

Using $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right)-\frac{\partial T}{\partial \phi}+\frac{\partial V}{\partial \phi}=a_{1\phi}\lambda$ gives:

$$\boxed{I\ddot{\phi}+K\phi-K\theta=0} \tag{4}$$

Solve the above four equations for the three generalized coordinates and the Lagrange multiplier.