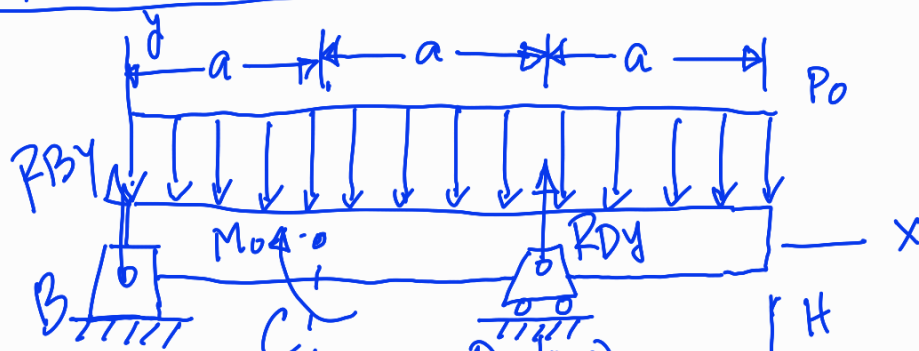


problem 6.1:



$$P_0 = 4 \text{ kN/m}$$

$$M_0 = 5 \text{ kN.m}$$

$$a = 2 \text{ m}$$

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$0 = -P_0 \cdot 3a + R_{By} + R_{Dy}$$

$$0 = -4 \cdot 3 \cdot 2 + R_{By} + R_{Dy}$$

$$\therefore R_{By} + R_{Dy} = 24 \text{ kN}$$

$$\sum M_D = 0$$

$$M_0 + R_{By} \cdot 2a -$$

$$P_0(3a) \frac{a}{2} = 0$$

$$5 + R_{By} \cdot 4 - 4(6) \frac{2}{2} = 0$$

$$R_{By} = \frac{24-5}{4}$$

$$R_{By} = \frac{19}{4} = 4.75 \text{ kN}$$

$$R_{Dy} = 19.25 \text{ kN}$$

Here:

$$V(x) = \begin{cases} R_{By} - P_0 x & [0 \leq x \leq 2a] \\ R_{By} - P_0 x + R_{Dy} & [2a \leq x \leq 3a] \end{cases}$$

$$\text{For } x=1: M(x) = 4.75 - \frac{4}{2} = 2.75 \text{ kNm}$$

$$x=2: M(x) = 4.75 - \frac{4}{2}(2) = 4.75 - 4 = 0.75$$

$$= 0.75 - 8 = -7.25$$

$$x=1.1875$$

$$\frac{d}{dx} M(x) = R_{By} - \frac{1}{2} \cdot 2 P_0 x$$

$$x=3: M(x) = 4.75(3) - \frac{4}{2}(3)^2 + 5 = 11.25 - 18 + 5 = -1.75$$

$$x=4: M(x) = 4.75(4) - \frac{4}{2}(4)^2 + 5 = 19 - 32 + 5 = -7.25$$

$$M(x) = R_{By}x - \frac{1}{2} P_0 x^2$$

$$[0 \leq x < a]$$

$$M(x) = R_{By}x - \frac{1}{2} P_0 x^2 + M_0$$

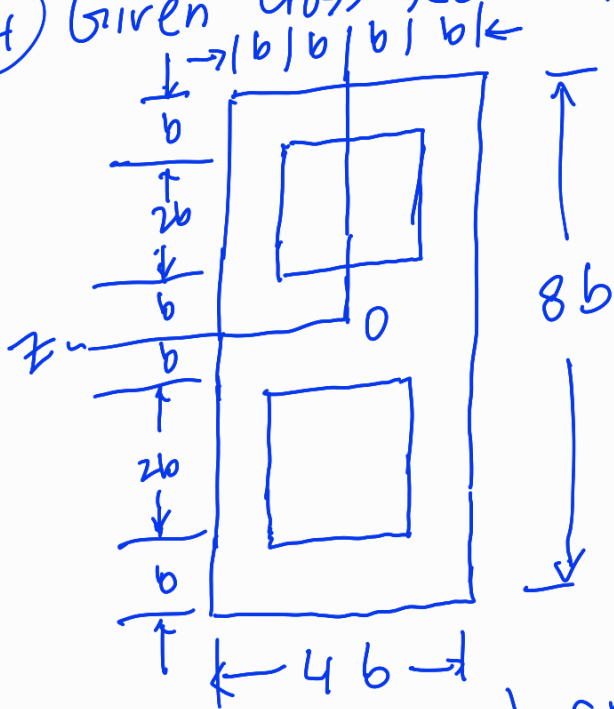
$$[a \leq x < 2a]$$

$$M(x) = \left(R_{By}x - \frac{1}{2} P_0 x^2 + M_0 \right) + R_{Dy}(x-2a)$$

$$\text{For } x=5: M(x) = 4.75(5) - \frac{4}{2} \cdot (5)^2 + 5 + 19.25(5-2 \times 2) = -2.0$$

$$\text{For } x=6: M(x) = 4.75(6) - \frac{4}{2}(6)^2 + 5 + 19.25(6-2 \times 2) = 0$$

(4) Given cross section of beam:



$$b = 0.02 \text{ m}$$

$$H = 8b$$

$$W = 4b$$

Centroidal second area moment; I

$$\text{Net Area, } A_{\text{total}} = 4b \times 8b = 32b^2$$

Second moment of area about neutral axis:

$$I = I_{\text{outer}} - 2 \left(I_{\text{square, centroid}} + A_{\text{square}} d^2 \right)$$

$$= \frac{4b(8b)^3}{12} - 2 \cdot \left[\frac{(2b)(2b)^3}{12} + (2b)(2b)(2b)^2 \right]$$

$$= \frac{2048b^4}{12} - 2 \cdot \left[\frac{16b^4}{12} + 16b^4 \right]$$

$$I = \frac{2048b^4}{12} - 2 \cdot \left(\frac{16b^4 + 192b^4}{12} \right) = \frac{1632b^4}{12} = 136b^4 = 2.176 \times 10^{-5} \text{ m}^4$$

$$\therefore I = 2.176 \times 10^{-5} \text{ m}^4$$

5) Maximum Magnitude of Flexural Normal Stress:

$$\sigma_{\max} = \frac{M_{\max} c}{I} = \frac{M \cdot (4b)}{136b^4} = \frac{M}{34b^3} = \frac{8000 \text{ Nm}}{34(0.02)^3 \text{ m}^3}$$

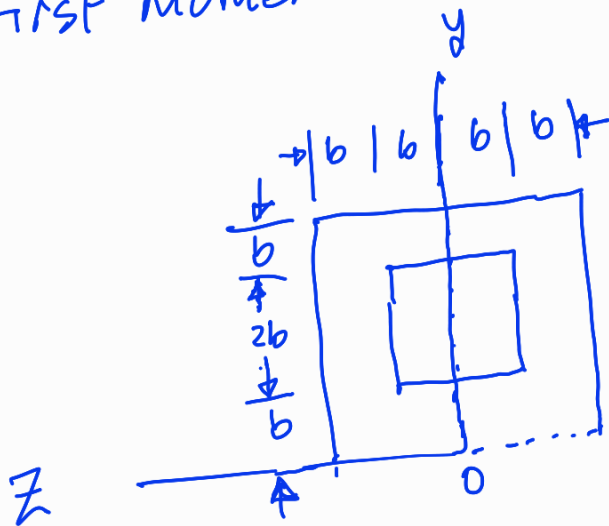
$$M_{\max} = -8 \text{ kN}\cdot\text{m} \text{ [at } x = 4 \text{ m]}$$

$$\sigma_{\max} = 29.41 \times 10^6 \text{ Pa} = 29.41 \text{ MPa} \approx 29.5 \text{ MPa}$$

6) Maximum magnitude of shear stress on the neutral plane:

$$\tau_{\max} = \frac{V_{\max} Q}{I t} \quad \left[V_{\max} = -11.25 \text{ kN at } x = 4.0 \text{ m} \right]$$

here, First moment of area above the neutral axis



$$\begin{aligned} W &= 4b \\ H &= 8b \end{aligned}$$

$$Q_{\text{top}} = A_{\text{top}} \bar{y}_{\text{top}}$$

$$\begin{aligned} A_{\text{top}} &= W \cdot \frac{H}{2} = 4b \cdot 4b = 8b^2 \\ \bar{y}_{\text{top}} &= \frac{H}{4} = 2b \end{aligned}$$

$$O_{\text{top}} = A_{\text{top, outer}} \cdot y_{\text{top, outer}} - A_{\text{top, hole}} \cdot d$$

$$= (16b^2) \cdot 2b - (2b)^2 \cdot 2b$$

$$= 32b^3 - 8b^3$$

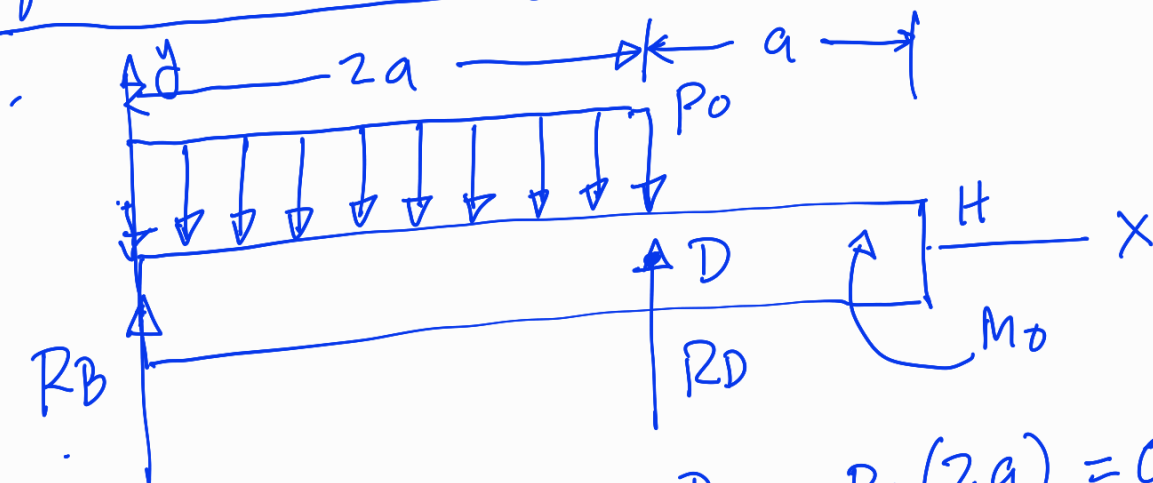
$$= 24b^3$$

$$\tau_{\text{max}} = \frac{(V_{\text{max}})_{\text{Q}}}{I \cdot t} = \frac{11.25 \times 10^3 \text{ N} \cdot 24 \cdot b^3 \text{ m}^3}{136b^4 \cdot 4b \text{ m}^5} = 1.241 \times 10^6 \text{ N/m}^2$$

$$\tau_{\text{max}} = 1.24 \text{ MPa}$$

Problem 6.2:

① Free Body Diagram (FBD):



$$+\uparrow \sum F_y = 0$$

$$+\circlearrowleft \sum M_B = 0$$

$$\therefore R_B + R_D - P_0(2a) = 0$$

$$\therefore -R_D(2a) + P_0(2a) \cdot a + M_0 = 0$$

$$R_D = \frac{2P_0 \cdot a^2 + M_0}{2a} = P_0 a + \frac{M_0}{2a}$$

$$R_B = P_0(2a) - R_D = 2P_0 a - P_0 a - \frac{M_0}{2a}$$

$$R_B = P_0 a - \frac{M_0}{2a}$$

$$R_D = P_0 a + \frac{M_0}{2a}$$

② Determinate / Indeterminate:

Since R_B and R_D unknown and we have

2 equations, The problem is

Statically determinate

(3) Deflection by Second order Integration:

For $0 \leq x \leq 2a$:

$$M_1(x) = R_B x - \frac{P_0 x^2}{2} \quad [0 \leq x \leq 2a]$$

For $2a \leq x \leq 3a$:

$$M_2(x) = R_B x - P_0 \cdot 2a(x-a) + R_D(x-2a)$$

$$R_D = P_0 a + \frac{M_0}{2a}$$

$$2a R_D = 2P_0 a^2 + M_0$$

$$\boxed{M_0 = 2R_D a - 2P_0 a^2}$$

$$= (R_B + R_D)x - 2P_0 a x + 2P_0 a^2 - 2R_D a$$

$$= 2P_0 a x - 2P_0 a x + 2P_0 a^2 - 2R_D a$$

$$= 2P_0 a^2 - 2R_D a$$

$$M_2(x) = -M_0 \quad [2a \leq x \leq 3a]$$

Using Double Integration method:

$$EI \psi'' + M(x) = 0$$

$$\psi_1''(x) = -\frac{M_1(x)}{EI}, \quad 0 \leq x \leq 2a$$

$$\psi_2''(x) = \frac{M_0}{EI}, \quad 2a \leq x \leq 3a$$

Integrating:

$$v_1''(x) = - \frac{R_B x - \frac{1}{2} P_0 x^2}{EI} \quad [0 \leq x \leq 2a]$$

$$v_1'(x) = \frac{1}{EI} \left[\frac{1}{2} P_0 \frac{x^3}{3} - R_B \frac{x^2}{2} \right] + C_1$$

$$v_1(x) = \frac{1}{EI} \left[\frac{1}{6} P_0 \cdot \frac{x^4}{4} - \frac{1}{2} R_B \frac{x^3}{3} \right] + C_1 x + C_2$$

$$v_1(x) = \frac{1}{EI} \left[\frac{P_0 x^4}{24} - \frac{R_B x^3}{6} \right] + C_1 x + C_2 \quad \dots \text{--- (1)}$$

$$v_2''(x) = \frac{M_0}{EI} \quad [2a \leq x \leq 3a]$$

$$v_2'(x) = \frac{M_0}{EI} x + C_3$$

$$v_2(x) = \frac{M_0}{EI} \frac{x^2}{2} + C_3 x + C_4 \quad \dots \text{--- (11)}$$

C_1, C_2, C_3 and C_4 are constants of integration and they are unknown.

Applying Physical Boundary Conditions:

a. Support at B:

$$v_1(0) = 0 \quad \therefore C_2 = 0$$

b. Support at D:

$$v_2(2a) = 0$$

$$\frac{M_0}{2EI} (2a)^2 + C_3 \cdot 2a + C_4 = 0$$

$$2 \frac{M_0 a^2}{EI} + 2a \cdot C_3 + C_4 = 0 \quad \dots \textcircled{III}$$

c. Continuity of displacement at $x = 2a$:

$$v_1(2a) = v_2(2a) \quad [C_2 = 0]$$

$$\frac{1}{EI} \left[\frac{1}{24} P_0 (2a)^4 - \frac{1}{6} R_B (2a)^3 \right] + C_1 (2a) = 0$$

$$C_1 = - \frac{1}{EI} \cdot \left[\frac{P_0 (2a)^3}{24} - \frac{R_B (2a)^2}{6} \right]$$

$$C_1 = \frac{1}{EI} \cdot \left[\frac{R_B \cdot 4a^2}{6} - \frac{P_0 \cdot 8a^3}{24} \right]$$

$$C_1 = \frac{1}{EI} \cdot \left[\frac{2}{3} R_B a^2 - \frac{P_0 \cdot a^3}{3} \right]$$

$$C_1 = \frac{a}{3EI} [2R_B a - P_0 a^2]$$

now,

$$R_B = P_0 a - \frac{M_0}{2a}$$

$$2aR_B = 2P_0 a^2 - M_0$$

$$\therefore C_1 = \frac{a}{3EI} (2P_0 a^2 - P_0 a^2 - M_0)$$

$$C_1 = \frac{(P_0 a^2 - M_0) a}{3EI}$$

$$C_1 = \frac{-M_0 a + a^3 P_0}{3EI}$$

d. Continuity of slope at $x=2a$:

$$v_1'(2a) = v_2'(2a)$$

$$\frac{1}{EI} \left[\frac{1}{6} P_0 (2a)^3 - \frac{1}{2} R_B (2a)^2 \right] + C_1 = \frac{M_0}{EI} 2a + C_3$$

$$\frac{1}{EI} \left[\frac{8P_0 a^3}{6} - \frac{4a^2 R_B}{2} \right] + \frac{a^3 P_0 - M_0 a}{3EI} = \frac{2a M_0}{EI} + C_3$$

$$\frac{1}{EI} \left[\frac{4}{3} P_0 a^3 - 2a^2 R_B \right] + \frac{a^3 P_0 - M_0 a}{3EI} = \frac{2M_0 a}{EI} + C_3$$

$$\frac{1}{3EI} [4Pa^3 - 6a^2 RB] + \frac{a^3 P_0 - Moa}{3EI} = \frac{2Mo a}{EI} + C_3$$

$$\frac{1}{3EI} [4Pa^3 - 3a(2a RB)] + \frac{a^3 P_0 - Moa}{3EI} = \frac{2Mo a}{EI} + C_3$$

$$\frac{1}{3EI} [4Pa^3 - 3a(2Pa^2 - Mo)] + \frac{a^3 P_0 - Moa}{3EI} = \frac{6Mo a}{3EI} + C_3$$

$$\frac{1}{3EI} [4Pa^3 - 6Pa^3 + 3Mo a + Pa^3 - Moa - 6Mo a] = C_3$$

$$C_3 = \frac{1}{3EI} [-Pa^3 - 4Mo a]$$

$$C_3 = - \left(\frac{Pa^3 + 4Mo a}{3EI} \right)$$

From (iii) substituting C_3 :

$$2 \frac{Mo a^2}{EI} + (2a) \left(- \frac{Pa^3 + 4Mo a}{3EI} \right) + C_4 = 0$$

$$C_4 = \frac{+2a^4 P_0 + 8Mo a^2 - 6Mo a^2}{3EI}$$

$$C_4 = \frac{2Mo a^2 + 2a^4 P_0}{3EI}$$

Substituting these constants into the equations:
 (i) and (ii):

$$\begin{aligned}
 v_1(x) &= \frac{1}{EI} \left[\frac{P_0 x^4}{24} - \frac{R_B x^3}{6} \right] + \frac{a^3 P_0 - M_0 a}{3 EI} x + 0 \\
 &= \frac{1}{EI} \left[\frac{P_0 x^4}{24} - \frac{\left(P_0 a - \frac{M_0}{2a} \right) x^3}{6} \right] + \frac{a^3 P_0 - M_0 a}{3 EI} x \\
 &= \frac{1}{24 EI} \left[P_0 x^4 - 4 \cdot P_0 a x^3 + \frac{4 M_0}{2a} x^3 + 8 a^3 P_0 x - 8 M_0 a x \right] \\
 &= \frac{1}{24 EI} \left[P_0 x^4 - 4 P_0 a x^3 + \frac{2 M_0 x^3}{a} + 8 P_0 a^3 x - 8 M_0 a x \right] \\
 &= \frac{1}{24 EI} \left[\frac{a P_0 x^4 - 4 P_0 a^2 x^3 + 2 M_0 x^3 + 8 P_0 a^4 x - 8 M_0 a^2 x}{a} \right] \\
 &= \frac{1}{24 EI a} \left[a P_0 x^4 - 4 P_0 a^2 x^3 + 2 M_0 x^3 + 8 P_0 a^4 x - 8 M_0 a^2 x \right] \\
 v_1(x) &= \frac{x \left[(8 P_0 a^4 - 8 M_0 a^2) + 2 x^2 (M_0 - 2 P_0 a^2) + P_0 a x^3 \right]}{24 EI a}
 \end{aligned}$$

$$\therefore v_1(x) = \frac{x \left[(8 P_0 a^4 - 8 M_0 a^2) + 2 x^2 (M_0 - 2 P_0 a^2) + P_0 a x^3 \right]}{24 EI a}$$

Similarly,

$$v_2(x) = \frac{M_0}{EI} \frac{x^2}{2} + (-x) \left(\frac{P_0 a^3 + 4M_0 a}{3EI} \right) + \left(\frac{2M_0 a^2 + 2a^4 P_0}{3EI} \right)$$

$$v_2(x) = \frac{3M_0 x^2 - 2P_0 a^3 x - 8M_0 a x + 4M_0 a^2 + 4a^4 P_0}{6EI}$$

$$v_2(x) = \frac{4M_0 a^2 + 3M_0 x^2 + 4a^4 P_0 - 2ax(4M_0 - P_0 a^2)}{6EI}$$

Now, for a complete solution: $[0 \leq x \leq 3a]$

$$v(x) = v_1(x) [0 \leq x \leq 2a] + v_2(x) [2a \leq x \leq 3a]$$

④ Rotation of the Beam Deflection curve:

At support D:

$$\theta_2'(2a) = \frac{M_0(2a)}{EI} + - \left(\frac{P_0 a^3 + 4M_0 a}{3EI} \right)$$

$$= \frac{6M_0 a - P_0 a^3 + 4M_0 a}{3EI}$$

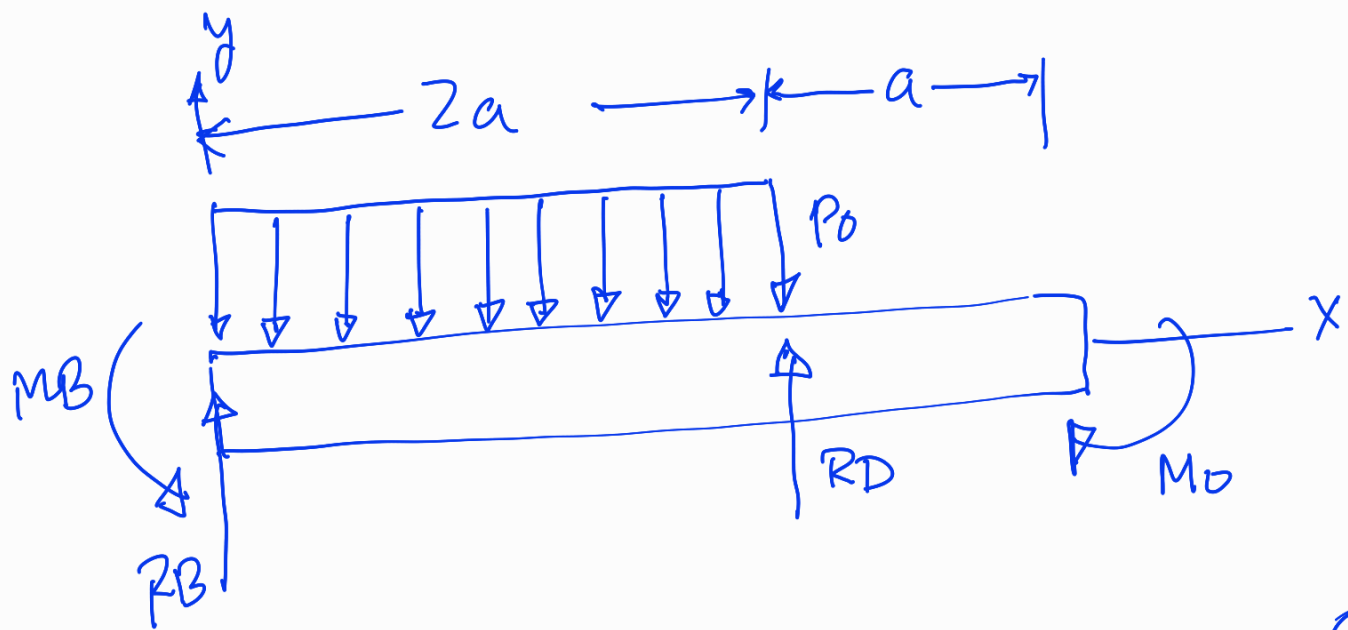
$$\theta_2'(2a) = \frac{a(2M_0 - a^2 P_0)}{3EI}$$

$\therefore$ Rotation at D:

$$\theta_D = \frac{a(2M_0 - a^2 P_0)}{3EI}$$

problem 6.3 :

① Free Body Diagram (FBD) :



$$+\uparrow \sum F_y = 0 \quad \therefore R_B + R_D = P_0 \cdot 2a \quad \dots \textcircled{I}$$

$$+\circlearrowleft \sum M_B = 0 \quad \therefore -R_D(2a) + P_0(2a) \cdot a + M_D - M_B = 0$$

$$M_D - M_B - 2a(R_D - P_0 a) = 0 \quad \dots \textcircled{II}$$

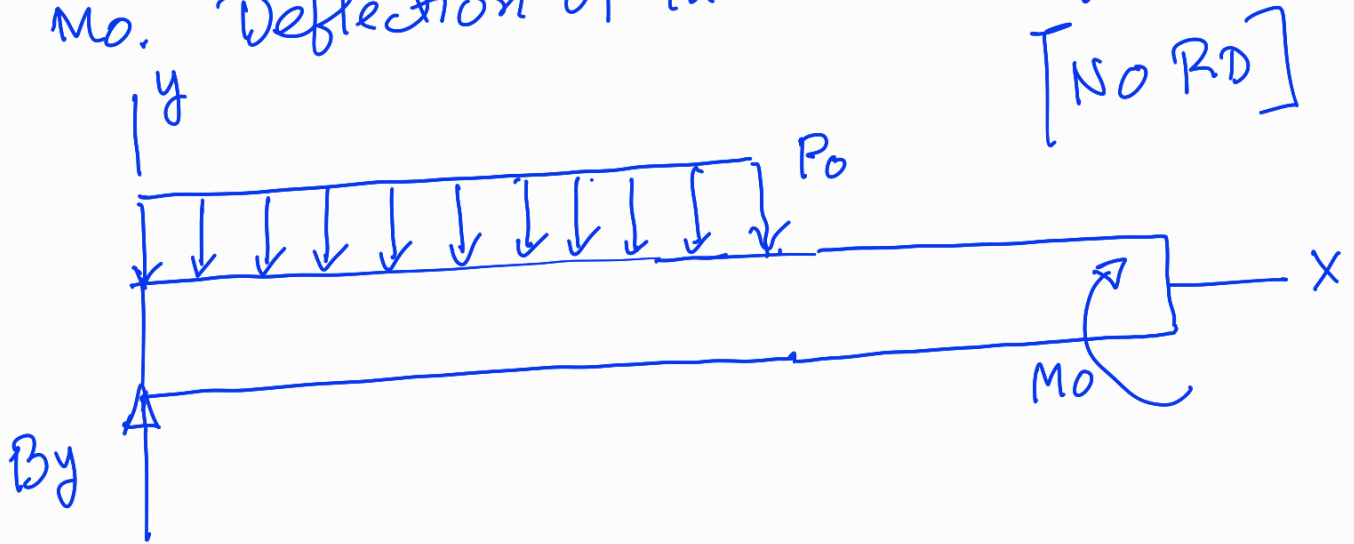
② Determinate/Indeterminate :

Since there are 2 equations, 3 unknowns, $(R_B, R_D \text{ and } M_B)$. The problem is statically indeterminate

③ Reactions acting at B and D:

Step 1 - Cantilever primary structure:

The vertical reaction R_D at D is redundant, Thus, the structure has distributed load P_0 and moment M_0 . Deflection of the beam, $v_{\text{free}}(x)$



Taking internal Bending Moment for the cantilever:

$$M_{\text{free}}(x) = - \left[M_0 + \frac{1}{2} P_0 (2a - x)^2 \right] \quad [0 \leq x \leq 2a]$$

$$M_{\text{free}}(x) = - M_0 \quad [2a \leq x \leq 3a]$$

Using the 2nd order integration approach:

$$EI v_{\text{free}}''(x) = - M_{\text{free}}(x)$$

With Fixed Boundary Conditions

$$v_{\text{free}}(0) = 0$$

$$v_{\text{free}}'(0) = 0$$

$$M_{\text{free}}(x) = -\left[M_0 + \frac{1}{2} P_0 (2a-x)^2\right]$$

$$v''_{\text{free}}(x) = -\frac{M_{\text{free}}(x)}{EI} = \frac{M_0 + \frac{1}{2} P_0 (2a-x)^2}{EI}$$

Integrating and imposing $v'(0) = 0$

$$v'_{\text{free}}(x) = \int \frac{M_0 + \frac{1}{2} P_0 (4a^2 - 4ax + x^2)}{EI} dx$$

$$v'_{\text{free}}(x) = \frac{1}{EI} \cdot \left[M_0 x + 2 \cdot P_0 a x^2 - 2 P_0 a \frac{x^2}{2} + \frac{P_0 x^3}{3} \right]$$

$$v'_{\text{free}}(x) = \frac{1}{EI} \cdot \left[(M_0 + 2P_0 a^2)x - P_0 a x^2 + \frac{P_0 x^3}{3} \right]$$

Integrating and imposing $v(0) = 0$

$$v_{\text{free}}(x) = \frac{1}{EI} \left[\frac{1}{2} (M_0 + 2P_0 a^2) x^2 - P_0 \frac{a x^3}{3} + \frac{P_0 x^4}{24} \right]$$

$$[0 \leq x \leq 2a]$$

Checking at $x = 2a$:

$$v_{\text{free}}(2a) = \frac{1}{EI} \left[\frac{1}{2} (M_0 + 2P_0 a^2) 4a^2 - \frac{P_0 a \cdot 8a^3}{3} + \frac{P_0 \cdot 16a^4}{24} \right]$$

$$\begin{aligned}
 &= \frac{1}{EI} \left[2M_0 a^2 + 4P_0 a^4 - \frac{8}{3} P_0 a^4 + \frac{2}{3} P_0 a^4 \right] \\
 &= \frac{1}{EI} \left[2M_0 a^2 + 4P_0 a^4 - 2P_0 a^4 \right] \\
 &= \frac{1}{EI} \left[2M_0 a^2 + 2P_0 a^4 \right]
 \end{aligned}$$

$$v_{\text{free}}(2a) = \frac{2a^2 (M_0 + P_0 a^2)}{EI}$$

Step 2: Unit-load deflection at D - $v_1(2a)$

Applying an upward point load $P=1$ at $x=2a$ on the primary cantilever. Similar double integration procedure $\rightarrow$

$$M_1(x) = 2a - x \quad [0 \leq x \leq 2a]$$

Integrate:

$$EI v_1''(x) = -M_1(x) \quad \text{with } v_1(0) = 0$$

$$v_1''(x) = \frac{-2a - x}{EI}$$

Integrating with x and $v_1'(0) = 0$

$$v_1'(x) = \frac{1}{EI} \left[\frac{x^2}{2} - 2ax \right]$$

Integrating again with BC: $v_1(0) = 0$ enforced:

$$v_1(x) = \frac{1}{EI} \left[\frac{x^3}{6} - ax^2 \right] \quad [0 \leq x \leq 2a]$$

$$v_1(2a) = \frac{1}{EI} \left[\frac{8a^3}{6} - a \cdot 4a^2 \right]$$

$$= \frac{1}{EI} \left[\frac{4}{3}a^3 - 4a^3 \right]$$

$$\boxed{v_1(2a) = -\frac{8}{3} \frac{a^3}{EI}}$$

Step 3 - Solving for redundant R_D :

The compatibility condition requires actual net deflection at $x = 2a$ to be 0.

$$v_{\text{total}}(2a) = v_{\text{free}}(2a) + R_D v_1(2a) = 0$$

$$R_D = - \frac{v_{\text{free}}(2a)}{v_1(2a)}$$

$$R_D = - \frac{\frac{2a^2(M_0 + a^2 P_0)}{EI}}{-\frac{8a^3}{3EI}} = \frac{2a^2(M_0 + a^2 P_0)}{1} \cdot \frac{3}{8a^3}$$

$$R_D = \frac{3(M_0 + a^2 P_0)}{4a}$$

Step 4: Finding R_B and M_B

From (i):

$$R_B = 2P_0a - R_D = 2P_0a - \frac{3(M_0 + a^2 P_0)}{4a}$$

$$R_B = \frac{8P_0a^2 - 3M_0 - 3a^2 P_0}{4a}$$

$$R_B = \frac{5P_0a^2 - 3M_0}{4a} = \frac{5P_0a}{4} - \frac{3M_0}{4a}$$

From (ii):

$$\begin{aligned} M_B &= P_0(2a) \cdot a + M_0 - 2aR_D \\ &= 2P_0a^2 + M_0 - 2a \cdot \frac{3(M_0 + a^2 P_0)}{4a} \\ &= 2P_0a^2 + M_0 - \frac{3}{2}(M_0 + a^2 P_0) \end{aligned}$$

$$M_B = \frac{1}{2} [P_0a^2 - M_0]$$

$$M_B = \frac{P_0 a^2 - M_0}{2}$$

④ Rotation of the beam deflection at support D and deflection of beam at H :

For the true indeterminate beam:

$$v(x) = v_{\text{free}}(x) + R_D v_1(x)$$

For $0 \leq x \leq 2a$:

$$v_{\text{free}}(x) = \frac{1}{EI} \left[\frac{1}{2} (M_0 + 2P_0 a^2) x^2 - \frac{1}{3} P_0 a x^3 + \frac{1}{24} P_0 x^4 \right]$$

$$v_1(x) = \frac{1}{EI} \left[\frac{x^3}{6} - a x^2 \right]$$

$$v(x) = \frac{1}{EI} \left[\frac{1}{2} (M_0 + 2P_0 a^2) x^2 - \frac{1}{3} P_0 a x^3 + \frac{1}{24} P_0 x^4 \right] + \frac{R_D}{EI} \left[\frac{x^3}{6} - a x^2 \right]$$

For $2a \leq x \leq 3a$:

$$v_{\text{free}}''(x) = \frac{M_0}{EI}$$

Integrating v'' from $2a$ to x . Using continuity of slope at $x=2a$.

$$v_{\text{free}}'(x) = v_{\text{free}}'(2a) + \int_{2a}^x v_{\text{free}}''(s) ds + \int_{2a}^x \frac{M_0}{EI} ds$$

$$= v_{\text{free}}'(2a) + \frac{M_0}{EI} (x-2a)$$

Integrating again:

$$v_{\text{free}}(x) = v_{\text{free}}(2a) + \int_{2a}^x v_{\text{free}}'(s) ds$$

$$= v_{\text{free}}(2a) + \int_{2a}^x \left[v_{\text{free}}'(2a) + \frac{M_0}{EI} (s-2a) \right] ds$$

$$= v_{\text{free}}(2a) + v_{\text{free}}'(2a)(x-2a) + \frac{M_0}{EI} \frac{(x-2a)^2}{2}$$

$$v_{\text{free}}(x) = v_{\text{free}}(2a) + v_{\text{free}}'(2a)(x-2a) + \frac{M_0}{2EI} (x-2a)^2$$

From earlier expression:

$$v'_{\text{free}}(x) = \frac{1}{EI} \left[(M_0 + 2P_0 a^2)x - P_0 a x^2 + \frac{P_0}{6} x^3 \right]$$

$$v'_{\text{free}}(2a) = \frac{1}{EI} \left[(M_0 + 2P_0 a^2)2a - P_0 a(2a)^2 + \frac{P_0}{6}(2a)^3 \right]$$

$$= \frac{1}{EI} \left[2aM_0 + 4P_0 a^3 - P_0 4a^3 + \frac{P_0}{6} \cdot 8a^3 \right]$$

$$= \frac{1}{EI} \left[2aM_0 + \frac{4}{3}P_0 a^3 \right]$$

$$\boxed{v'_{\text{free}}(2a) = \frac{1}{EI} \left(2aM_0 + \frac{4}{3}P_0 a^3 \right)}$$

$$v_{\text{free}}(x) = v_{\text{free}}(2a) + \frac{1}{EI} \left(2aM_0 + \frac{4}{3}P_0 a^3 \right) (x-2a) + \frac{M_0}{2EI} (x-2a)^2$$

$$= v_{\text{free}}(2a) + \frac{1}{EI} \left[2aM_0(x-2a) + \frac{4}{3}P_0 a^3(x-2a) + \frac{M_0}{2EI} (x-2a)^2 \right]$$

$2a \leq x \leq 3a$:

$$v_{\text{free}}(x) = v_{\text{free}}(2a) + \frac{1}{EI} \left[\frac{M_0}{2} \{x^2 - (2a)^2\} + \frac{4}{3}P_0 a^3(x-2a) \right]$$

$$v_1(x) = v_1(2a) + \frac{1}{EI} \left[-\frac{1}{2} \{x^2 - (2a)^2\} \right] \quad [x \geq 2a]$$

$$v_{\text{free}}(x) = \frac{1}{EI} \left[2a^2(M_0 + aP_0) + \left(2aM_0 + \frac{4}{3}P_0a^3\right)(x-2a) + \frac{M_0}{2}(x-2a)^2 \right]$$

$$(x-2a)^2 = x^2 - 4ax + 4a^2$$

$$v_{\text{free}}(x) = \frac{1}{EI} \left[\frac{M_0}{2}x^2 - \frac{M_0}{2}4ax + \frac{M_0}{2}4a^2 + 2aM_0 + 2P_0a^2 + 2aM_0x - 4a^2M_0 + \frac{4}{3}P_0a^3x - \frac{8}{3}P_0a^4 \right]$$

$$= \frac{1}{EI} \left[\frac{M_0}{2}x^2 - 2M_0ax + 2M_0a^2 + 2M_0a^2 + 2P_0a^4 + 2M_0ax - 4M_0a^2 + \frac{4}{3}P_0a^3x - \frac{8}{3}P_0a^4 \right]$$

$$= \frac{1}{EI} \left[\frac{M_0x^2}{2} - \frac{2}{3}P_0a^4 + \frac{4}{3}P_0a^3x \right]$$

$$v_{\text{free}}(x) = \frac{1}{EI} \left(\frac{M_0x^2}{2} + \frac{4}{3}P_0a^3x - \frac{2}{3}P_0a^4 \right) \quad [2a \leq x \leq 3a]$$

$$v_{\text{free}}(2a) = \frac{1}{EI} \left\{ \frac{M_0(2a)^2}{2} + \frac{4}{3}P_0a^3(2a) - \frac{2}{3}P_0a^4 \right\}$$

$$= \frac{1}{EI} \left[2M_0a^2 + \frac{8}{3}P_0a^4 - \frac{2}{3}P_0a^4 \right]$$

$$= \frac{1}{EI} [2M_0a^2 + 2P_0a^4]$$

$$v_{\text{free}}(2a) = \frac{2a^2(M_0 + aP_0)}{EI}$$

$$v_{free}(x) = \frac{1}{EI} \left[\frac{1}{2} (M_0 + 2P_0 a^2) x^2 - \frac{1}{3} P_0 a x^3 + \frac{1}{24} P_0 x^4 \right],$$

$$v_{free}(2a) = \frac{2a^2 (M_0 + \tilde{a} P_0)}{EI} \quad [0 \leq x \leq 2a]$$

$$v_{free}(x) = \frac{1}{EI} \left[\frac{M_0}{2} x^2 + \frac{4}{3} P_0 a^3 x - \frac{2}{3} P_0 a^4 \right] \quad [2a \leq x \leq 3a]$$

$$v'_{free}(2a) = \frac{2a M_0 + \frac{4}{3} P_0 a^3}{EI}$$

$$v_1'(x) = -\frac{M_1(x)}{EI} = 0 \quad [2a \leq x \leq 3a]$$

$$v_1'(x) = v_1'(2a) = -\frac{2a^2}{EI}$$

$$v_1(x) = v_1(2a) + v_1'(2a)(x-2a)$$

$$= -\frac{8a^3}{3EI} + \left(-\frac{2a^2}{EI}\right)(x-2a)$$

$$= \frac{-8a^3 - 6a^2x + 12a^3}{3EI}$$

$$= \frac{4a^3 - 6a^2x}{3EI}$$

$$= \frac{1}{EI} \left[-2a^2x + \frac{4}{3}a^3 \right]$$

$$v_1(x) = \begin{cases} \frac{1}{EI} \left(-ax^2 + \frac{1}{6}x^3 \right), & 0 \leq x \leq 2a \\ \frac{1}{EI} \left(-2a^2x + \frac{4}{3}a^3 \right), & 2a \leq x \leq 3a \end{cases}$$

$$v(x) = v_{\text{free}}(x) + v_1(x)$$

$$v(x) = \frac{1}{EI} \left[\frac{m_0}{2} x^2 - \frac{q}{6} (9m_0 + P_0 a^2) x + a^2 \left(m_0 + \frac{1}{3} P_0 a^2 \right) \right]$$

$$[2a \leq x \leq 3a]$$

Again,

$$v(x) = \frac{1}{EI} \left[\frac{1}{2} (m_0 + 2P_0 a^2) x^2 - \frac{1}{3} P_0 a x^3 + \frac{1}{24} P_0 x^4 \right] +$$

$$\frac{P_0 D}{EI} \left[-a x^2 + \frac{1}{6} x^3 \right] \quad [0 \leq x \leq 2a]$$

These are the equations for deflection at support D.

At Cantilever tip: ($x=3a$)

$$v_{free}(3a) = \frac{1}{EI} \left[\frac{M_0}{2} (3a)^2 + \frac{4}{3} P_0 a^3 \cdot (3a) - \frac{2}{3} P_0 a^4 \right]$$

$$= \frac{1}{EI} \left[\frac{M_0}{2} 9a^2 + 4 P_0 a^4 - \frac{2}{3} P_0 a^4 \right]$$

$$v_{free}(3a) = \frac{1}{EI} \left(\frac{9}{2} M_0 a^2 + \frac{10}{3} P_0 a^4 \right)$$

Unit-load deflection:

$$\begin{aligned} v_1(3a) &= \frac{1}{EI} \left(\frac{4}{3} a^3 - 2a^2 x \right) \\ &= \frac{1}{EI} \left[\frac{4}{3} a^3 - 2a^2 (3a) \right] \\ &= \frac{1}{EI} \left[\frac{4}{3} a^3 - 6a^3 \right] \end{aligned}$$

$$v_1(3a) = - \frac{14a^3}{3EI}$$

Rotation at support D ($x=2a$)

$$\theta_D = v'(2a) = v'_{\text{free}}(2a) + R_D v'_1(2a)$$

$$v'(2a) = \frac{1}{EI} \left(2aM_0 + \frac{4}{3}P_0a^3 \right) + \frac{3(M_0 + a^2P_0)}{4a} \left(-\frac{2a^2}{EI} \right)$$

$$= \frac{1}{EI} \left[2aM_0 + \frac{4}{3}P_0a^3 - 2a^2 \cdot \frac{3(M_0 + a^2P_0)}{4a} \right]$$

$$= \frac{1}{EI} \left(2aM_0 + \frac{4}{3}P_0a^3 - \frac{3}{2}M_0a - \frac{3}{2}a^3P_0 \right)$$

$$= \frac{1}{EI} \left(\frac{1}{2}aM_0 - \frac{1}{6}P_0a^3 \right)$$

$$v'(2a) = \frac{a}{6EI} (3M_0 - P_0a^3)$$

$$\theta_D = \frac{a (3M_0 - P_0a^3)}{6EI}$$

Deflection at the tip H :

$$v(3a) = v_{\text{free}}(3a) + R_D v_1(3a)$$

$$v(3a) = \frac{1}{EI} \left(\frac{9}{2} M_0 a^2 + \frac{10}{3} P_0 a^4 \right) + \frac{3(M_0 + a^2 P_0)}{4a} \cdot \frac{1}{EI} \left(-\frac{14}{3} a^3 \right)$$

$$= \frac{1}{EI} \left[\frac{9}{2} M_0 a^2 + \frac{10}{3} P_0 a^4 - \frac{14}{3} a^3 \cdot \frac{3(M_0 + a^2 P_0)}{4a} \right]$$

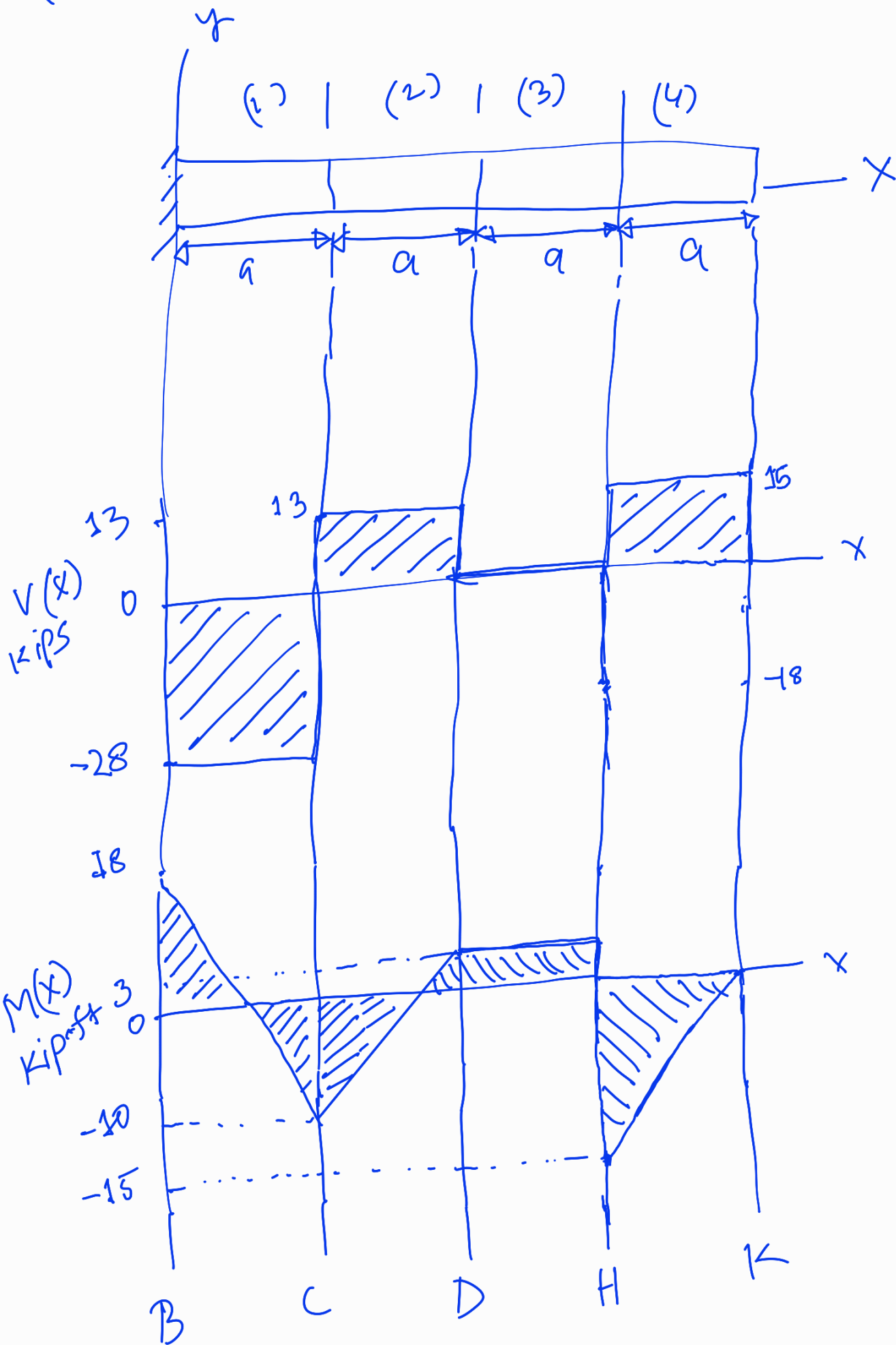
$$= \frac{1}{EI} \left[\frac{9}{2} M_0 a^2 + \frac{10}{3} P_0 a^4 - \frac{7}{2} a^2 (M_0 + a^2 P_0) \right]$$

$$= \frac{1}{EI} \left(M_0 a^2 - \frac{1}{6} P_0 a^4 \right)$$

$$v(3a) = \frac{a^2}{EI} \left(M_0 - \frac{P_0 a^2}{6} \right)$$

$$\boxed{v_H = \frac{a^2}{6EI} (6M_0 - P_0 a^2)}$$

Problem 6.4:



$$V = \frac{dM}{dx} = \frac{\Delta M}{\Delta x}$$

Maximum Shear stress,

$$V_{BC} = -28 \text{ kips.}$$

Maximum Compressive Bending moment,

$$M_H = -15 \text{ kip-ft}$$

Maximum Tensile Bending moment:

$$M_B = 18 \text{ kip-ft}$$

Maximum Compressive flexural stress.

$$(\sigma_{\max})_{\text{comp}} = \frac{M_H c}{I}$$

Maximum Tensile flexural stress

$$(\sigma_{\max})_{\text{ten}} = \frac{M_B c}{I}$$

Maximum shear stress:

$$\tau_{\max} = \frac{V_{\max} Q}{I t} = \frac{(V_{BC}) Q}{I t}.$$

6.4.1 · d) Location A

6.4.2 a) Location B

6.4.3 a) segment BC.