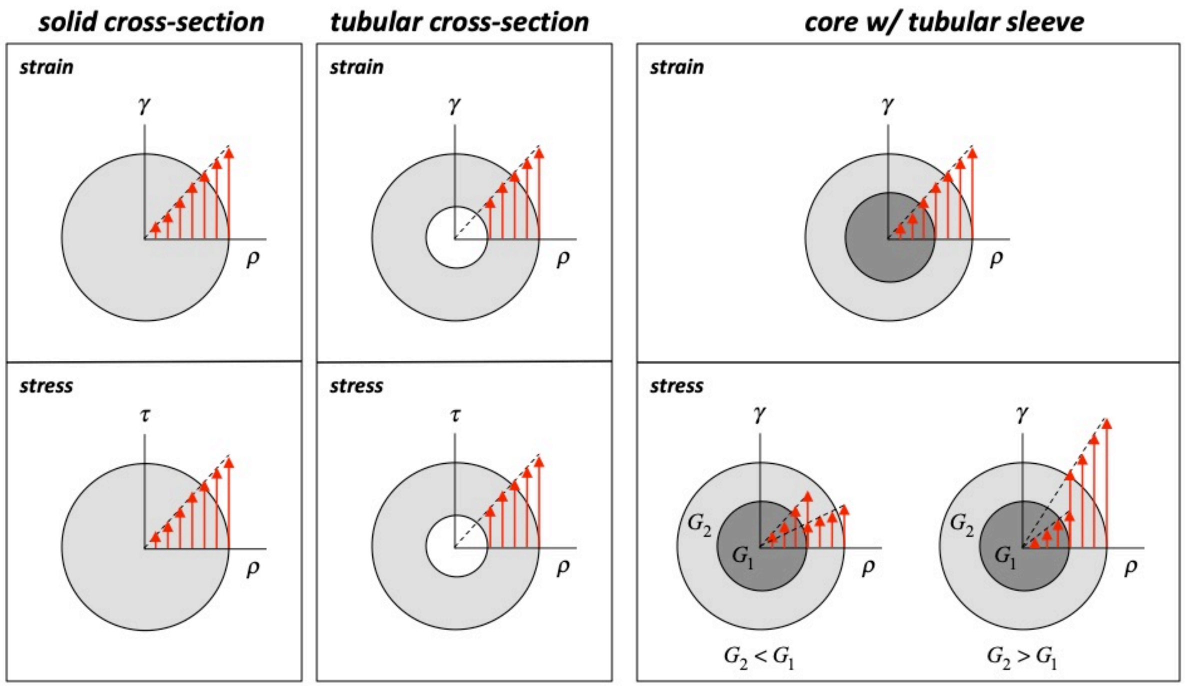
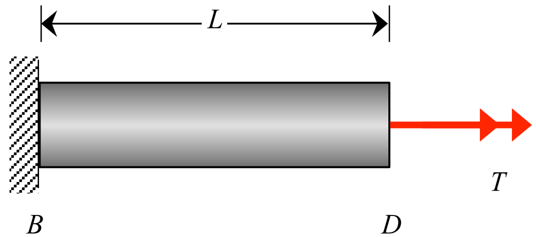
Summary: torsion stresses in shafts

Consider an axial torque T acting on a shaft with a circular cross section.

- *STRAIN*: The shear strain, γ, varies linearly with radius, *ρ*, through the cross-section of the shaft, regardless of the material makeup of the cross-section.
- *STRESS*: Across annular regions on the cross-section where the material makeup is a constant, the shear stress, τ, varies linearly with radius, *ρ*, through the cross-section of the shaft: $\tau = G\gamma = T\rho / I_P$ where I_P is the polar area moment of the cross section.
- *STRAIN/STRESS DISTRIBUTIONS*:



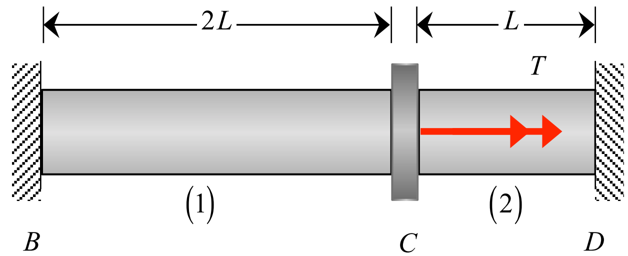
Summary: determinate shafts



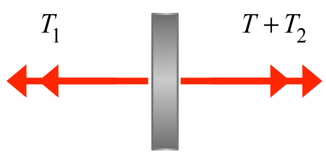
Consider an axial torque T acting on a shaft with a circular cross section.

- *STRAIN*: Linear distribution across the entire cross-section.
- *STRESS*: Linear distribution across an annulus of constant material properties: $\tau = G\gamma = T\rho / I_P$
- *ANGLE OF TWIST:* $\Delta\phi = \phi_D - \phi_B = \int_0^L \frac{T}{GI_P} dx = \frac{TL}{GI_P}$

Summary: indeterminate shafts



Consider an axial torque T acting on a shaft with a circular cross section. To solve for torque in each element, use the four-step plan:



1. *EQUILIBRIUM*: $(1)\quad \sum M = T_2 + T - T_1 = 0$

2. *TORQUE/ROTATION*: $(2)\quad \Delta\phi_1 = \frac{T_1(2L)}{GI_P}$

   $(3)\quad \Delta\phi_2 = \frac{T_2 L}{GI_P}$

3. *COMPATIBILITY:* $(4)\quad \phi_C = \phi_B + \Delta\phi_1 = \Delta\phi_1$

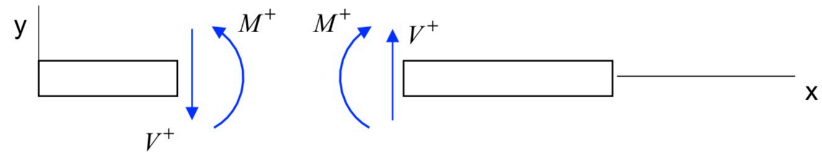
   $(5)\quad \phi_D = \phi_C + \Delta\phi_2 = \Delta\phi_1 + \Delta\phi_2 = 0$

4. *SOLVE:* $(2),(3),(5) \;\Rightarrow\; 2\frac{T_1 L}{GI_P} + \frac{T_2 L}{GI_P} = 0 \;\Rightarrow\; T_2 = -2T_1 \quad (6)$

   $(1),(6) \;\Rightarrow\; -2T_1 + T - T_1 = 0 \;\Rightarrow\; T_1 = \frac{1}{3}T \;\Rightarrow\; T_2 = -\frac{2}{3}T$

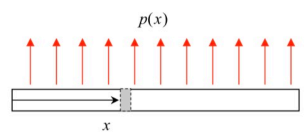
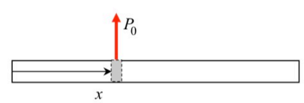
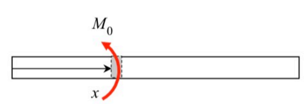
Summary: shear force/bending moment diagrams

SIGN CONVENTIONS:



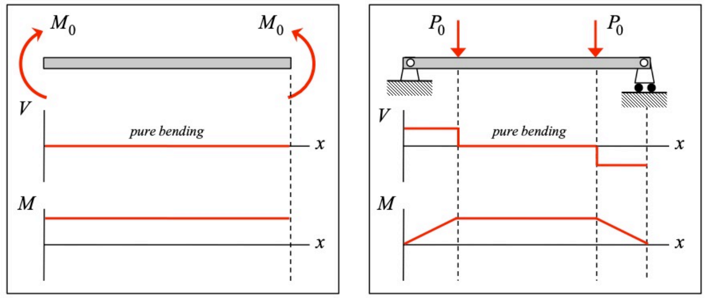
positive bending moment and shear force at cut in beam

FROM EQUILIBRIUM:

applied loading	*key relationship(s)*
	$\frac{dV}{dx} = p(x)$ $\frac{dM}{dx} = V(x)$
	$V(x^+) = V(x^-) + P_0$
	$M(x^+) = M(x^-) - M_0$

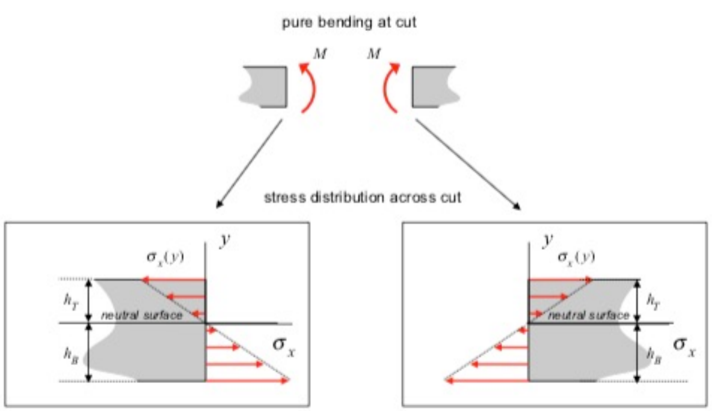
Summary: flexural stresses in pure bending in beams

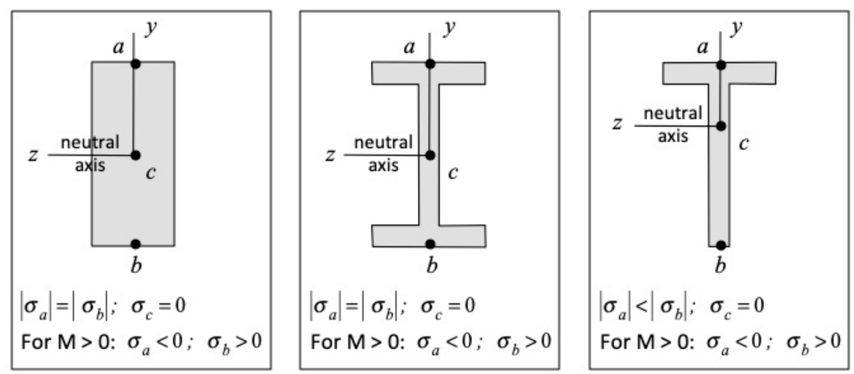
- *Pure bending*: locations on a beam for which the shear force is zero. Examples:



- *Flexural s*tresses in pure bending:

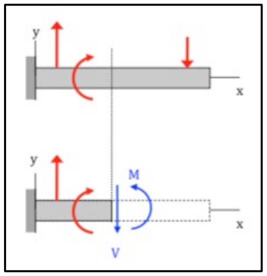
$$\sigma = -\frac{My}{I}$$ (linearly-varying on the cross-section with y measured from *neutral surface*)





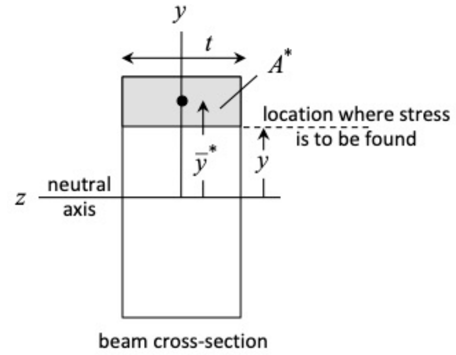
- Where on the cross-section is the flexural stress the greatest?

Summary: shear stresses in beams



- *General loading*: For a general resultant pair of V and M at a cross-section, the normal stress is approximately that of pure bending, $\sigma = -\frac{My}{I}$, and the shear stress is given by:

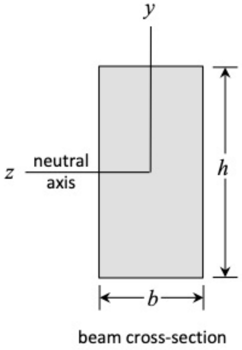
$$\tau = \frac{VA^* \overline{y}^*}{It}$$



with A^* and $\overline{y}^*$ are the area and centroid of the cross-section above "*y*", respectively, and *t* is the depth thickness of the beam at "*y*".

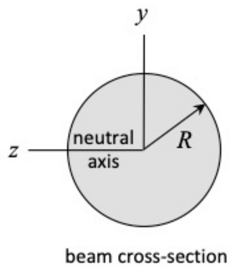
- Special cases:
  - Rectangular cross-section:

$$\tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{V}{bh}$$

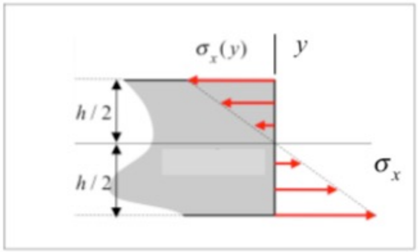


  - Circular cross-section:

$$\tau_{max} = \frac{4}{3}\frac{V}{A} = \frac{4}{3}\frac{V}{\pi R^2}$$



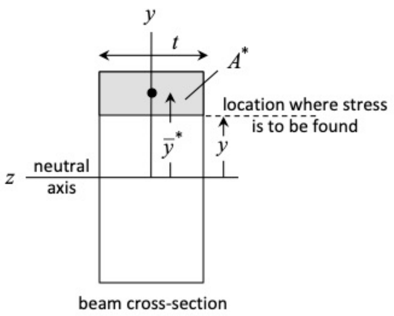
Summary: general stresses in beams



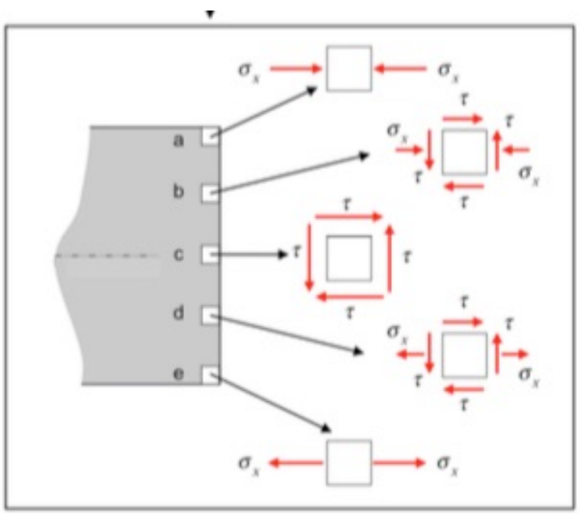
- *General loading*: For a typical point on a beam cross-section there are two components of stress:

  flexural stress: $\sigma = -\dfrac{My}{I}$

  shear stress: $\tau = \dfrac{VA^* \overline{y}^*}{It}$



- *Stress distributions*: Each component varies over the height dimension *y* (measured from neutral plane):
  - The flexural stress σ varies linearly in *y*, with σ taking on a value of zero at the neutral axis and maximum magnitude values at the top and bottom surfaces. The top and bottom locations are of opposite signs (tension and compression).
  - The distribution of the shear stress τ depends on the shape of the cross-section. τ is zero at the top and bottom surfaces, always. The maximum magnitude of τ occurs at (or near) the neutral axis. <u>*The direction of τ is governed only by the direction of V.*</u>



Summary: Beam deflection by integration (indeterminate)

FUNDAMENTAL EQUATIONS

differential form	***integral form***
$M(x) = \frac{EI}{\rho} \approx EI\frac{d\theta}{dx}$	$\theta(x) = \theta(x_1) + \frac{1}{EI}\int_{x_1}^{x} M(s)\,ds$
$tan\theta \approx \theta = \frac{dv}{dx}$	$v(x) = v(x_1) + \int_{x_1}^{x} \theta(s)\,ds$

METHOD

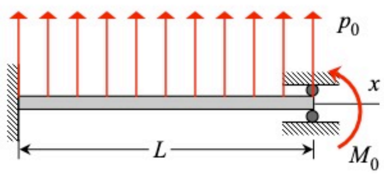
- Draw FBD of entire structure and write down equilibrium equations in terms of reactions.
- Divide beam into sections based on changes in supports or loadings.
- For each section:
  - Make cut through section, and determine *M(x)*.
  - Integrate *M(x)/EI* to find θ*(x)*.
  - Integrate θ*(x)* to find *v(x)*.
  - Enforce boundary conditions on θ and *v*.
  - Match θ and *v* across boundaries of sections.
- Solve for unknown reactions using boundary conditions and equilibrium equations.

specific to indeterminate beams

Summary: Beam deflection by superposition (indeterminate)

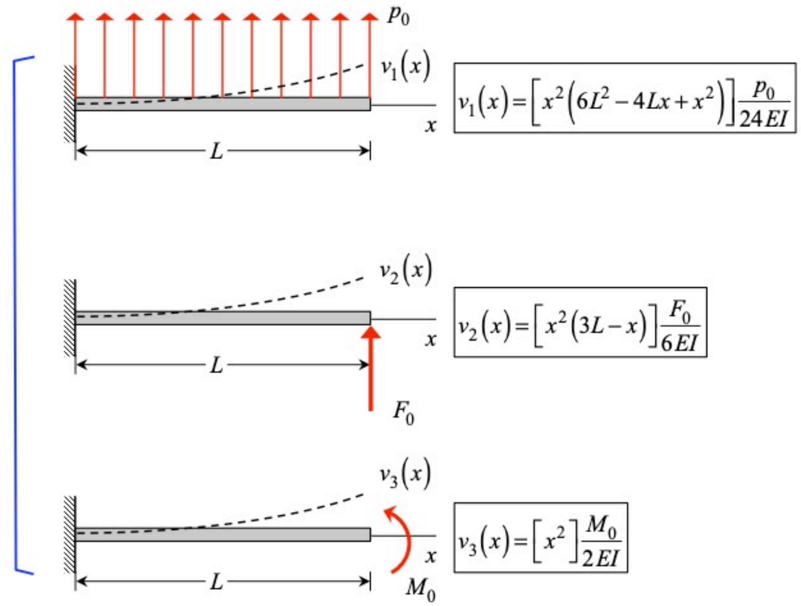
GOAL: Break down complicated loading on a beam into simple loading components and used published deflection formulas to determine the beam deflection through linear superposition.

EXAMPLE:



$$v(x) = v_1(x) + v_2(x) + v_3(x)$$

=



METHOD

- Each component system must be in equilibrium.
- For piece-wise defined loadings and intermediate supports, the superposed solutions must be written in a piece-wise way.
- The components chosen are not unique; there are typically many ways to solve.
- For the above <u>indeterminate</u> beam, the reaction force F_0 is found by setting $v(L) = 0$.

Using Castigliano's 2nd Theorem

Notation for generalized displacements and generalized loads

- Displacements and forces: Δ and P
- Shaft rotations and torques: ϕ and T
- Beam rotations and bending moments: θ and M
- Redundant loads: R (forces, torques, and/or bending moments)
- Dummy loads: F_d (forces, torques, and/or bending moments)

Finding reactions for indeterminate structures

1. Draw FBD(s) and write down the equilibrium equations.
2. Determine the number of redundant loads N_R, and choose your redundant loads:
$$R_i; \quad i = 1, 2, \dots, N_R$$
3. Using the equilibrium equations, write the non-redundant loads in terms of the redundant loads.
4. Write strain energy function U *in terms of only the redundant and applied loads*.
5. Apply Castigliano's theorem for indeterminate structures:
$$\frac{\partial U}{\partial R_i} = 0\,; \quad i = 1, 2, \dots, N_R$$
6. Solve Castigliano equations, along with equilibrium equations, to find the external reactions.

Finding displacements

1. Draw FBD(s) and write down the equilibrium equations. If no load at point or direction of desired displacement, then add dummy load F_d.
2. Is structure indeterminate? If so, use *Castigliano's theorem for indeterminate structures* (shown to the left) to find external reactions FIRST.
3. Write the strain energy function U for the structure. Substitute in the external reactions found either from equilibrium (determinate) or from Castigliano and equilibrium (indeterminate). *At this point, U should be in terms of only applied loads, and possibly dummy loads*.
4. Apply Castigliano's theorem for determining the generalized displacements:
$$\Delta_i = \frac{\partial U}{\partial P_i}$$
$$\phi_i = \frac{\partial U}{\partial T_i}$$
$$\theta_i = \frac{\partial U}{\partial M_i}$$
Set any dummy load $F_d = 0$ at this point.

ME 323 – Mechanics of Materials
Midterm #2
March 30th, 2022

PURDUE UNIVERSITY | School of Mechanical Engineering

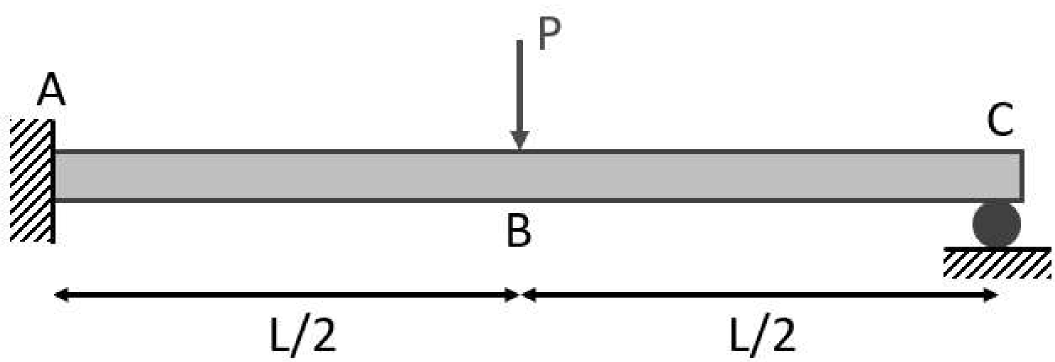
Name (Print) ____

(Last) **(First)**

PROBLEM # 3 (25 points)

Cantilever ABC has a Young's modulus of E and a second area moment of I. The cantilever is supported by a roller at C and a force of P is applied at B (L/2). Assume that deformation energy due to shear is negligible.

a) Determine the reactions using Castigliano's Second Theorem.
b) Determine the angle at point C using Castigliano's Second Theorem.



Name (Print) ____

(Last) **(First)**

PROBLEM # 3 CONT.

PURDUE UNIVERSITY | School of Mechanical Engineering

Name (Print) ____
(Last) **(First)**

PROBLEM # 3 CONT.

PROBLEM #3 (25 points)

A torque T is applied to a gear-shaft system and is transmitted through rigid gears B and C to a fixed end E as shown in Fig. 3(a). The shafts (1) and (2) are tightly fit to each other. Frictionless bearings are used to support the shafts. The geometry and material property of the shafts and gears are listed in the following table.

	Size	Length	Shear modulus
Shaft (1)	Outer diameter = 2*d* Inner diameter = *d*	*L*	2*G*
Shaft (2)	Diameter = *d*	*L*	*G*
Shaft (3)	Diameter = *d*	*L*/2	*G*
Shaft (4)	Diameter = *d*	*L*	*G*
Gear B	Diameter = 1.5*d*	Negligible	Rigid
Gear C	Diameter = 3*d*	Negligible	Rigid

(a) Determine the torque carried by each shaft.
(b) Determine the angle of twist at the free ends A and D.
(c) Consider the cross section *aa*' for the shafts (1) and (2), show the magnitude of the shear stress as a function of the distance from the center on Fig. 3(b). Mark the critical values in the diagram.
(d) Consider the points M and N on the cross section *aa*', shown in Fig. 3(c). Sketch the stress states at M and N on the stress elements on Fig. 3(d).

Express all your answers in terms of d, L, G, T, π.

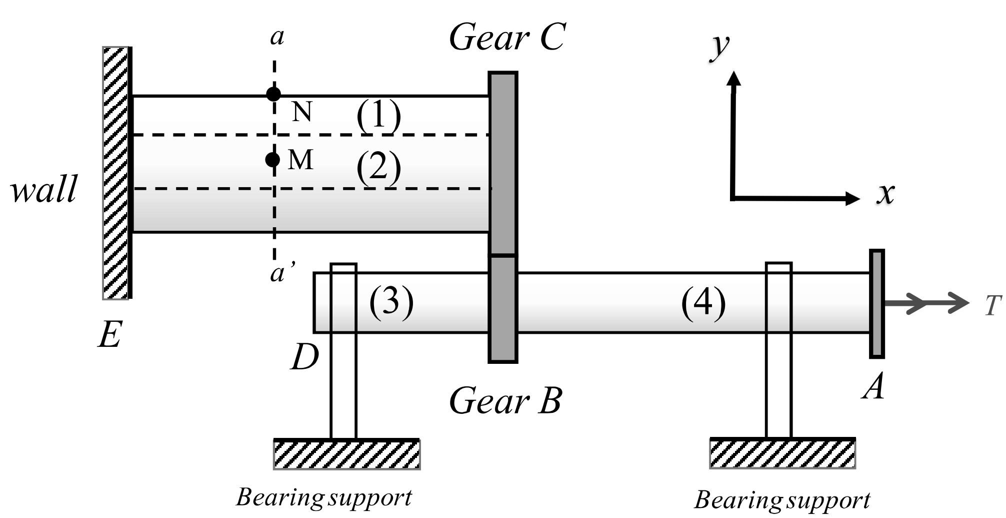


Fig. 3(a)

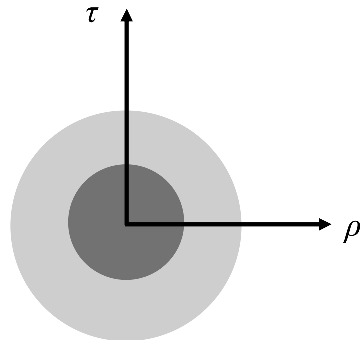


Fig. 3(b)

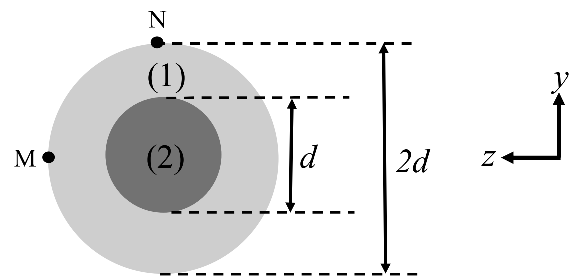


Fig. 3(c)

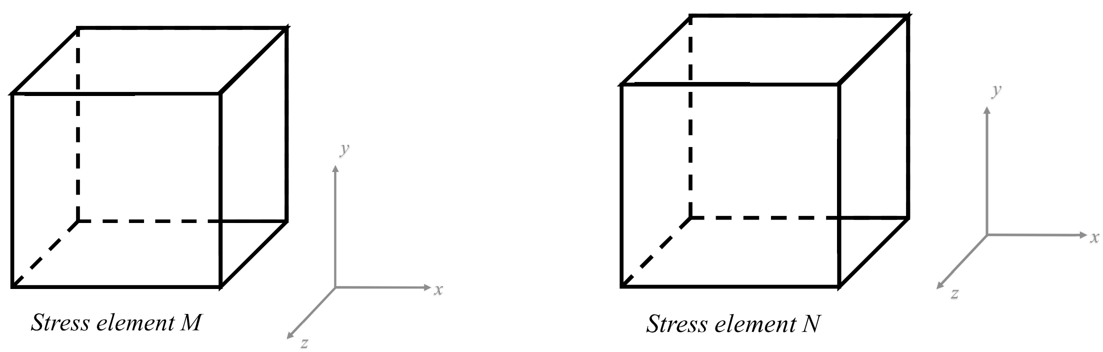


Fig. 3(d)

ME 323 – Mechanics of Materials
Exam 2
April 3, 2024

PURDUE UNIVERSITY | School of Mechanical Engineering

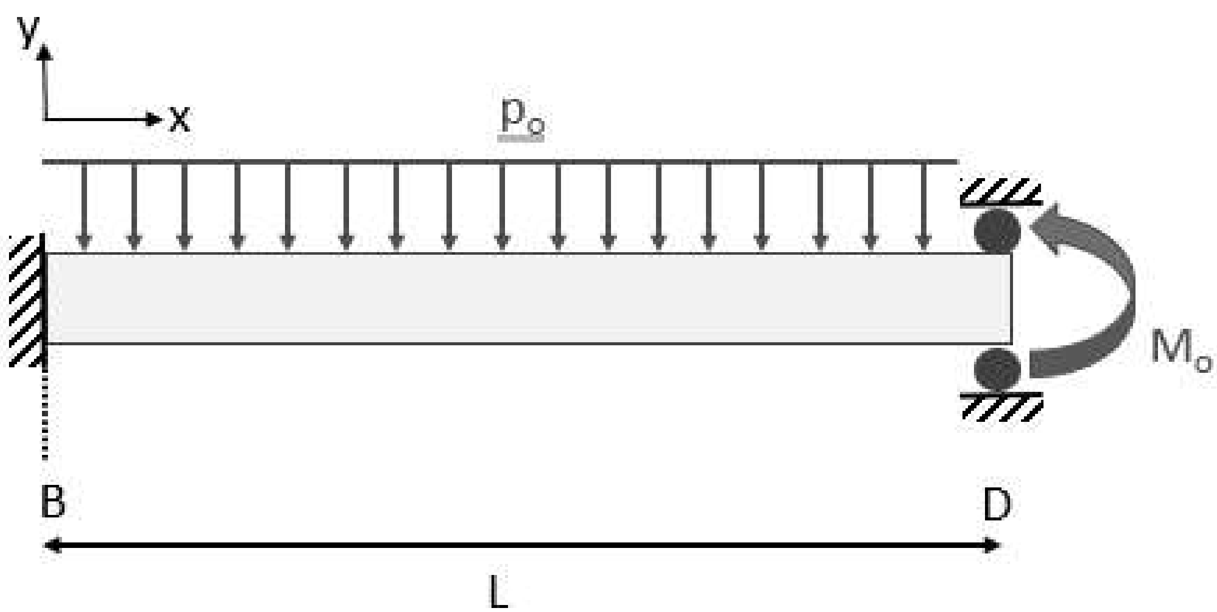
Name (Print) ____

(Last) **(First)**

PROBLEM #1 (25 points)

The beam BD has a distributed load p_o acting between B and D and a concentrated moment M_o applied at D as shown. The beam is fixed at B and is supported by a roller at D. E and I are constant along the beam. p_o has a value of 16P/L and M_o has a value of 2PL. Use **the second-order integration method** to answer the following:

$p_o = 16P/L$ $\quad\quad$ $M_o = 2PL$



a) Draw the FBD for the beam BD.

Name (Print) ____

(Last) (First)

b) Write the equilibrium equations for the beam BD.

c) Calculate the reactions on the beam at B and D in terms of *P and L*.

PURDUE UNIVERSITY | School of Mechanical Engineering

Name (Print) ____

(Last) **(First)**

d) Find the equation for the vertical displacement, $v(x)$ throughout the beam in terms of P, L, E, and I.

Name (Print) ____

(Last) (First)

e) Find the slope at point D in terms of *P, L, E, and I*.

PROBLEM #4 – PART A (6 points)

Beam (i) and (ii) are identical cylindrical beams except that beam (i) is made of steel and beam (ii) is made of aluminum. $E_{steel} > E_{aluminum}$.

Beam (a) Steel

Beam (b) Aluminum

(a) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$|\tau_{max,a}| < |\tau_{max,b}|$

$|\tau_{max,a}| = |\tau_{max,b}| = 0$

$|\tau_{max,a}| = |\tau_{max,b}| \neq 0$

$|\tau_{max,a}| > |\tau_{max,b}|$

(b) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$|\sigma_{max,a}| < |\sigma_{max,b}|$

$|\sigma_{max,a}| = |\sigma_{max,b}| = 0$

$|\sigma_{max,a}| = |\sigma_{max,b}| \neq 0$

$|\sigma_{max,a}| > |\sigma_{max,b}|$

(c) Circle the correct relationship between the maximum deflection $v(x)$ in the two beams (**1 point**).

$|v_{max,a}| < |v_{max,b}|$

$|v_{max,a}| = |v_{max,b}| = 0$

$|v_{max,a}| = |v_{max,b}| \neq 0$

$|v_{max,a}| > |v_{max,b}|$

(d) The diameter of the original beams is D. If the diameter is doubled to 2D, how will the new deflection of the new beam (v^*_{max}) with diameter of 2D compare to the deflection of the original beam (v_{max}) with diameter of D (**3 points**):

$v^*_{max} = v_{max}$

$v^*_{max} = 2v_{max}$

$v^*_{max} = 4v_{max}$

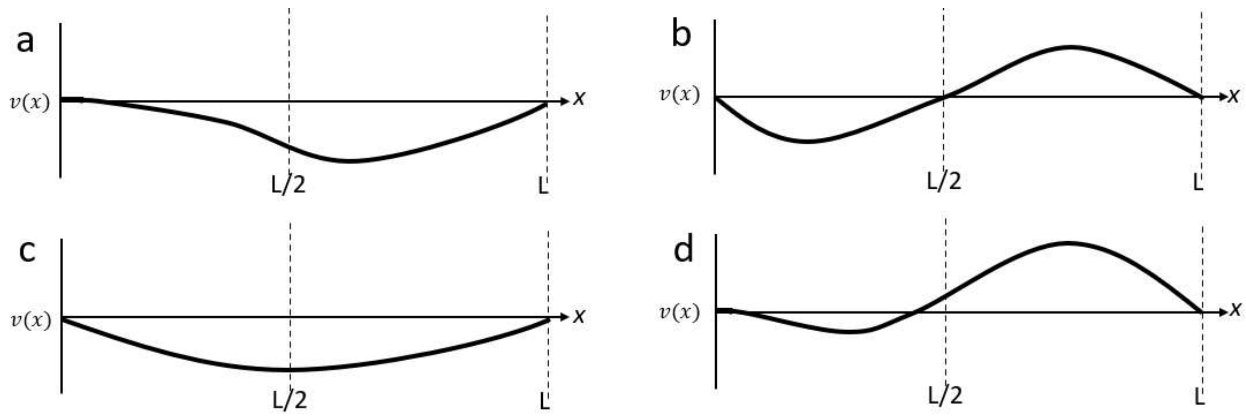
$v^*_{max} = 8v_{max}$

$v^*_{max} = 16v_{max}$

 Name (Print)________________________________

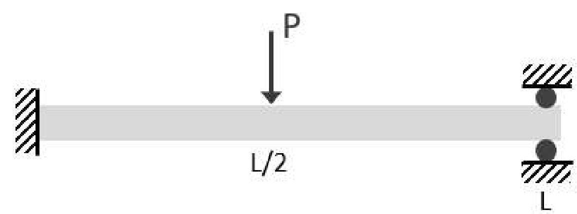
PROBLEM 4 – PART B (6 points)

Figures a-d indicate the deflection curve along four different beams.



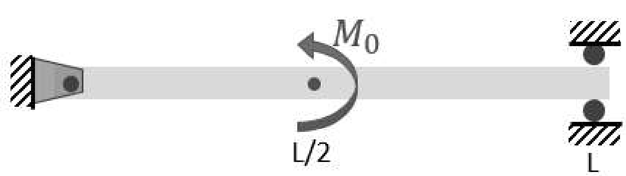
(i) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a b c d



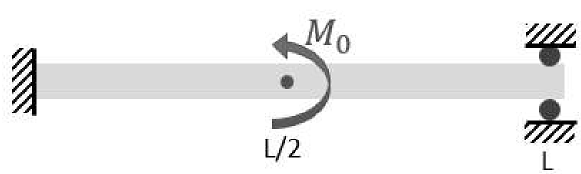
(ii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a b c d



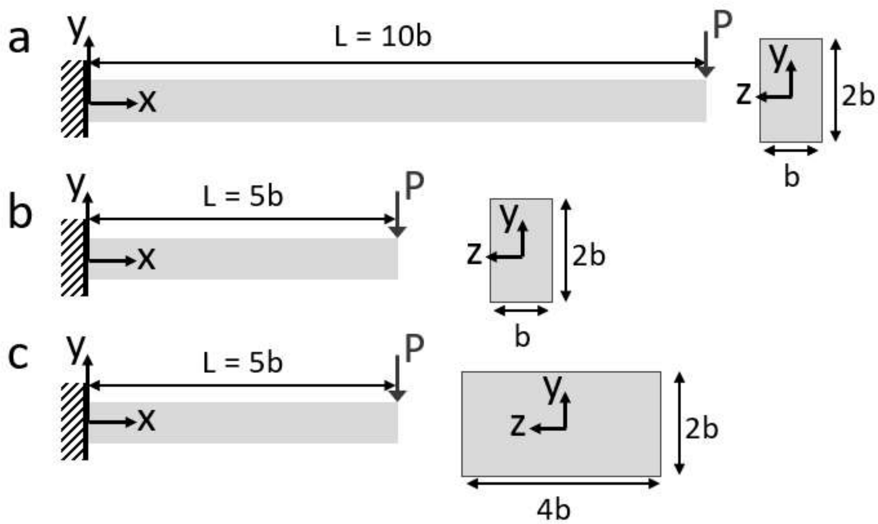
(iii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a b c d



PROBLEM 4 – PART C (2 points)

Based on the assumptions used when deriving the equation for shear stress on a beam cross-section ($\tau = VQ/It$), choose the correct ranking for the accuracy of the shear stress predicted by this equation for the three beams shown below:

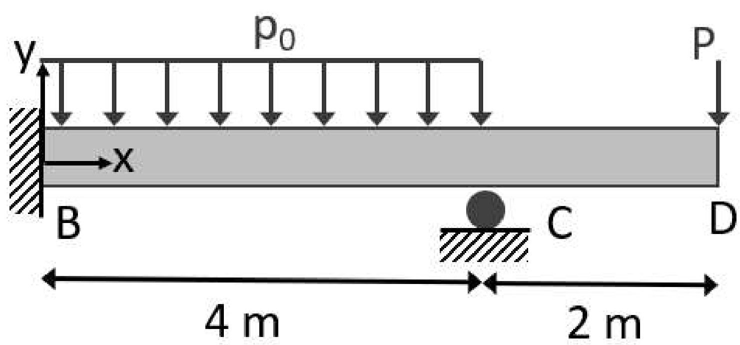


	Option 1	Option 2	Option 3	Option 4	Option 5
Most accurate	a	a	b	c	All have the same accuracy
	b	c	a	b	
Least accurate	c	b	c	a	

 Name (Print)______________________________________

PROBLEM 4 – PART D (6 points)

A beam is loaded with a distributed load from 0 to 4 m and a point load at 6 m.



Circle the value(s) that will be zero at x = 0m (**2 points**):

V(0) M(0) θ(0) *v*(0)

Circle the value(s) that will be zero at x = 4m (**2 points**):

V(4) M(4) θ(4) *v*(4)

Circle the value(s) that will be zero at x = 6m (**2 points**):

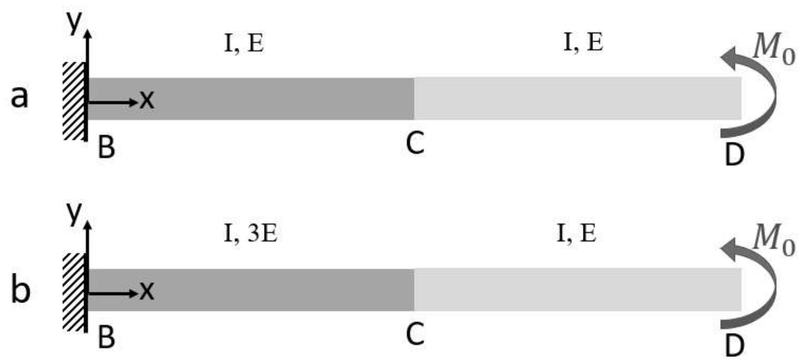
V(6) M(6) θ(6) *v*(6)

Exam 2

March 29, 2023 Name (Print)____________________________________

PROBLEM 4 – PART E (5 points)

A simple cantilever is composed of two sections with an applied moment at the end.



(i) **(3 points)** In beam (a), the two sections both have the same Young's moduli of E. In beam (b), one of the sections has a Young's modulus of 3E, while one has a Young's modulus of E. How does the total strain energy of these two beams compare?:

$U_{total,a} > U_{total,b}$

$U_{total,a} = U_{total,b}$

$U_{total,a} < U_{total,b}$

(ii) **(2 points)** Circle the loading condition below (c to f) that would be used if we want to calculate the deflection at point C in the y-direction.