Problem 5.1 (10 points)

A simply-supported beam with a circular cross-section is loaded as shown in **Figure 5.1(i)**. The beam cross-section at location K is shown to the right in **Figure 5.1 (ii)**. Point O on the cross-section is on the neutral axis of the beam.

a) Determine the shear force and bending moment resultants on the cross-section at K.

b) Determine distribution of normal stress on the cross-section of the beam at K as a function of y, the perpendicular distance from the neutral axis.

c) Determine the maximum (magnitude) of the normal stress on the cross-section at K.

d) Determine the shear stress on the cross-section at K on the neutral axis.

Leave your answers in terms of, at most: P, a and b.

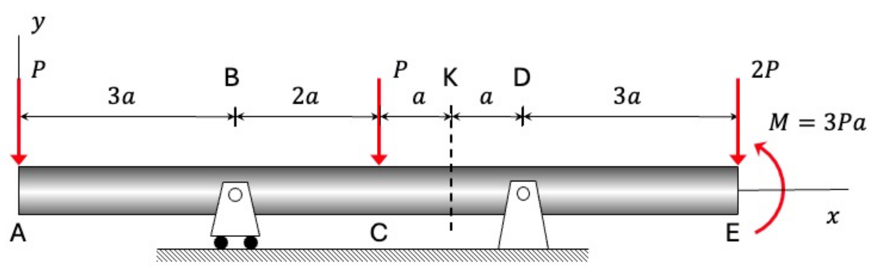


Figure 5.1 (i).

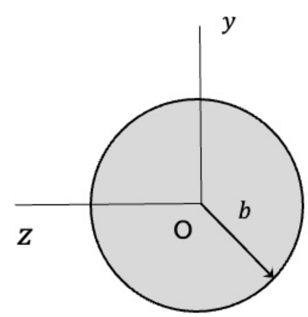


Figure 5.1 (ii).

Solution:

(a) FBD:

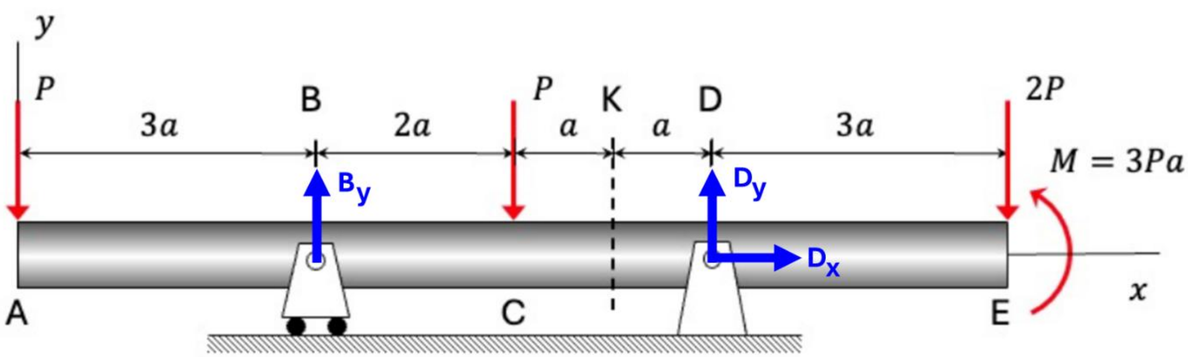


Fig. 1.

From the equilibrium equations for the entire beam (Fig. 1):

$$\sum F_x = 0 \Rightarrow D_x = 0.$$

$$\sum F_y = 0 \Rightarrow D_y + B_y - P - P - 2P = 0 \Rightarrow D_y + B_y = 4P. \qquad (1)$$

$$\sum M_D = 0 \Rightarrow 3Pa - 2P(3a) + P(2a) - B_y(4a) + P(7a) = 0 \Rightarrow B_y = \frac{3}{2}P. \qquad (2)$$

From (1) and (2), $D_y = \frac{5}{2}P$.

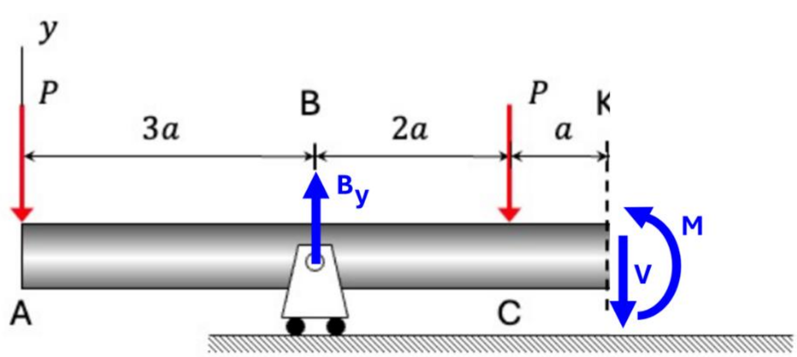


Fig. 2.

From the equilibrium of the left segment (Fig. 2), the shear force and bending moment resultants at K are:

$$\sum F_y = 0 \Rightarrow -P + \frac{3}{2}P - P - V = 0 \Rightarrow \; V = -\frac{1}{2}P.$$

$$\sum M_k = 0 \Rightarrow M + P(6a) - \frac{3}{2}P(3a) + P(a) = 0 \Rightarrow M = -\frac{5}{2}Pa.$$

(b) Normal stress distribution on the cross-section

$\sigma = -\frac{Mc}{I}$ where $M = -\frac{5}{2}Pa, I = \frac{1}{4}\pi b^4$ and $c = y$. Therefore,

$$\sigma = -\frac{Mc}{I} = -\frac{\left(-\frac{5}{2}Pa\right)y}{\frac{1}{4}\pi b^4} = \frac{10pay}{\pi b^4}.$$

(c) Maximum (magnitude) of the normal stress at K

At $y = b$:

$$|\sigma|_{max} = \frac{10pa}{\pi b^3}.$$

(d) Shear stress on the cross-section at k

$$\tau = \frac{VQ}{It},$$

where $V = -\frac{1}{2}P, I = \frac{1}{4}\pi b^4, t = 2b$. The first moment: $Q = A_{\text{semi}}\bar{y} = \left(\frac{1}{2}\pi b^2\right)\left(\frac{4b}{3\pi}\right) = \frac{2}{3}b^3$. Hence:

$$\tau = \frac{VQ}{It} = \frac{(0.5P)\left(\frac{2}{3}b^3\right)}{\left(\frac{1}{4}\pi b^4\right)(2b)} = \frac{2}{3}\frac{P}{\pi b^2}.$$

Problem 5.2 (10 points)

Consider the T-section beam cross-section in **Figure 5.2** below.

a) Locate the neutral axis of the beam cross-section.

b) Determine the second area moment of the cross-section with respect to the neutral axis using the parallel axis theorem.

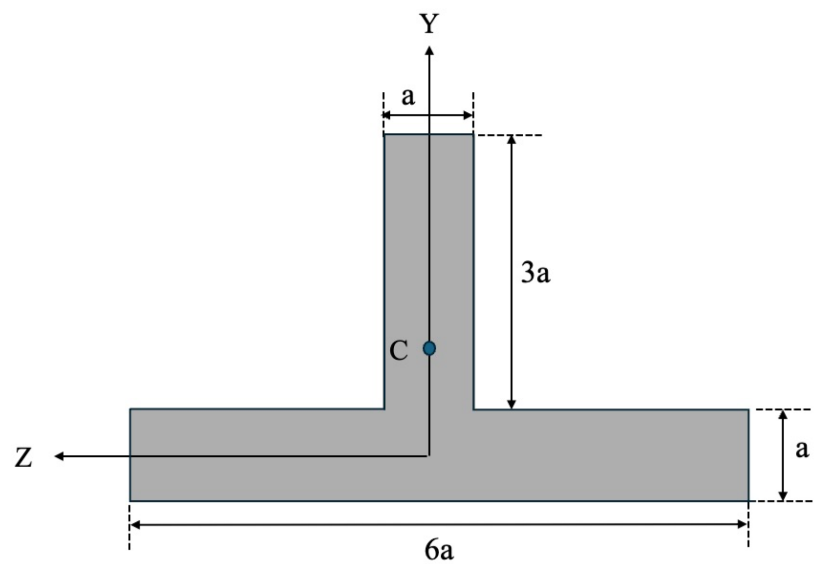


Figure 5.2

Solution:

(a) Neutral axis of the beam cross-section:

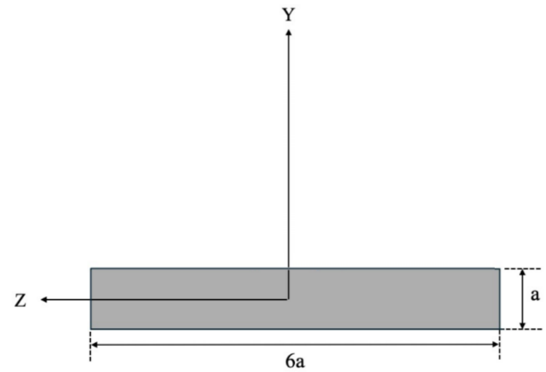


Fig. 3: Flange schematic.

From the flange schematic shown in Fig. 3:

$A_1 = (a)(6a) = 6a^2.$

$\bar{y}_1 = 0.$

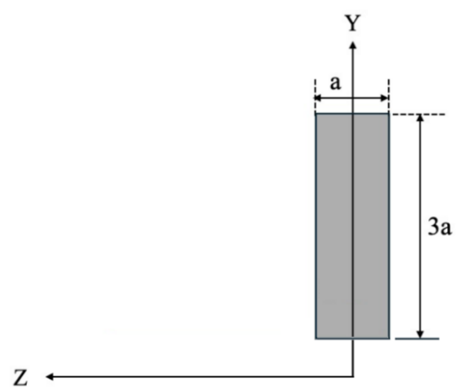


Fig. 4: Web schematic.

From the web schematic shown in Fig. 4:

$A_2 = (a)(3a) = 3a^2.$

$\bar{y}_2 = \frac{a}{2} + \frac{3a}{2} = 2a.$

Therefore, the y-coordinate of the neutral axis of the entire cross-section is:

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = \frac{(6a^2)(0) + (3a^2)(2a)}{6a^2 + 3a^2} = \frac{2}{3}a.$$

(b) Second moment of area about the neutral axis

$$I_{\text{total,n}} = I_{\text{flange,n}} + I_{\text{web,n}},$$

and by the the parallel-axis theorem,

$$I_{\text{flange,n}} = I_{\text{flange,c}} + A_{\text{flange}} {d_{\text{flange,n}}}^2 = \frac{1}{12}(6a)(a^3) + (6a)(a)\left(\frac{2}{3}a - 0\right)^2,$$

$$I_{\text{web,n}} = I_{\text{web,c}} + A_{\text{web}} {d_{\text{web,n}}}^2 = \frac{1}{12}(3a)^3(a) + (3a)(a)\left(2a - \frac{2}{3}a\right)^2.$$

Thus,

$$\begin{aligned} I_{\text{total,n}} &= I_{\text{flange,n}} + I_{\text{web,n}} \\ &= \frac{1}{12}(6a)(a^3) + (6a)(a)\left(\frac{2}{3}a\right)^2 + \frac{1}{12}(3a)^3(a) + (3a)(a)\left(2a - \frac{2}{3}a\right)^2 \\ &= \frac{43}{4}a^4. \end{aligned}$$

Problem 5.3 (10 points)

A rectangular, hollow cross-section beam is supported by a roller pin at point A and pinned at point C. A distributed load (force/length) acts of the beam with this load having a constant value of w between points A and B, and is linearly decreasing between B and C, as shown in **Figure 5.3**. The dimensions of the beam cross-section are also shown below in the figure.

a) Determine the reactions on the beam at A and C.

b) Determine the internal shear force and bending moment resultants at location D on the beam.

c) Determine the second area moment of the beam cross section about the neutral axis.

d) Determine the normal stress and shear stress at point K on the beam cross-section at location D on the beam.

e) Show the normal stress and shear stress acting at cross-section D on the stress element at K provided below.

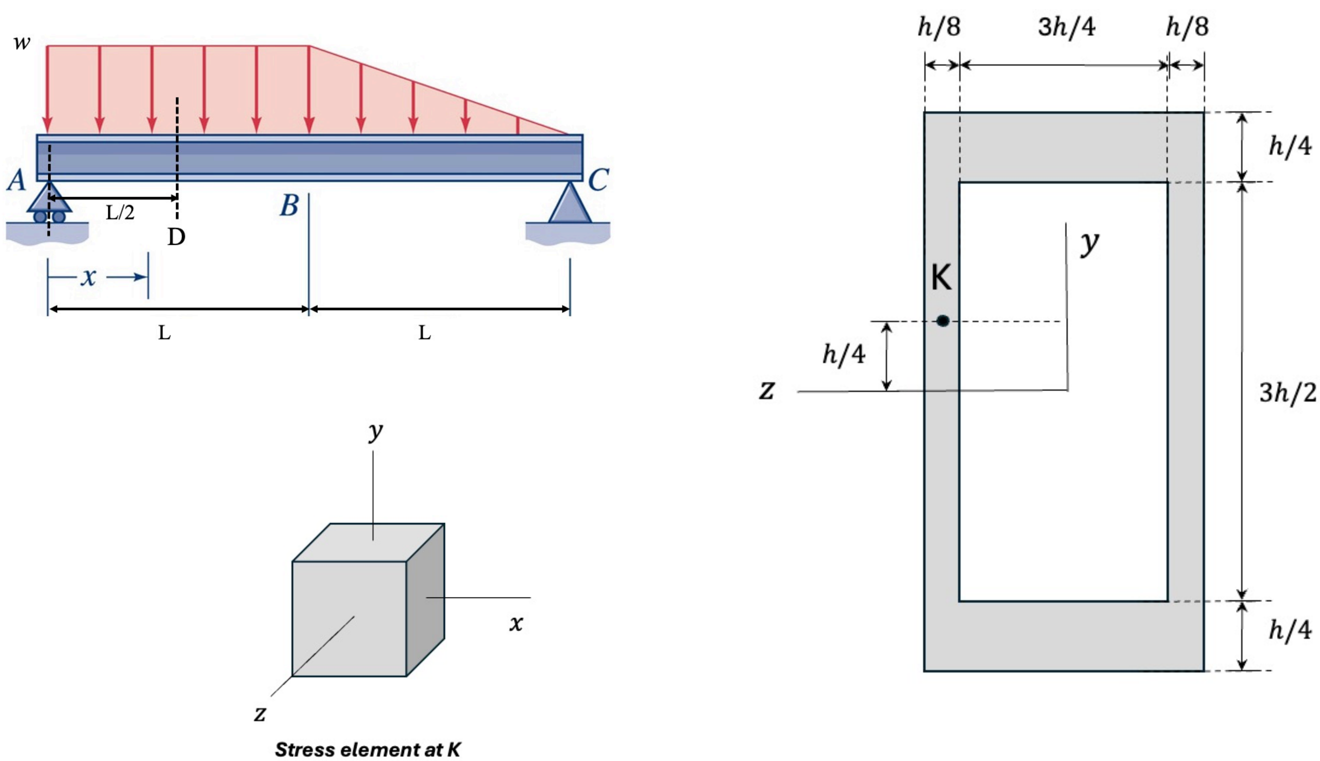


Figure 5.3

Solution:

(a) FBD:

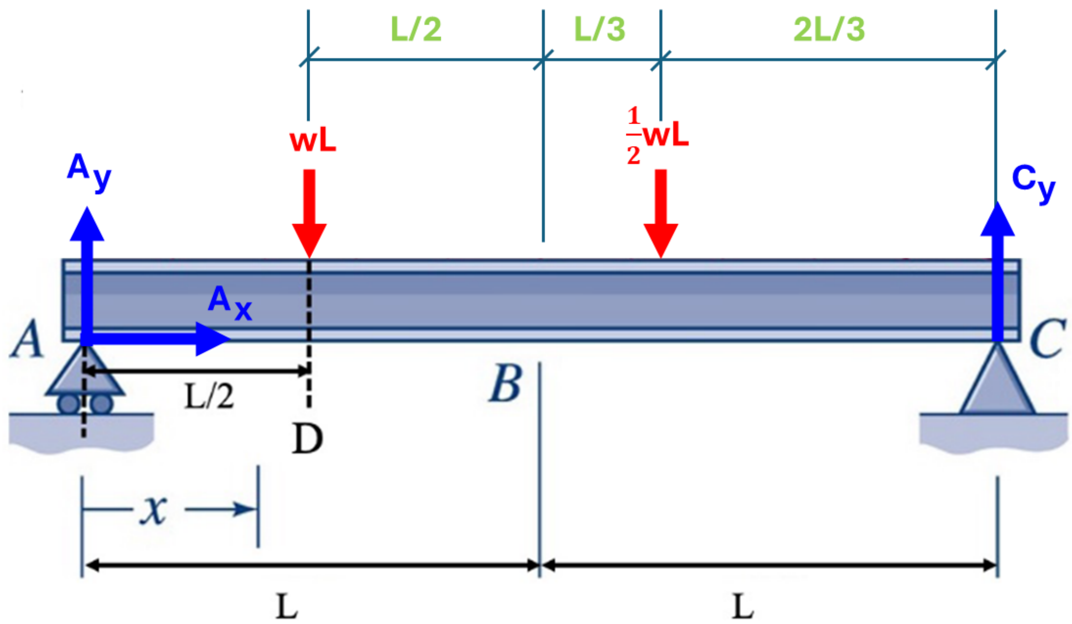


Fig. 5.

From equilibrium:

$\sum F_x = 0 \Rightarrow A_x = 0.$

$$\sum F_y = 0 \Rightarrow A_y + C_y - wL - 0.5wL = 0 \Rightarrow A_y + C_y = 1.5wL. \qquad (1)$$

$$\sum M_A = 0 \Rightarrow C_y(2L) - wL(0.5L) - (0.5wL)(\frac{4}{3}L) = 0 \Rightarrow C_y = \frac{7}{12}wL. \qquad (2)$$

From (1) and (2), $A_y = \frac{11}{12}wL.$

(b) Internal shear and bending moment at location D:

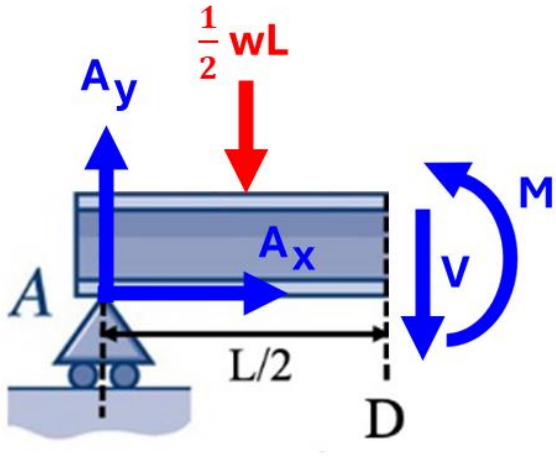


Fig. 6.

For the left segment:

$$\sum F_y = 0 \Rightarrow \frac{11}{12} wL - \frac{1}{2} wL - V = 0 \Rightarrow \ V = \frac{5}{12} wL.$$

$$\sum M_D = 0 \Rightarrow M + \frac{1}{2} wL \left(\frac{L}{4}\right) - \frac{11}{12} wL \left(\frac{L}{2}\right) = 0 \Rightarrow M = \frac{1}{3} wL^2.$$

(c) By vertical symmetry the neutral axis coincides with the Z-axis (y=0). From Fig. 7, treating the section as an outer rectangle minus an inner void rectangle,

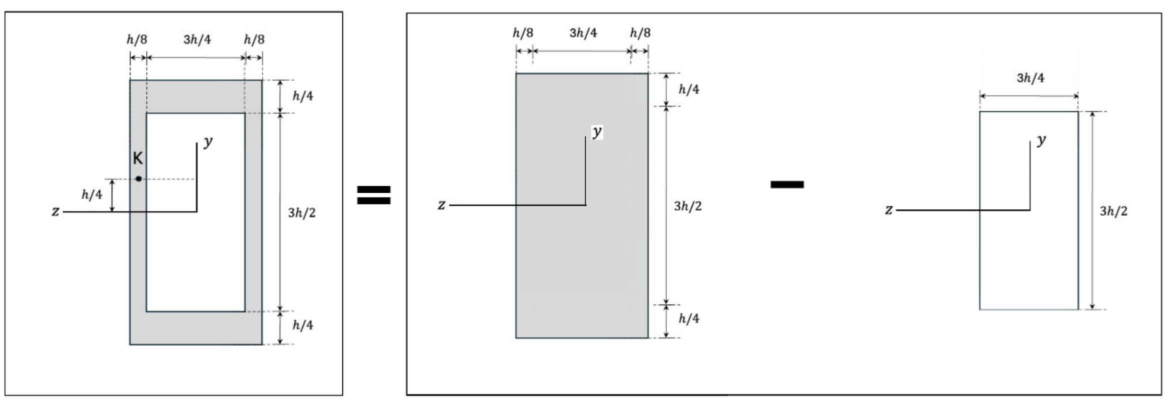


Fig. 7.

$$I_{\text{total,n}} = I_{\text{outer,n}} - \ I_{\text{inner,n}} = \frac{1}{12}(h)(2h)^3 - \frac{1}{12}\left(\frac{3}{4}h\right)\left(\frac{3}{2}h\right)^3 = \frac{175}{384} h^4.$$

(d) Normal and shear stresses at point K

Normal stress:

$$\sigma = -\frac{Mc}{I}, M = \frac{1}{3}wL^2, I = \frac{175}{384}h^4, c = \frac{h}{4} \Rightarrow \sigma = -\frac{Mc}{I} = -\frac{\left(\frac{1}{3}wL^2\right)\frac{h}{4}}{\frac{175}{384}h^4} = -\frac{32}{175}\frac{wL^2}{h^3}.$$

Shear stress:

$$\tau = \frac{VQ}{It}, V = \frac{5}{12}wL, I = \frac{175}{384}h^4, t = \frac{h}{8}.$$

Treating the section as outer minus inner rectangles (Fig. 8), the first moment above K is

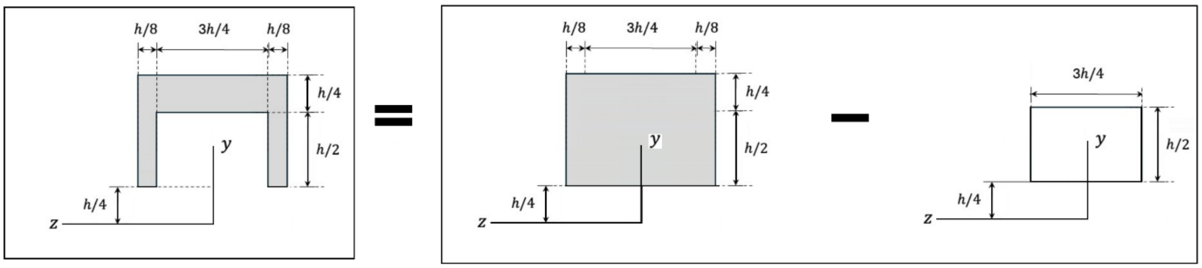


Fig. 8.

$$Q = A\bar{y} = A_o\bar{y}_o - A_i\bar{y}_i = \left((h)\left(\frac{3h}{4}\right)\left(\frac{h}{4}+\frac{3h}{8}\right)\right) - \left(\left(\frac{3h}{4}\right)\left(\frac{h}{2}\right)\left(\frac{h}{4}+\frac{h}{4}\right)\right) = \frac{9}{32}h^3.$$ Hence,

$$\tau = \frac{VQ}{It} = \frac{\left(\frac{5}{12}wL\right)\left(\frac{9}{32}h^3\right)}{\left(\frac{175}{384}h^4\right)\left(\frac{h}{8}\right)} = \frac{72}{35}\frac{wL}{h^2}.$$

(e)

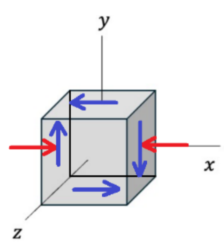


Fig. 9.

Problem 5.4 (10 points)

A shear force V and bending moment M act at a cross-section of a trapezoidal cross-sectioned beam. Consider five points (i), (ii), (iii), (iv) and (v) on the beam cross-section, as shown in **Figure 5.4**. Match up the state of stress at each of these five points with stress elements (a) through (o) shown below.

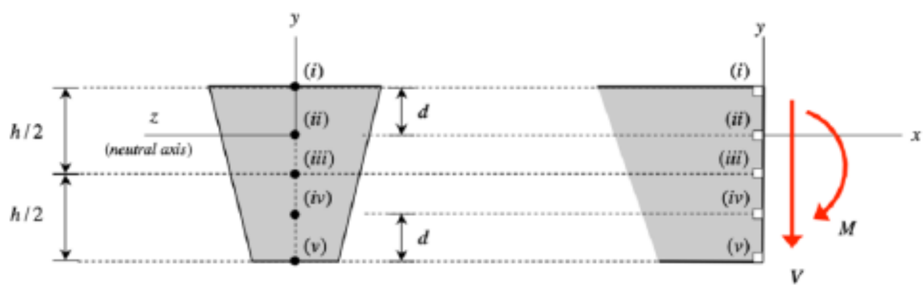
The state of stress at point (i) is:

The state of stress at point (ii) is:

The state of stress at point (iii) is:

The state of stress at point (iv) is:

The state of stress at point (v) is:



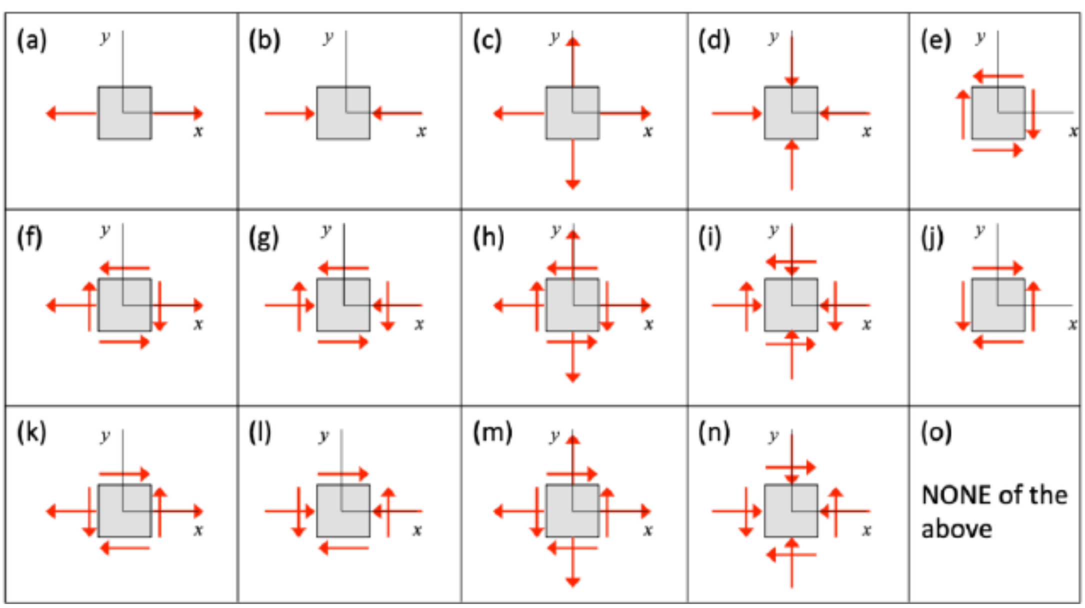


Figure 5.4

Solution:

The state of stress at point (i) is: (a)

The state of stress at point (ii) is: (e)

The state of stress at point (iii) is: (g)

The state of stress at point (iv) is: (g)

The state of stress at point (v) is: (b)