ME 323: Mechanics of Materials **Homework 6**

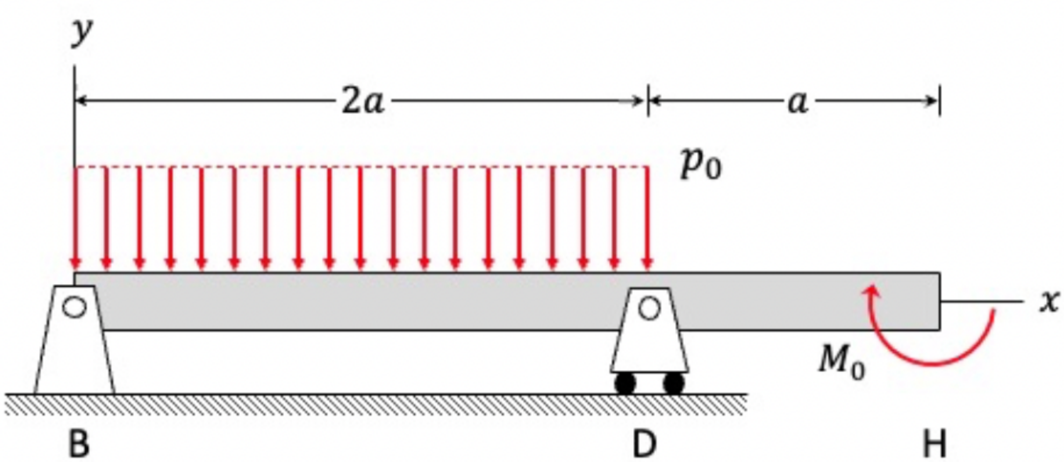
Spring 2026 **Due: March 6, 2026**

Problem 1 (10 points):

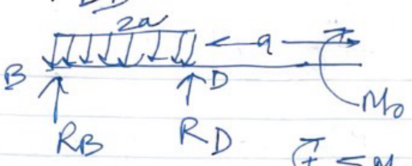
The simply-supported beam whose cross-section has a 2nd area moment of I and is made up of a material with a Young's modulus of E shown above experiences a constant line load of p_0 along section BD and a concentrated couple M_0 at end H.

1. Draw the free body diagram (FBD) of the beam. Based on this FBD, write down the equilibrium equations for the beam.
2. State whether the beam is determinate or indeterminate. If determinate, solve for the reactions.
3. Using the 2nd-order integration approach, determine the deflection of the beam $v(x)$ over the range of $0 \leq x \leq 3a$. Clearly identify the segments along the beam for which the different segments of the deflection curve are valid.
4. Determine the rotation (slope) of the beam deflection curve at support D.

Leave your answers in terms of, at most: E, I, a, p_0 and M_0.



① 1. FBD



Equilibrium
y dr.ⁿ

$\uparrow + \sum F_y = 0$

$R_B + R_D - P_0(2a) = 0$

$\circlearrowleft + \sum M_B = 0$

$M_0 - R_D(2a) + P_0(2a)a = 0$

2. Since two unknowns R_B and R_D and two equations ⟹ statically determinate

$R_B + R_D = 2P_0 a$

$R_D(2a) = P_0 2a^2 + M_0$

$$\boxed{R_D = P_0 a + \frac{M_0}{2a}}$$

$R_B = 2P_0 a - R_D = 2P_0 a - \left(P_0 a + \frac{M_0}{2a}\right)$

$$\boxed{R_B = P_0 a - \frac{M_0}{2a}}$$

3. $M_1 = R_B x - \frac{P_0 x^2}{2}$ (for $0 \le x \le 2a$)

$M_2 = R_B x - P_0 2a(x-a) + R_D(x-2a)$ (for $2a \le x \le 3a$)

$= (R_B + R_D)x - 2R_D a - 2P_0 a x + 2P_0 a^2$ (with $R_B + R_D = 2P_0 a$)

$= 2P_0 a^2 - 2R_D a = 2P_0 a^2 - 2a\left(P_0 a + \frac{M_0}{2a}\right)$

$M_2 = -M_0$

$v_1'' = \frac{-M_1}{EI}$ for $0 \le x \le 2a$

$v_1'' = \frac{-1}{EI}\left[R_B x - P_0 \frac{x^2}{2}\right]$

integrate $v_1' = \frac{-1}{EI}\left[R_B \frac{x^2}{2} - \frac{P_0 x^3}{6}\right] + C_1$ — ⓐ

$C_1, C_2 \rightarrow$ const

Integrate again

$$v_1 = \frac{1}{EI}\left[\frac{P_0 x^4}{24} - \frac{R_B x^3}{6}\right] + C_1 x + C_2 \quad \text{--- (b)}$$

$$v_2'' = \frac{-M_2}{EI} = \frac{-(-M_0)}{EI} \quad [\text{for } 2a \le x \le 3a]$$

$$v_2'' = \frac{M_0}{EI}$$

Integ

$$v_2' = \frac{M_0}{EI} x + C_3 \quad \text{--- (c)}$$

$C_3, C_4 \rightarrow$ const.

$$v_2 = \frac{M_0}{EI}\frac{x^2}{2} + C_3 x + C_4 \quad \text{--- (d)}$$

BC (i) $v_1(0) = 0 \Rightarrow C_2 = 0$

(ii) $v_1(2a) = v_2(2a)$. Continuity

$$\frac{1}{EI}\left[\frac{P_0 (2a)^4}{24} - \frac{R_B}{6}(2a)^3\right] + C_1 (2a) = 0$$

[Since $v_2(2a) = 0$]

$$C_1 = \frac{1}{EI}\left[\frac{2}{3} R_B a^2 - \frac{P_0 a^3}{3}\right]$$

$$C_1 = \frac{1}{3EI}\left[2\left(P_0 a - \frac{M_0}{2a}\right)a^2 - P_0 a^3\right]$$

$$C_1 = \frac{P_0 a^3 - M_0 a}{3EI}$$

(iii) $v_2(2a) = 0$

$$\frac{M_0}{EI}\frac{(2a)^2}{2} + C_3 (2a) + C_4 = 0$$

$$\frac{2M_0 a^2}{EI} + 2a C_3 + C_4 = 0 \quad \text{--- (e)}$$

(iv) $v_1'(2a) = v_2'(2a)$ Continuity of slope

$$\frac{1}{EI}\left[\frac{1}{6}P_0(2a)^3 - \frac{1}{2}R_B(2a)^2\right] + C_1 = \frac{M_0}{EI}2a + C_3$$

$$\frac{1}{EI}\left[\frac{4}{3}P_0a^3 - 2R_Ba^2\right] + \frac{P_0a^3 - M_0a}{3EI} = \frac{M_0 2a}{EI} + C_3$$

$$C_3 = \frac{1}{3EI}\left[4P_0a^3 - 6R_Ba^2 + P_0a^3 - M_0a - 6M_0a\right]$$

$R_B \rightarrow P_0a - M_0/2a$

$$C_3 = \frac{1}{3EI}\left[4P_0a^3 - 6P_0a^3 + 3M_0a + P_0a^3 - M_0a - 6M_0a\right]$$

$$C_3 = \frac{1}{3I}\left[-P_0a^3 - 4M_0a\right]$$

Substituting C_3 in eq^n (c)

$$\frac{2M_0a^2}{EI} + 2a\frac{1}{3EI}\left[-P_0a^3 - 4M_0a\right] + C_4 = 0$$

$$C_4 = \frac{2}{3EI}\left[2M_0a^2 + a^4P_0\right]$$

All constants C_1, C_2, C_3, C_4 are calculated

$$v_1 = \frac{1}{EI}\left[\frac{P_0x^4}{24} - \frac{R_Bx^3}{6}\right] + \left(\frac{a^3P_0 - M_0a}{3EI}\right)x + 0$$

$$= \frac{1}{EI}\left[\frac{P_0x^4}{24} - \left(P_0a - \frac{M_0}{2a}\right)\frac{x^3}{6}\right] + \left(\frac{a^3P_0 - M_0a}{3EI}\right)x$$

$$= \frac{1}{24EI}\left[\frac{aP_0x^4 - 4P_0a^2x^3 + 2M_0x^3 + 8P_0a^4x - 8M_0a^2x}{a}\right]$$

$$v_1 = \frac{1}{24EIa}\left[aP_0x^4 - 4P_0a^2x^3 + 2M_0x^3 + 8P_0a^4x - 8M_0a^2x\right]$$

from eq^n (d)

$$v_2 = \frac{M_0}{2EI}x^2 + \frac{1}{3EI}\left[-P_0a^3 - 4M_0a\right]x + \frac{2}{3EI}\left[M_0a^2 + a^4P_0\right]$$

$$v_2 = \frac{3M_0x^2 - 2P_0a^3x - 8M_0ax + 4M_0a^2 + 4a^4P_0}{6EI}$$

Total $v = v_1 + v_2$

4. Rotation (slope) at support D.

$$\theta_D = v_2'(2a) = \frac{M_0}{EI}(2a) + \frac{1}{3EI}\left(-P_0a^3 - 4M_0a\right)$$

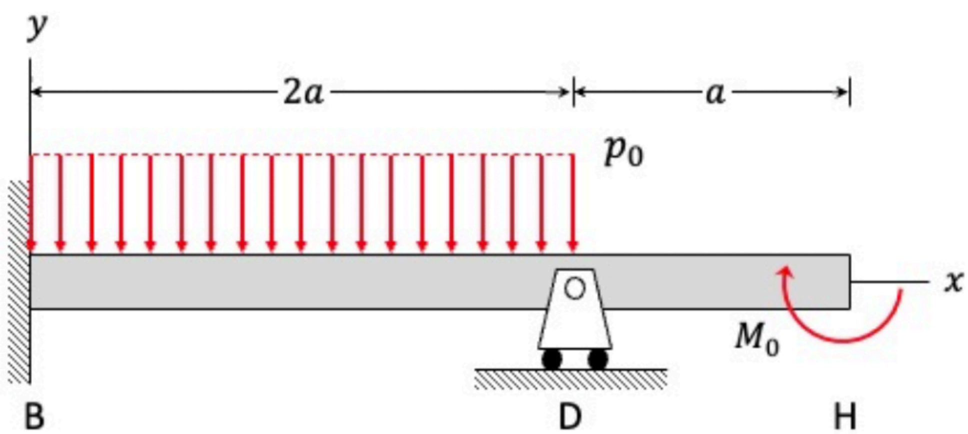
$$\boxed{v_2'(2a) = \frac{2M_0a - P_0a^3}{3EI}}$$

Problem 2 (10 points):

The propped cantilevered beam whose cross-section has a 2nd area moment of I and is made up of a material with a Young's modulus of E shown above experiences a constant line load of p_0 along section BD and a concentrated couple M_0 at end H.

1. Draw the free body diagram (FBD) of the beam. Based on this FBD, write down the equilibrium equations for the beam.
2. State whether the beam is determinate or indeterminate.
3. Using the 2nd-order integration approach, determine the reactions acting on the beam at B and D.
4. Determine the rotation (slope) of the beam deflection curve at support D and the deflection of the beam at H.

Leave your answers in terms of, at most: E, I, a, p! and M!.



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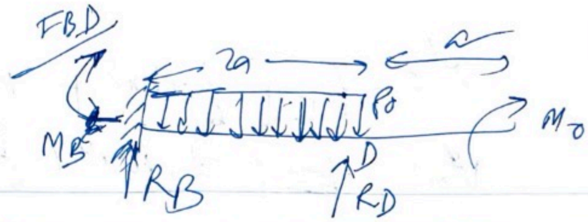
FBD



a) $+\uparrow \sum F_y = 0$

$R_B + R_D = p_0(2a)$ ——— (A)

$\sum M = 0$

$-M_B + R_D(2a) - p_0 2a \cdot a - M_0 = 0$ ——— (B)

b) 3 unknowns (R_B, R_D, M_D), Two eqn.
Statically indeterminate

c) Segment BD $0 \le x \le 2a$

$M - M_B - R_B x + \frac{1}{2} p_0 x^2 = 0$

$EIv'' = \frac{M}{EI} = M_B + R_B x - \frac{1}{2} p_0 x^2$

Segment DH $2a \le x \le 3a$ $\quad M - M_B - R_B x - R_D(x-2a) + 2p_0 a(x-a) = 0$

$EIv'' = (R_B + R_D - 2p_0 a)x + (M_B + 2p_0 a^2 - 2R_D a)$

$EIv'' = M_B + 2p_0 a^2 - 2R_D a$

Now, BD $\quad v'' = \frac{1}{EI}\left[-\frac{1}{2}p_0 x^2 + R_B x + M_B\right]$

$v' = \frac{1}{EI}\left[-\frac{1}{6}p_0 x^3 + \frac{R_B x^2}{2} + M_B x + C_1\right]$

$v = \frac{1}{EI}\left(-\frac{1}{24}p_0 x^4 + \frac{R_B x^3}{6} + \frac{1}{2}M_B x^2 + C_1 x + C_2\right.$

$v(0) = 0 \Rightarrow C_2 = 0$

$v'(0) = 0 \Rightarrow C_1 = 0$

$$v' = \frac{1}{EI}\left[-\frac{1}{6}p_0 x^3 + \frac{R_B x^2}{2} + M_B x\right]$$

$$v = \frac{1}{EI}\left[-\frac{1}{24}p_0 x^4 + \frac{1}{6}R_B x^3 + \frac{1}{2}M_B x^2\right]$$

$$v(2a) = 0$$

$$\frac{1}{EI}\left(-\frac{2}{3}p_0 a^4 + \frac{4}{3}R_B a^3 + 2M_B a^2\right) = 0$$

$$R_B = \frac{1}{2}p_0 a - \frac{3}{2}\frac{M_B}{a} \quad \text{— (C)}$$

$$R_B + R_D = 2p_0 a \Rightarrow R_D = \frac{3}{2}p_0 a + \frac{3}{2}\frac{M_B}{a} \quad \text{— (D)}$$

$$-M_B - 2p_0 a^2 + 2a\left(\frac{3}{2}p_0 a + \frac{3}{2}\frac{M_B}{a}\right) - M_0 = 0$$

$$-M_B - 2p_0 a^2 + 3p_0 a^2 + 3M_B - M_0 = 0$$

$$\boxed{M_B = \frac{1}{2}M_0 - \frac{1}{2}p_0 a^2}$$

Substituting M_B in (C) and (D)

$$R_B = \frac{1}{2}p_0 a - \frac{3}{2}\left(\frac{1}{2}M_0 - \frac{1}{2}p_0 a^2\right)$$

$$\boxed{R_B = \frac{5}{4}p_0 a - \frac{3}{4}\frac{M_0}{a}}$$

$$R_D = \frac{3}{2}p_0 a + \frac{3}{2}\frac{M_B}{a} = \frac{3}{2}p_0 a + \frac{3}{2}\left(\frac{1}{2}M_0 - \frac{1}{2}p_0 a^2\right)$$

$$\boxed{R_D = \frac{3}{4}p_0 a + \frac{3}{4}\frac{M_0}{a}}$$

d) $v' = \frac{1}{EI}\left[-\frac{1}{6}P_0 x^3 + \frac{R_B x^2}{2} + M_B x\right]$

$v'(2a) = \frac{1}{EI}\left[-\frac{1}{6}P_0(2a)^3 + \left(\frac{5}{4}P_0 a - \frac{3}{4}\frac{M_0}{a}\right)\frac{(2a)^2}{2} + \left(\frac{1}{2}M_0 - \frac{1}{2}P_0 a^2\right)2a\right]$

$= \frac{1}{EI}\left[-\frac{4}{3}P_0 a^3 + \frac{5}{2}P_0 a^3 + M_0 a - \frac{3}{2}M_0 a - P_0 a^3\right]$

$$\boxed{\theta_D = v'(2a) = \frac{1}{EI}\left[\frac{1}{6}P_0 a^3 - \frac{1}{2}M_0 a\right]}$$

Segment DH, $EIv'' = M_B + 2P_0 a^2 - 2R_D \cdot a$

$= \frac{1}{2}M_0 - \frac{1}{2}P_0 a^2 + 2P_0 a^2 - \frac{3}{2}P_0 a^2 - \frac{3}{2}M_0$

$v'' = \frac{1}{EI}\left[-M_0\right]$

$v' = \frac{1}{EI}\left[-M_0 x + C_3\right]$

$v = \frac{1}{EI}\left[-\frac{M_0 x^2}{2} + C_3 x + C_4\right]$

Continuity of slope: $v'(2a) = \frac{1}{EI}\left[-2M_0 a + C_3\right] = \frac{1}{EI}\left[\frac{1}{6}P_0 a^3 - \frac{1}{2}M_0 a\right]$

$C_3 = \frac{1}{6}P_0 a^3 + \frac{3}{2}M_0 a$

$v(2a) = 0 \Rightarrow \frac{1}{EI}\left[-\frac{M_0 \cdot 4a^2}{2} + C_3(2a) + C_4\right] = 0$

$-2M_0 a^2 + \frac{1}{3}P_0 a^4 + 3M_0 a^2 + C_4 = 0$

$C_4 = -\frac{1}{3}P_0 a^4 - M_0 a^2$

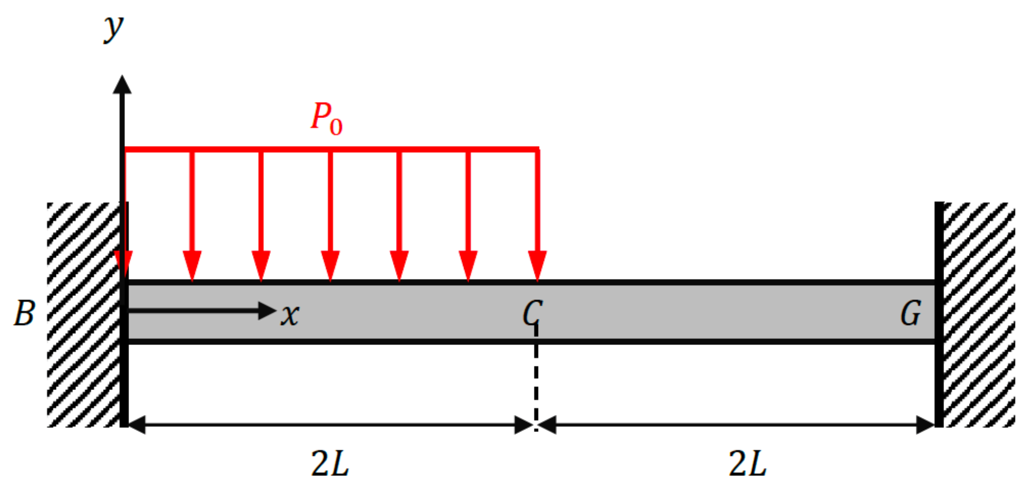
$v_H = v(3a) = \frac{1}{EI}\left[-\frac{1}{2}M_0(3a)^2 + \left(\frac{1}{6}P_0 a^3 + \frac{3}{2}M_0 a\right)3a + \left(-\frac{1}{3}P_0 a^4 - M_0 a^2\right)\right]$

$= \frac{1}{EI}\left[-\frac{9}{2}M_0 a^2 + \frac{1}{2}P_0 a^4 + \frac{9}{2}M_0 a^2 - \frac{1}{3}P_0 a^4 - M_0 a^2\right]$

$$\boxed{v_H = \frac{1}{EI}\left[-M_0 a^2 + \frac{1}{6}P_0 a^4\right]}$$

Problem 3 (10 points):

The beam shown below has Young's modulus E and cross-sectional second area moment I. Determine the reaction forces and moments at ends B and G using the second-order integration method.



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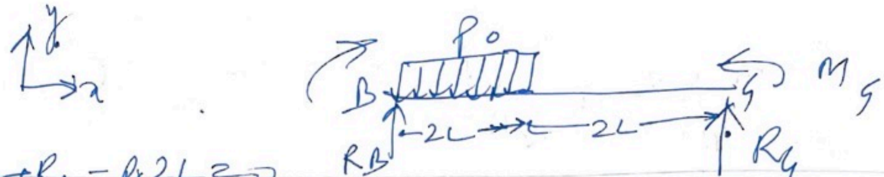


Eqm

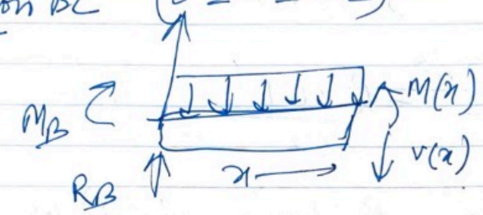
$\Sigma F_y = 0$: $R_B + R_4 - P_0 2L = 0$

$\Sigma M_B = 0$ $M_B + P_0 2L^2 - R_4 (4L) - M_4) = 0$

Four unknowns (R_B, R_4, M_B, M_4)

Two eqn. → indeterminate

Section BL $(0 \le x \le 2L)$



$\Sigma M = 0$ $M_B - M(x) + R_B x - \frac{P_0 x^2}{2} = 0$

$M(x) = M_B + R_B x - \frac{P_0 x^2}{2}$

Slope $EI v_1' = M_B x + R_B \frac{x^2}{2} - \frac{P_0 x^3}{6} + C$

Deflecti $EI v_1 = \frac{M_B x^2}{2} + \frac{R_B x^3}{6} - \frac{P_0 x^4}{24} + C_1 x + C_2$

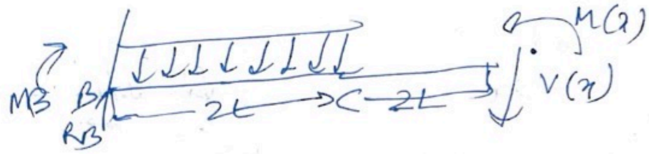
$v_1(0) = 0 \Rightarrow C_2 = 0$

$v_1'(0) = 0 \Rightarrow C_1 = 0$

$\therefore v_1 = \frac{1}{EI}\left(\frac{M_B x^2}{2} + \frac{R_B x^3}{6} - \frac{P_0 x^4}{24}\right)$

Section CG

$(2L \le x \le 4L)$



$$EI\frac{d^2v_2}{dx^2} = M_2(x) = M_B + R_B x - 2P_0L(x-L)$$

Slope $EIv_2' = M_B x + R_B\frac{x^2}{2} + 2P_0L^2x - P_0Lx^2 + C_3$

Deflection $EIv_2 = M_B\frac{x^2}{2} + R_B\frac{x^3}{6} + P_0L^2x^2 - \frac{P_0Lx^3}{3} + C_3x + C_4$

BC $v_1'(2L) = v_2'(2L)$

~~$\frac{M_B(4L^2)}{2} + \frac{R_B 8L^3}{6} - \frac{P_0 16L^4}{24} = M_B$~~

$$M_B(2L) + R_B\frac{(2L)^2}{2} - \frac{P_0(2L)^3}{6} = M_B(2L) + R_B\frac{(2L)^2}{2} + 2P_0L^2(2L) - P_0L(2L)^2 + C_3$$

$$C_3 = -\frac{4}{3}P_0L^3$$

BC $v_1(2L) = v_2(2L)$

$$M_B\frac{(2L)^2}{2} + \frac{R_B}{6}(2L)^3 - \frac{P_0(2L)^4}{24} = M_B\frac{(2L)^2}{2} + R_B\frac{(2L)^3}{6} + P_0L^2(2L)^2 - \frac{P_0L(2L)^3}{3} + \left(-\frac{4}{3}P_0L^3\right)(2L) + C_4$$

$$C_4 = \frac{2}{3}P_0L^4$$

BC $v_2'(4L) = 0$

$$M_B(4L) + R_B\frac{(4L)^2}{2} + 2P_0L^2(4L) - P_0L(4L)^2 - \frac{4}{3}P_0L^3 = 0$$

$$4M_BL + 8R_BL^2 = \frac{28}{3}P_0L$$

$$4M_B + 8R_B L = \frac{28}{3} P_o L^2 \quad \text{—— (A)}$$

BC $\quad v_2(4L) = 0$

$$M_B\left(\frac{16L^2}{2}\right) + R_B \frac{64L^3}{6} + P_o L^2 (4L)^2 - \frac{P_o L}{3}(4L)^3 + \left(-\frac{4}{3}P_o L^3\right) 4L$$

$$+ \frac{2}{3} P_o L^4 = 0$$

$$8M_B L^2 + \frac{32}{3} R_B L^3 = 10 P_o L^4$$

$$8M_B + \frac{32}{3} L R_B = 10 P_o L^2 \quad \text{—— (B)}$$

Eq^n (A) & (B) $\quad 4M_B + 8R_B L = \frac{28}{3} P_o L^2$

Two eq^n; Two variables M_B and R_B $\quad 8M_B + \frac{32}{3} R_B L = 10 P_o L^2$

Multiplying (A) by 2 and subtracting from (B)

$$R_B = \frac{26}{16} P_o L \Rightarrow \boxed{R_B = \frac{13}{8} P_o L}$$

eq^n (A) $\quad 4M_B + 13 P_o L^2 = \frac{28}{3} P_o L^2$

$$\boxed{M_B = -\frac{11}{12} P_o L^2}$$

$$R_B + R_4 = 2P_o L$$

$$\boxed{R_4 = \frac{3}{8} P_o L}$$

~~$M_B - M_4 + -4P_o L^2 - R_4 4L = 0$~~

$$M_2(x) = M_B + R_B x - 2P_o L(x - L)$$

$$M_4 = M_2(4L) = M_B + R_B(4L) - 2P_o L(4L - L)$$

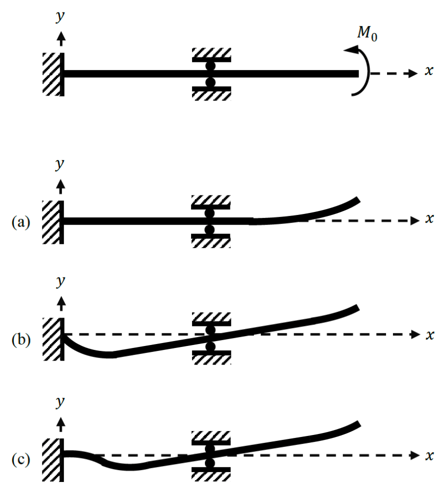
$$= -\frac{11}{12} P_o L^2 + \frac{13}{2} P_o L^2 - 6P_o L^2$$

$$\boxed{M_4 = -\frac{5}{12} P_o L^2}$$

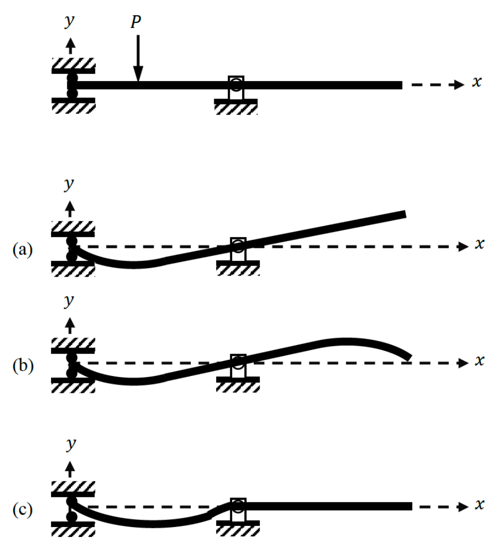
Problem 4 (2.5 + 2.5 points)

Identify the schematic that represents the deflection curve of the following beams (each 2.5 points) :

Beam (A)



Beam (B)



Solution 4

Beam A : c

Beam B : a