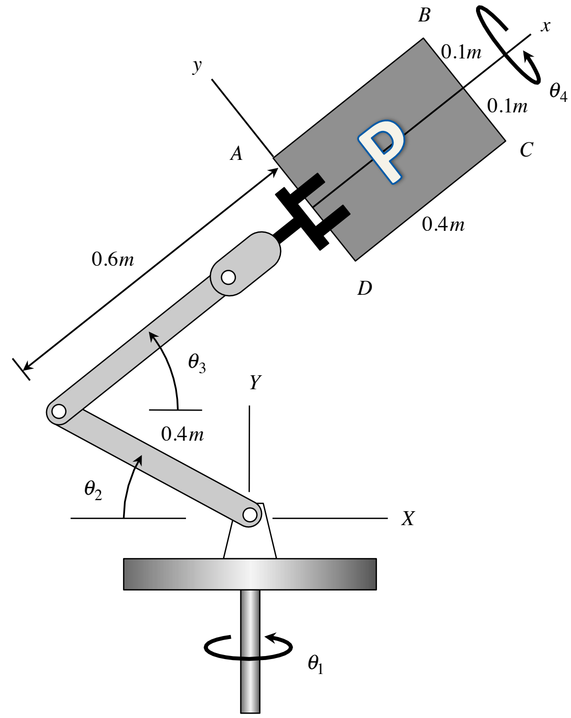
ME562 – Spring 2020
Purdue University
West Lafayette, IN

Homework Set No. 2 - SOLUTION

Problem 2.1 – 30 points



Consider the robotic arm shown above holding a rectangular sign ABCD in its end effector. Let the XYZ axes be fixed in space, and the xyz axes attached to the sign. When $\theta_1 = \theta_3 = \theta_4 = 0$, the xyz and XYZ axes are aligned. With $\theta_2 = 90°$ held fixed, consider the three rotations through which the robotic arms is taken:

$\theta_1 :\ 0 \rightarrow 180°$

$\theta_3 :\ 0 \rightarrow 90°$

$\theta_4 :\ 0 \rightarrow -90°$

for two different orders of rotation:

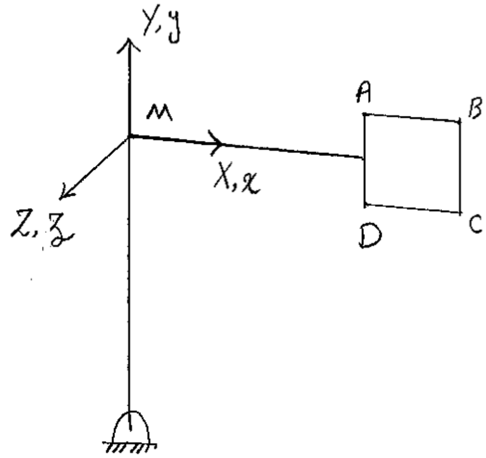
a) θ_1 followed by θ_3 followed by θ_4

b) θ_4 followed by θ_3 followed by θ_1

For each of the two rotation orders a) and b) above, do the following:

i. Determine the final space-fixed coordinates of point B on the sign.
ii. Make a sketch of the final orientation of the sign.
iii. Determine the Euler axis of rotation and the Euler angle of rotation for the sign.

2)



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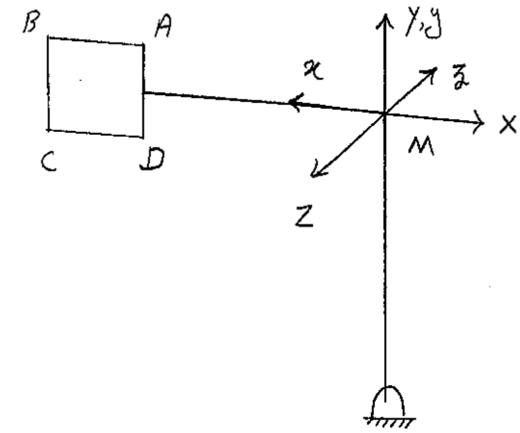

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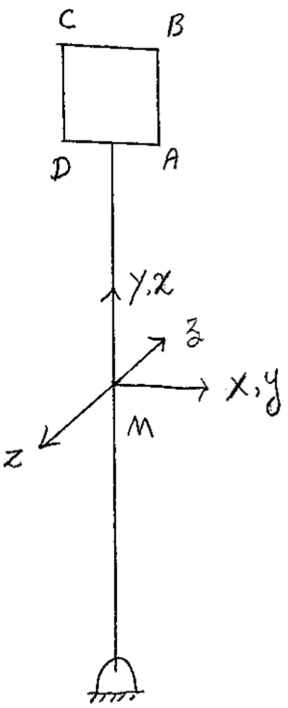
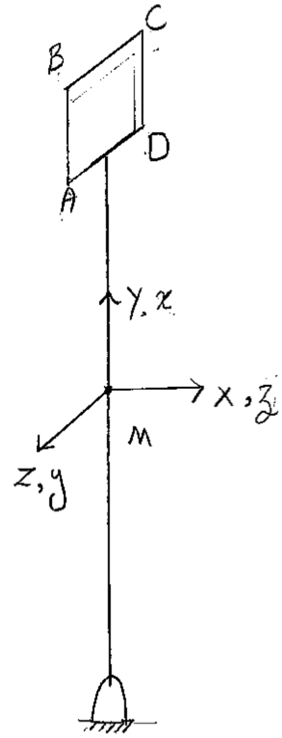




$$R_1 = R_y(180^\circ) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad R_2 = R_z(90^\circ) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 = R_x(-90^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

θ_1 followed by θ_3 followed by θ_4: $[T_{A \to B}] = R_3 \cdot R_2 \cdot R_1$

$$[T_{B\to A}] = [T_{A\to B}]' = R_1' \cdot R_2' \cdot R_3'$$

$$\Rightarrow \quad [T_{B\to A}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\{r_{B/M}\}^B = \begin{Bmatrix} 1 \\ 0.1 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \{r_{B/M}\}^A = [T_{B\to A}] \{r_{B/M}\}^B$$

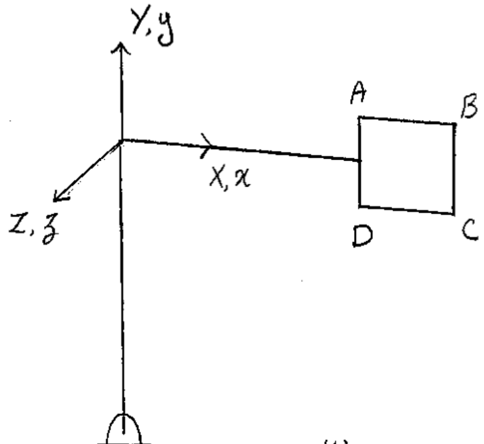
$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0.1 \end{Bmatrix}$$

$$\{r_B\}^A = \{r_{B/M}\}^A + \{r_M\}^A = \begin{Bmatrix} 0 \\ 1 \\ 0.1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0.4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.4 \\ 0.1 \end{Bmatrix}$$

$$\cos\phi = \frac{tr\,[T_{B\to A}] - 1}{2} = \frac{-1}{2} \Rightarrow \phi = 120^\circ$$

$$(-[I] + [T_{B\to A}])\{e\}^A = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad u_1 = u_2 = u_3 \quad \Rightarrow \quad \{e\}^A = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$



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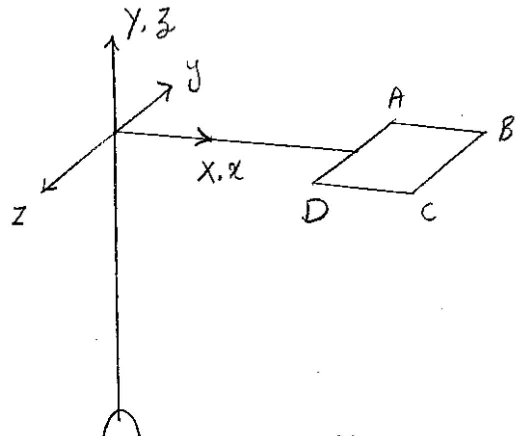

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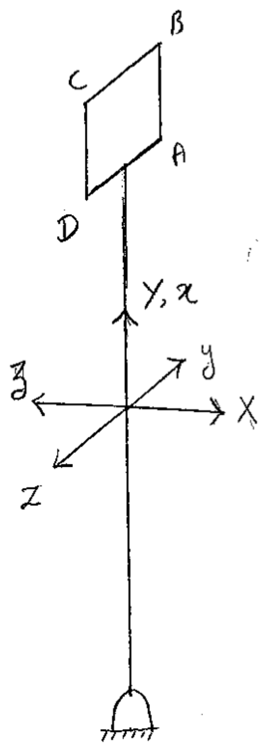

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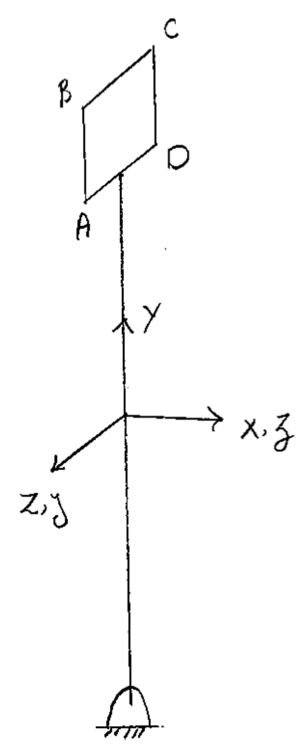


(4)

$$R_1 = R_x(-90^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad , \quad R_2 = R_y(-90^\circ) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$R_3 = R_x(180^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

θ_4 followed by θ_3 followed by θ_1 : $[T_{A \to B}] = R_3 \cdot R_2 \cdot R_1$

$$[T_{B\to A}] = [T_{A\to B}] = R_1 R_2 R_3$$

$$\Rightarrow \quad [T_{B\to A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\{r_{B/M}\}^B = \begin{Bmatrix} 1 \\ 0.1 \\ 0 \end{Bmatrix} \Rightarrow \{r_{B/M}\}^A = [T_{B\to A}]\{r_{B/M}\}^B$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0.1 \end{Bmatrix}$$

$$\{r_B\}^A = \{r_{B/M}\}^A + \{r_M\}^A = \begin{Bmatrix} 0 \\ 1 \\ 0.1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0.4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1.4 \\ 0.1 \end{Bmatrix}$$

$$\cos\phi = \frac{tr[T_{B\to A}] - 1}{2} = \frac{-1}{2} \Rightarrow \phi = 120^\circ$$

$$(-[I] + [T_{B\to A}])\{e\}^A = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow u_1 = u_2 = u_3 \quad \Rightarrow \quad \{e\}^A = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Problem 2.2 – 20 points

Reconstider the robotic arm described in Problem 2.1. Here the arm goes through the following rotations:

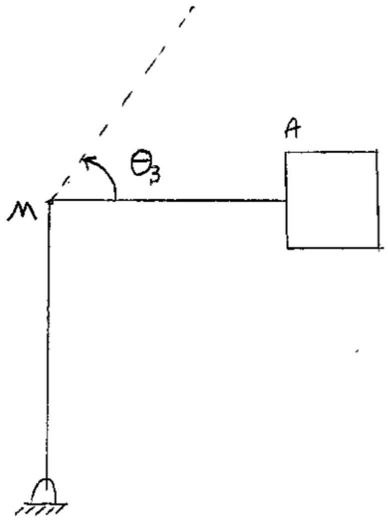
$$\theta_3 : \ 0 \rightarrow 50^\circ$$

$$\theta_4 : \ 0 \rightarrow 75^\circ$$

while $\theta_1 = 0$ and $\theta_2 = 90^\circ$ are held fixed. As a result of these rotation:

a) Determine the distance through which point A on the sign moves.
b) Determine the distance between the initial and final positions of point A on the sign.
c) Determine the direction angles for edge AB of the sign. Verify that these three direction angles are consistent with the usual constraint on direction angles in 3-D space.

1) $\theta_3 : 0 \to 50$



$$R_{\theta_3} = \sqrt{0.6^2 + 0.1^2} = 0.6083\ m$$

$$S_1 = R_{\theta_3} \cdot \theta_3 = (0.6083) \cdot (50 \tfrac{\pi}{180}) = 0.5308\, m$$

$\theta_4 : 0 \to 75$

$$R_{\theta_4} = 0.1$$

$$S_2 = R_{\theta_4} \cdot \theta_4 = (0.1) \cdot (75 \tfrac{\pi}{180}) = 0.1309\ m$$

Total distance $= S_1 + S_2 = 0.6619\ m$

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$$\theta_3 : 0 \to 50 \Rightarrow R_1 = R_z(50^\circ) = \begin{bmatrix} \cos 50^\circ & \sin 50^\circ & 0 \\ -\sin 50^\circ & \cos 50 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\theta_4 : 0 \to 75 \Rightarrow R_2 = R_x(75) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 75 & \sin 75 \\ 0 & -\sin 75 & \cos 75 \end{bmatrix}$$

$$[T_{A\to B}] = R_2 \cdot R_1 \Rightarrow [T_{B\to A}] = [T_{A\to B}]^T = R_1^T \cdot R_2^T$$

$$\Rightarrow [T_{B\to A}] = \begin{bmatrix} 0.643 & -0.766 & 0 \\ 0.766 & 0.643 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.259 & -0.966 \\ 0 & 0.966 & 0.259 \end{bmatrix} = \begin{bmatrix} 0.643 & -0.198 & 0.74 \\ 0.766 & 0.167 & -0.621 \\ 0 & 0.966 & 0.259 \end{bmatrix}$$

$$\Rightarrow \{r_{A/M}\}^A = [T_{B\to A}] \{r_{A/M}\}^B = [T_{B\to A}] \begin{Bmatrix} 0.6 \\ 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.366 \\ 0.476 \\ 0.097 \end{Bmatrix}$$

Distance between the initial and final position : $\Delta \vec{r}_A = \{r_{A/M}\}^A - \{r_{A/M}\}^B = \begin{Bmatrix} -0.234 \\ +0.376 \\ +0.097 \end{Bmatrix}$

$$|\Delta \gamma_A| = \sqrt{(-0.234)^2+(0.376)^2+(0.097)^2} = 0.453\ m$$

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$$\vec{\gamma}_{AB} = \vec{\gamma}_B - \vec{\gamma}_A = \vec{\gamma}_{B/M} - \vec{\gamma}_{A/M}$$

$$\left\{\vec{\gamma}_{B/M}\right\}^A = [T_{B\to A}]\left\{\vec{\gamma}_{B/M}\right\}^B = [T_{B\to A}]\begin{Bmatrix} 1 \\ 0.1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.623 \\ 0.783 \\ 0.097 \end{Bmatrix}$$

$$\Rightarrow \quad \vec{\gamma}_{AB} = \begin{Bmatrix} 0.623 \\ 0.783 \\ 0.097 \end{Bmatrix} - \begin{Bmatrix} 0.366 \\ 0.476 \\ 0.097 \end{Bmatrix} = \begin{Bmatrix} 0.257 \\ 0.307 \\ 0 \end{Bmatrix} \Rightarrow |\vec{\gamma}_{AB}| = 0.4$$

$$\alpha = \cos^{-1}\frac{\vec{\gamma}_{AB}\cdot I}{|\vec{\gamma}_{AB}|} = 50^\circ \quad , \quad B = \cos^{-1}\frac{\vec{\gamma}_{AB}\cdot \hat{J}}{|\vec{\gamma}_{AB}|} = 39.93$$

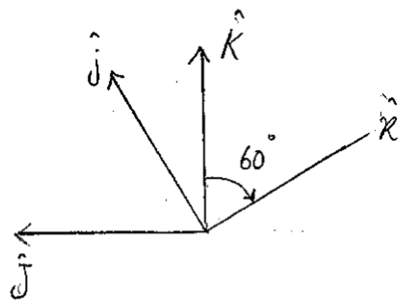
$$\gamma = \cos^{-1}\frac{\vec{\gamma}_{AB}\cdot \hat{k}}{|\vec{\gamma}_{AB}|} = 90^\circ$$

$$\cos\alpha^2 + \cos B^2 + \cos\gamma^2 = 1$$

Problem 2.3 – 10 points

Reconsider the robotic arm described in Problem 2.1 where here the lower joint is locked at $\theta_2 = 90°$. When $\theta_3 = 30°$ and $\theta_4 = 60°$ it is known that $\dot{\theta}_3 = 2\ rad/sec$ and $\dot{\theta}_4 = -3\ rad/sec$. At this position, determine the angular velocity vector of the sign and the velocity vector of point B on the sign. Express your answers in terms of both the space-fixed and body-fixed coordinates.

$\omega = 2\hat{K} - 3\hat{i}$



$\hat{K} = \cos 60° \hat{k} + \sin 60° \hat{j}$

$\Rightarrow \omega = 2[\frac{1}{2}\hat{k} + \frac{\sqrt{3}}{2}\hat{j}] - 3\hat{i} \Rightarrow \omega = -3\hat{i} + \sqrt{3}\hat{j} + \hat{k}$

$$\Rightarrow \{\omega\}^B = \begin{Bmatrix} -3 \\ \sqrt{3} \\ 1 \end{Bmatrix} \Rightarrow [\omega]^B = \begin{bmatrix} 0 & -1 & \sqrt{3} \\ 1 & 0 & 3 \\ -\sqrt{3} & -3 & 0 \end{bmatrix}$$

$$\left\{\frac{d\vec{r}_B}{dt}\right\}^B = [\omega]^B_\times \{r_B\}^B$$

$$= \begin{bmatrix} 0 & -1 & \sqrt{3} \\ 1 & 0 & 3 \\ -\sqrt{3} & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 1 \\ -2.032 \end{bmatrix}$$

$\{\omega\}^A = [T_{B\to A}]\{\omega\}^B$ where $[T_{B\to A}] = [T_{A\to B}]^T = [R_x(60°) \cdot R_z(30°)]^T$

$$\Rightarrow \{\omega\}^A = \begin{bmatrix} 0.866 & -0.25 & +0.433 \\ 0.5 & 0.433 & -0.75 \\ 0 & 0.866 & 0.5 \end{bmatrix} \begin{Bmatrix} -3 \\ \sqrt{3} \\ 1 \end{Bmatrix} = \begin{Bmatrix} -2.598 \\ -1.5 \\ 2 \end{Bmatrix}$$

$$\left\{\frac{d\vec{r}_B}{dt}\right\}^A = [T_{B\to A}]\left\{\frac{d\vec{r}_B}{dt}\right\}^B = \begin{bmatrix} 0.866 & -0.25 & 0.433 \\ 0.5 & 0.433 & -0.75 \\ 0 & 0.866 & 0.5 \end{bmatrix} \begin{Bmatrix} -0.1 \\ 1 \\ -2.032 \end{Bmatrix} = \begin{Bmatrix} -1.2165 \\ 1.907 \\ -0.15 \end{Bmatrix}$$