15 min.

Problem 7.1 (10 points)

Three different shafts made up of the same material with shear modulus G are connected through connectors B and C. Shaft (1) is a hollow shaft with outer and inner diameters of *3d* and *d* while shafts (2) and (3) are rigid shafts with diameters of *2d* and *d* respectively. This system of shafts is subjected to external torques T_0 (applied on connector B) and T (applied on connector C). The configuration of the system is shown in Figure 1. Assume shafts and connectors are weightless.

- **a)** Write down the strain-energy equation of the system.
- **b)** Calculate the static rotations of connectors B and C using *Castigliano's theorem*.

Write your answer in terms of, at most, T_0, T, L, d, and G.

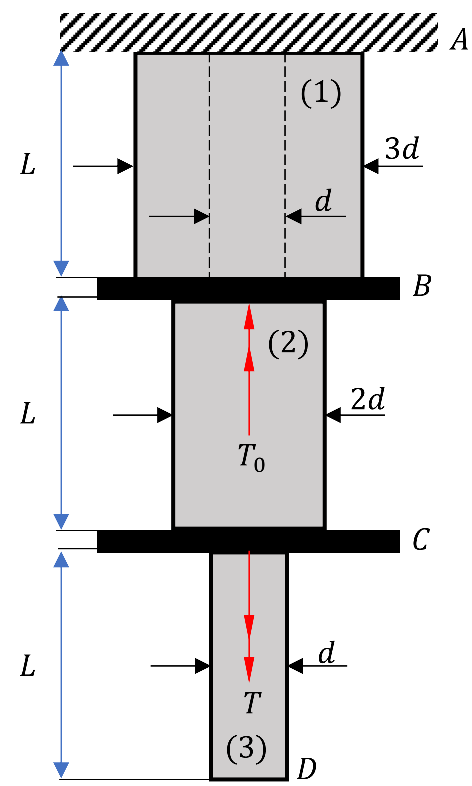
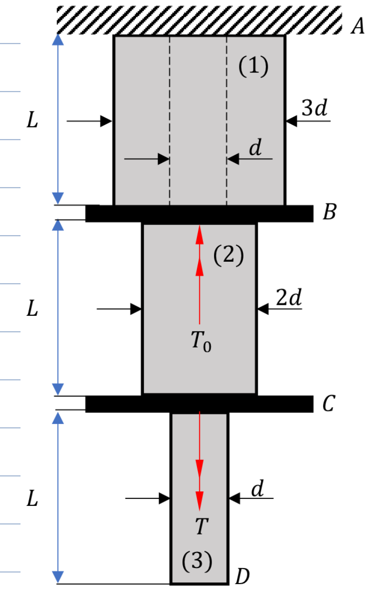


Figure 1

Given: T_0, T, L, d, G

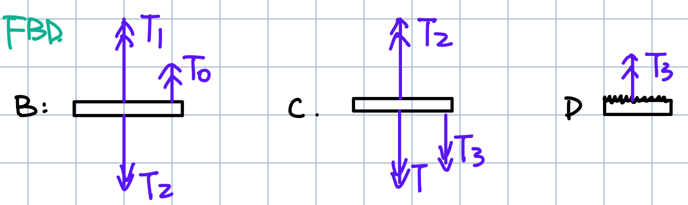
Find: U, ϕ_B, ϕ_C



Solution:

1. coor. $\downarrow x^+$

2. FBD



3. Equilibrium Equation

$T_2 - T_0 - T_1 = 0$ ①

$T + T_3 - T_2 = 0$ ②

$T_3 = 0$ ③

unknown: T_1 T_2 T_3.

3 unknown. 3 Eq.

4. Energy Method.

Solve for T_1 T_2 T_3

$T_3 = 0$.

$T_2 = T$

$T_1 = T_2 - T_0 = T - T_0$

$$J_1 = \frac{\pi}{2}\left[\left(\frac{3d}{2}\right)^4 - \left(\frac{d}{2}\right)^4\right] = \frac{80\pi d^4}{32} = \frac{5}{2}\pi d^4$$

$$J_2 = \frac{\pi}{2} d^4$$

$$J_3 = \frac{\pi}{2}\cdot\left(\frac{d}{2}\right)^4 = \frac{\pi}{32} d^4$$

Strain Energy function.

$$U = \frac{1}{2}\frac{T_1^2 L}{J_1 G} + \frac{1}{2}\frac{T_2^2 L}{J_2 G} + \frac{1}{2}\frac{T_3^2 L}{J_3 G}$$

$$= \frac{L}{\cancel{2}G\pi d^4}\left[\frac{\cancel{2}}{5}T_1^2 + \cancel{2}T_2^2 + \overset{16}{\cancel{32}}T_3^2\right]$$

$$= \frac{L}{G\pi d^4}\left[\frac{1}{5}T^2 + \frac{1}{5}T_0^2 - \frac{2}{5}TT_0 + T^2\right]$$

$$= \frac{L}{G\pi d^4}\left[\frac{6}{5}T^2 - \frac{2}{5}TT_0 + \frac{1}{5}T_0^2\right]$$

ϕ_B (same direction as T_0)

$$\phi_B = \frac{\partial U}{\partial T_0} = \frac{L}{G\pi d^4}\left[-\frac{2}{5}T + \frac{2}{5}T_0\right] = \boxed{\frac{2L(T_0 - T)}{5G\pi d^4} \Uparrow}$$

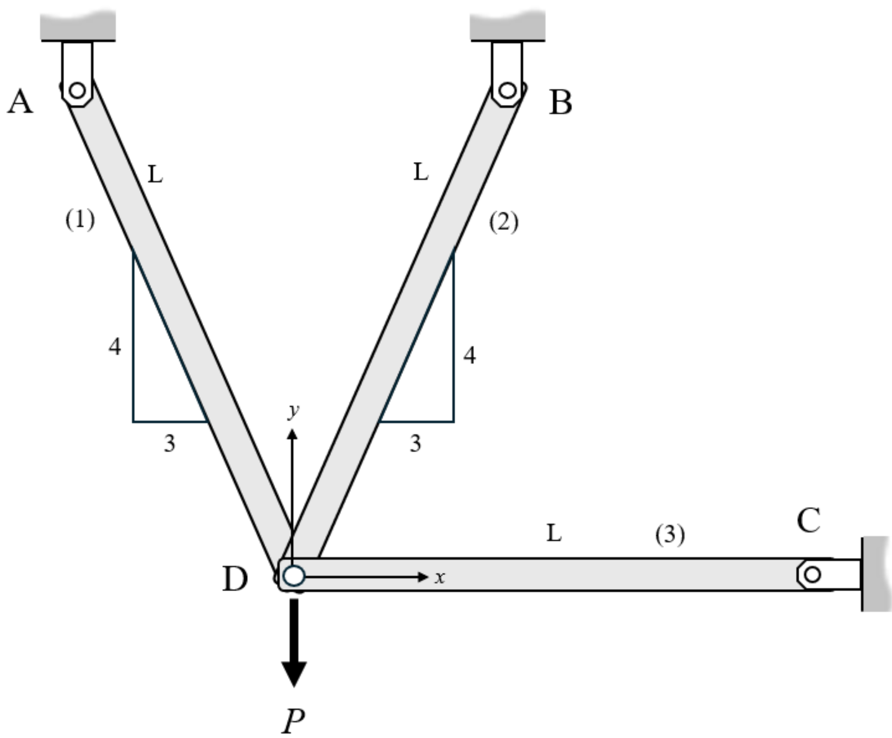
ϕ_C (Same direction as T)

$$\phi_C = \frac{\partial U}{\partial T} = \frac{L}{G\pi d^4}\left[\frac{12}{5}T - \frac{2}{5}T_0\right] = \boxed{\frac{L(12T - 2T_0)}{5G\pi d^4} \Downarrow}$$

15 min.

Problem 7.2 (10 points)

Consider the truss depicted below. All members have a cross-sectional area A, length L, and Young's modulus E. A load P is applied at D as shown. Use energy methods to solve this problem.



Follow the steps provided below to determine the vertical displacement of pin D:

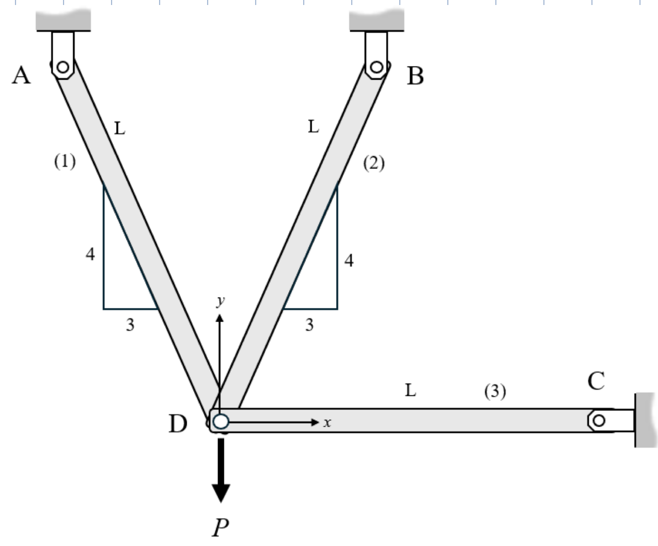
(a) Draw the necessary free body diagram(s).
(b) Write down the corresponding equilibrium equations.
(c) Determine whether the structure is determinate or indeterminate.
(d) State your choice for redundant load(s), if needed.
(e) State your choice for dummy load(s), if needed.
(f) Determine the elastic strain energy of beams 1, 2, and 3 in terms of P, E, L, A, and the redundant load(s).
(g) Determine the axial loads in the three elements using Castigliano's second theorem.
(h) Use Castigliano's second theorem to determine the vertical displacement of pin D.

Express your results in terms of P, E, L, and A.

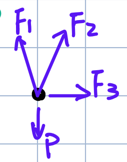
Given: A. E. L. P.

Find: F_1, F_2, F_3. v_D.

Solution: 1. coor (given)



2. FBD



3. Equilibrium Equation.

F_x : $-\frac{3}{5}F_1 + \frac{3}{5}F_2 + F_3 = 0$

F_y : $\frac{4}{5}F_1 + \frac{4}{5}F_2 - P = 0$

3 unknowns: F_1 F_2 . F_3 .

2 Equation ⇒ Indeterminate.

4. Energy method.

$U = U_1 + U_2 + U_3$

$= \frac{1}{2} \cdot \frac{F_1^2 L}{EA} + \frac{F_2^2 L}{EA} + \frac{F_3^2 L}{EA}$

$= \frac{L}{2EA}(F_1^2 + F_2^2 + F_3^2)$

choose F_3 as redundant force.

② → $F_1 = \frac{5}{4}P - F_2$

① → $-\frac{3}{4}P + \frac{3}{5}F_2 + \frac{3}{5}F_2 + F_3 = 0$

$\frac{6}{5}F_2 = \frac{3}{4}P - F_3$

$F_2 = \frac{5}{8}P - \frac{5}{6}F_3$

$F_1 = \frac{5}{4}P - \frac{5}{8}P + \frac{5}{6}F_3$

$= \frac{5}{8}P + \frac{5}{6}F_3$

Write U w/ Redundant load.

$U = \frac{L}{2EA}[(\frac{5}{8}P + \frac{5}{6}F_3)^2 + (\frac{5}{8}P - \frac{5}{6}F_3)^2 + F_3^2]$

$= \frac{L}{2EA}[\frac{25}{32}P^2 + \frac{25}{18}F_3^2 + F_3^2]$

$= \frac{L}{2EA}[\frac{25}{32}P^2 + \frac{43}{18}F_3^2]$

w/ Redundant load, $\frac{\partial U}{\partial F_3} = 0$

$\frac{\partial U}{\partial F_3} = \frac{L}{2EA} \cdot \frac{43}{18} \cdot 2F_3 = 0$ ⇒ $F_3 = 0$

Find F_1, F_2 from Equilibrium Eq:

$F_1 = \frac{5}{8}P$, $F_2 = \frac{5}{8}P$

For vertical displacemen v_D,
v_D in same direction as P
no dummy load needed.

Find $v_D = \frac{\partial U}{\partial P}$

$v_D = \frac{\partial U}{\partial P} = \frac{L}{\cancel{2}EA} \cdot \frac{25}{32} \cdot \cancel{2}P = \frac{25PL}{32EA}$

$v_D = \frac{25PL}{32EA}$ ↓

40 min.

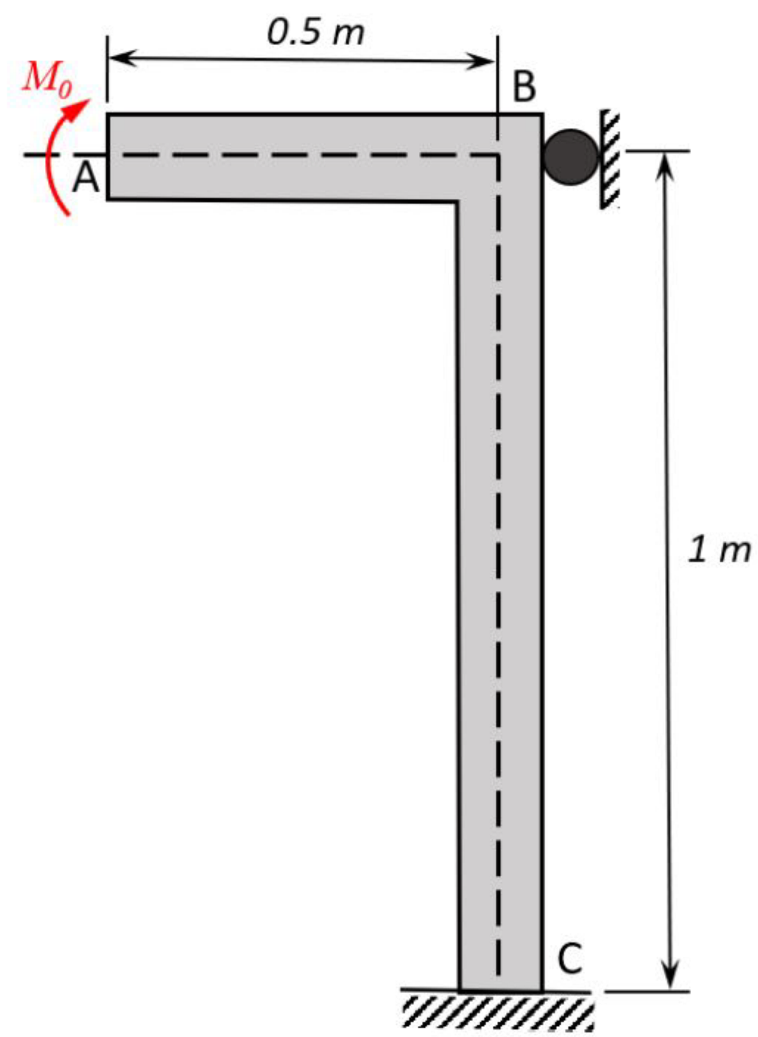
Problem 7.3 (10 points)

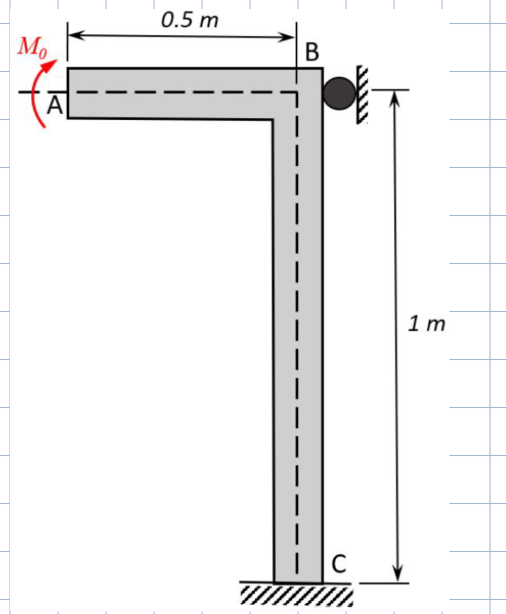
Two segments, AB and BC, with a thin walled hollow circular cross section of outer diameter a and inner diameter 0.8a, are welded together at B to form the L-shaped ABC shown in the figure below. Shear strain energy due to bending can be neglected.

Use the following data in your analysis: E = 280 GPa, G = 120 GPa, a = 20 mm, M = 1000 Nm

(a) Determine the reactions.

(b) Use Castigliano's Second Theorem to determine the vertical displacement at point A, v_A .





Given: thin wall circ. $d_o = 20mm$, $d_i = 16mm$.

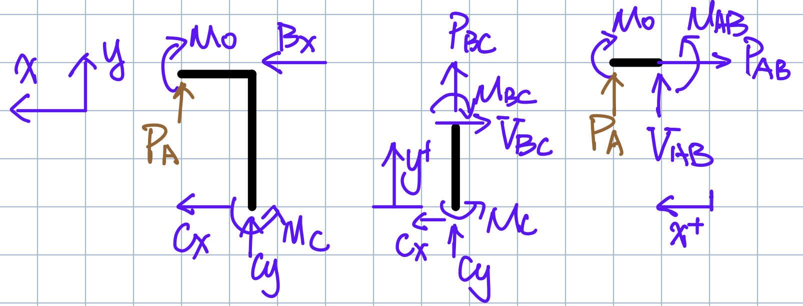
$E = 280\,GPa = 2.8\times10^{11}\,Pa$.

$G = 120\,GPa = 1.2\times10^{11}\,Pa$

$M_o = 10^3\,N\cdot m$. Ignor U_V

Find: C_x, C_y, M_C, B_x, δ_{Ay}

Solution: 1. coor and FBD.



* P_A is dummy Load @ A since need to find δ_A in y direction.

2. Equilibrium. Eq.

ABC: $B_x + C_x = 0$

$C_y + P_A = 0$

$M_C - M_o + (1m)B_x - P_A\cdot(0.5m) = 0$

BC segment: $C_y + P_{BC} = 0$

$C_x - V_{BC} = 0$

$M_C - M_{BC} - C_x\cdot y$

AB segment: $-P_{AB} = 0$

$P_A + V_{AB} = 0$

$-M_o + M_{AB} - P_A(0.5m - x) = 0$

3. Internal Reaction

Statically Indeterminate.

Redudant support: C_x.

$B_x = -C_x$,

$C_y = -P_A$

$M_C = M_o - (-C_x) = M_o + C_x\cdot(1m) + 0.5P_A$

Seg BC: $P_{BC} = -C_y$.

$M_{BC} = -C_x\cdot y + M_o + C_x\cdot(1m)$

$M_{BC}(y) = -C_x y + (10^3 + C_x + 0.5P_A)$

Sec AB: $P_{AB} = 0$.

$M_{AB}(x) = M_o + P_A(0.5m - x)$

$M_{AB}(x) = -P_A x + (0.5P_A + 10^3)$

$A = \pi[(10^{-2}m)^2 - (8\times10^{-3})^2] = 36\pi\times10^{-6}m^2 = 1.13\times10^{-4}m^2$

$I = \frac{\pi}{2}[(10^{-2}m)^4 - (8\times10^{-3}m)^4] = 2952\pi\times10^{-12}m^4 = 9.273\times10^{-9}m^4$

4. Strain Energy

seg. BC: $(P_{BC}, M_{BC}(y))$

$$U_{BCP} = \frac{C_y^2\cdot L}{2EA} = \frac{C_y^2}{2\times2.8\times10^{11}\cdot1.13\times10^{-4}} = 0.158\cdot10^{-7}P_A^2$$

$$U_{BCM} = \frac{1}{2EI}\int_0^{1m}(-C_x y + C_x + 10^3 + 0.5P_A)^2 dx$$

$$= \frac{\frac{1}{3}C_x^2 + 0.5C_xP_A + 10^3C_x + 0.25(P_A + 10^3)^2}{2\times2.8\times10^{11}\times9.273\times10^{-9}}$$

$$= 6.42\times10^{-5}C_x^2 + 0.1926C_x + 9.63\times10^{-5}C_xP_A + 4.81\times10^{-5}P_A^2 + 0.1926P_A + 192.6$$

Seg. AB: $(P_{AB}, M_{AB}(x))$

$U_{ABP} = 0 \quad (P_{AB} = 0)$

$$U_{ABM} = \frac{1}{EI}\int_0^{0.5m}[-P_A x + (0.5P_A + 10^3)]^2 dx$$

$$= \frac{1}{2\times2.8\times10^{11}\times9.273\times10^{-9}}\left[\frac{1}{24}P_A^2 + 250P_A + 5\times10^5\right]$$

$$= 8.024\times10^{-6}P_A^2 + 4.814\times10^{-2}P_A + 96.29$$

Total U

$$U = U_{BCP} + U_{BCM} + \cancel{U_{ABP}}^{\,0} + U_{ABM}.$$

$$= 6.42\times10^{-5}C_x^2 + 9.628\times10^{-5}C_xP_A + 0.1926C_x + 5.632\times10^{-5}P_A^2 + 0.2407P_A + 288.86$$

Continue to next page.

5. Reactions:

Redundant support: C_x. w/ $P_A = 0$

$$\frac{\partial U}{\partial C_x} = 0 \Rightarrow \frac{\partial U}{\partial C_x} = 2 \times 6.42 \times 10^{-5} C_x + 0.1926 = 0 \Rightarrow C_x = -1500\,N.$$

w/ Equilibrium Eq and $P_A = 0$.

$B_x = -C_x = 1500\ N$

$C_y = 0\ N$.

$M_C = M_0 - B_x \cdot 1m = 1000\ N\cdot m - 1500N \cdot 1m = -500\ N\cdot m$

Reaction @ B: $\leftarrow$ 1500N

@ C: $\rightarrow$ 1500N

$\circlearrowleft$ 500Nm

6. Displacement @ A.

$$V_A \text{ in } y = \frac{\partial U}{\partial P_A} = 9.628 \times 10^{-5} C_x + 2 \cdot 5.632 \times 10^{-5} \cancel{P_A}^{\to 0} + 0.2407$$

$$= 9.628 \times 10^{-5} \cdot (-1500N) + 0.2407$$

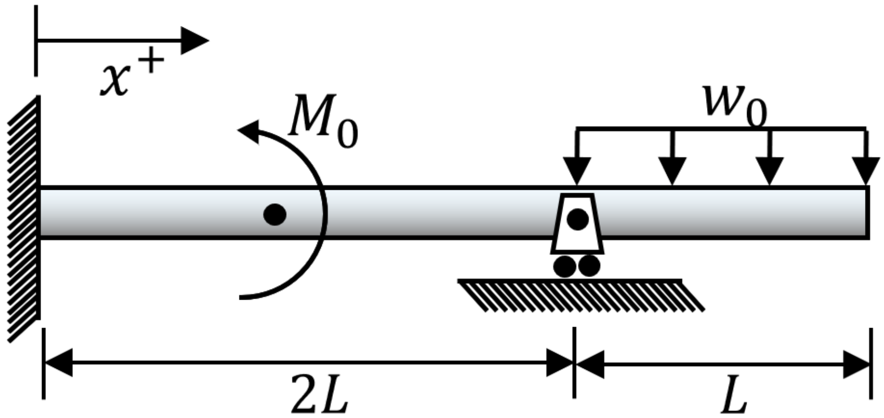
$$= 0.09627\,m$$

V_A in same direction as P_A.

$V_A \uparrow 0.09627\,m$

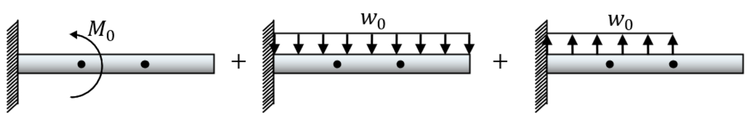
3 min.

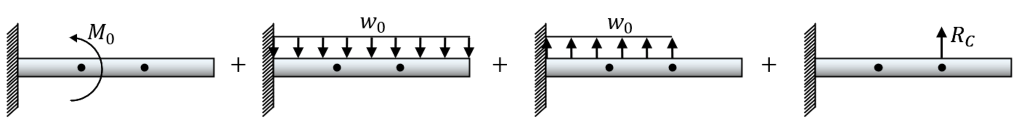
Problem 7.4 (5 points)

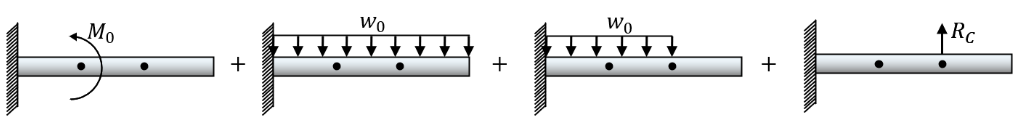


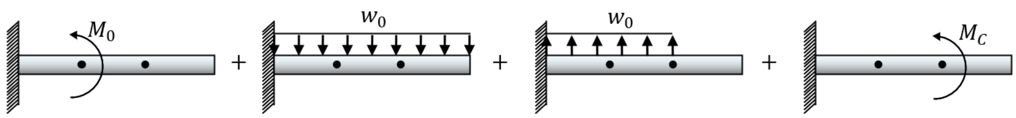
Problem 7.4.1 (2.5 points)

Refer to the figure above. For the given loads and supports, which of the following is the correct breakdown of the system if the deflection was to be calculated with the Superposition principle?

A) 

B) 

C) 

D) 

Problem 7.4.2 (2.5 points)

Which of the following represent the correct set of boundary conditions applied to the beam?

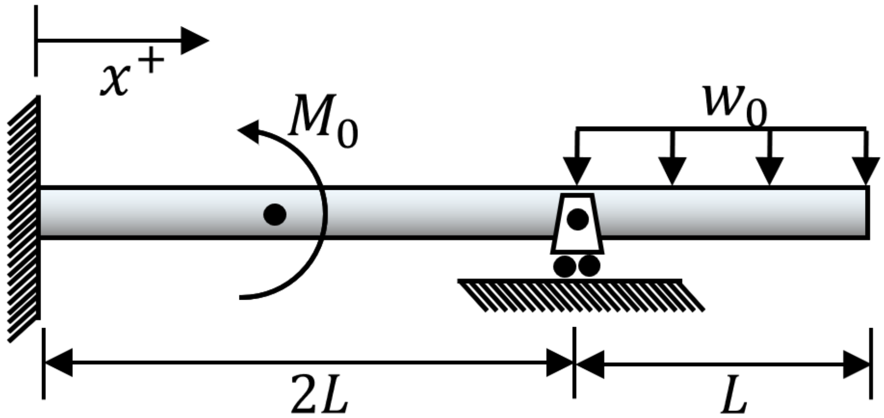
A) $v(0) = 0,\ v'(0) = 0,\ v(2L) = 0,\ v'(2L) = 0$

B) $v(0) = 0,\ v'(0) = 0,\ v(2L) = 0$

C) $v(0) = 0,\ v'(0) = 0,\ v'(2L) = 0$

D) $v(0) = 0,\ v'(0) = 0$

Problem 7.4 (5 points)



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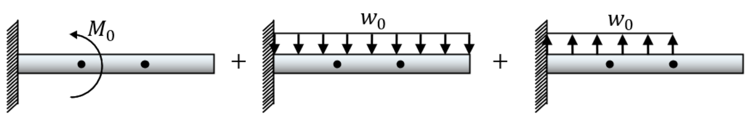
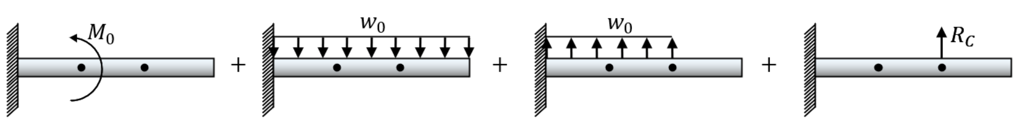
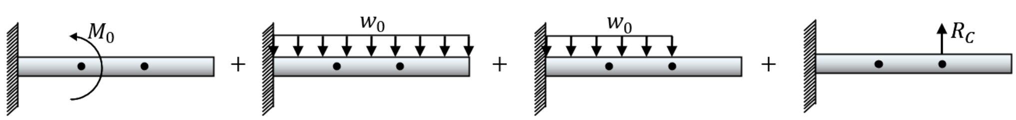
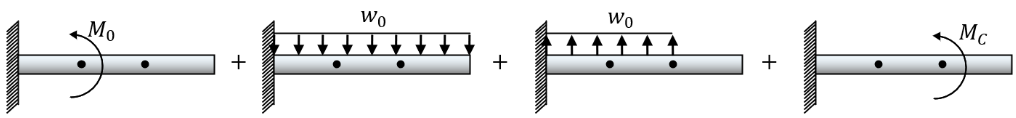
A) 

B) 

C) 

D) 

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Which of the following represent the correct set of boundary conditions applied to the beam?

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B) $v(0) = 0,\ v'(0) = 0,\ v(2L) = 0$

C) $v(0) = 0,\ v'(0) = 0,\ v'(2L) = 0$

D) $v(0) = 0,\ v'(0) = 0$

w/ $x=0$, $v=0$, $v'=0$

w/ $x=2L$, $v=0$. v' unknown.