Problem 11.1 (10 Points)

The elbow shown below is fixed to the ground at the center of the coordinate system. Two loads 75 N and 150 N are applied at the free end in the y and z directions, respectively. If the elbow has a circular cross-section with a diameter of 20 cm, find:

(a) The internal reactions at a cross-section perpendicular to the y-axis at point A ($y = 1$ m). Classify the forces as either axial or shear forces, and the moments as either bending or torsion.

(b) The stresses induced (magnitude and direction) in the stress elements M and N on the cross-section at A, shown in Fig. 11.1 (b), due to each reaction calculated in part (a). Use the three-dimensional stress elements in Fig. 11.1 (c).

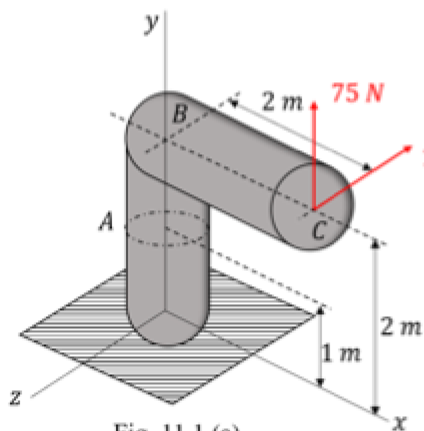


Fig. 11.1 (a)

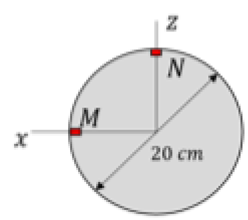


Fig. 11.1 (b)

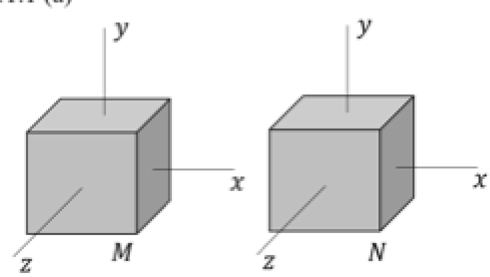
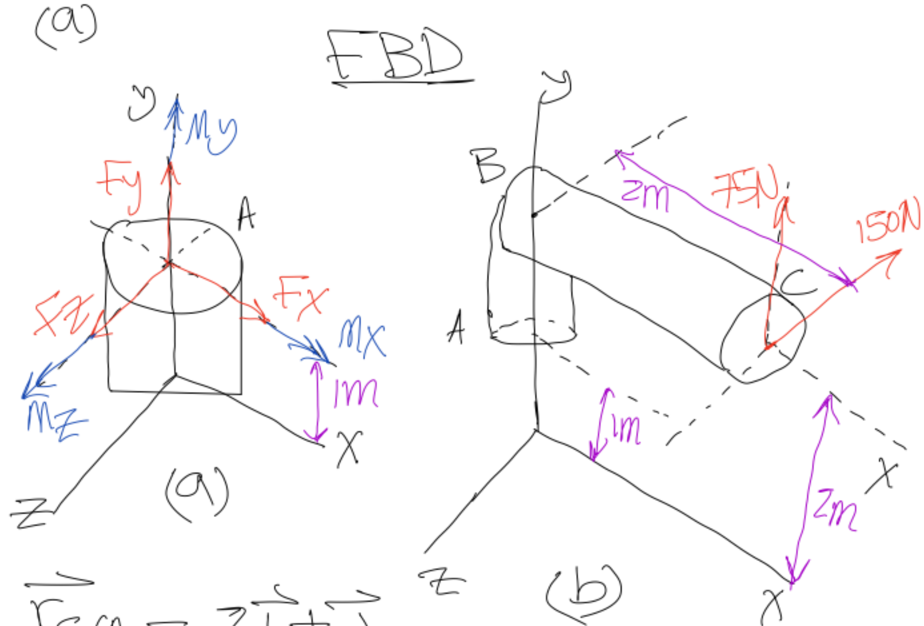


Fig. 11.1 (c)

Solution:

(a)

FBD



$\vec{r}_{C/A} = 2\vec{i} + \vec{j}$

from (a),

$\vec{F}_A = F_x\vec{i} + F_y\vec{j} + \vec{F_z}$

$\vec{M}_A = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$

From (b),

$\sum \vec{F} = 0$

$-\vec{F}_A + 75\vec{j} - 150\vec{k} = 0$

$\therefore \vec{F}_A = 75\vec{j} - 150\vec{k}$ (N)

$$\sum \vec{M} = 0$$

$$-\vec{M}_A + \vec{r}_{C/A} \times (75\vec{j} - 150\vec{k}) = 0$$

$$\Rightarrow \vec{M}_A = (2\vec{i} + \vec{j}) \times (75\vec{j} - 150\vec{k})$$

$$\therefore \vec{M}_A = -150\vec{i} + 300\vec{j} + 150\vec{k} \quad (Nm)$$

	Reaction at A	Type of reaction
1	$75\vec{j}$ N	Axial force
2	$-150\vec{k}$ N	shear force
3	$-150\vec{i}$ Nm	Bending moment
4	$300\vec{j}$ Nm	Torsion
5	$150\vec{k}$ Nm	Bending moment

(b)

For the circular cross section,

$$A = \frac{\pi (0.2)^2}{4} = 0.0314\, m^2$$

$$I = \frac{\pi (0.2)^4}{64} = 7.854 \times 10^{-5}\, m^4$$

$$J = I_P = \frac{\pi (0.2)^4}{32} = 1.571 \times 10^{-4}\, m^4$$

1. $75\vec{j}$ N (axial force)

$$\sigma_{M,1} = \sigma_{N,1} = \sigma_y = \frac{F_y}{A} = \frac{75}{0.0314} = 2388.53\ Pa$$

2. $-150\vec{k}$ N (shear force in z)

$$\tau_{M,1} = \tau_{yz} = \frac{4F_z}{3A} = \frac{4(-150)}{3(0.0314)}$$

$$= -6369.42 \text{ Pa}$$

$$\tau_{N,1} = 0$$

3. $-150\vec{i}$ Nm (Bending moment)

$$\sigma_{N,2} = \sigma_y = \left|\frac{M_x R}{I}\right| = \frac{(150)(0.1)}{7.854\times10^{-5}}$$

$$= 1.91\times10^{5} \text{ Pa}$$

$$\sigma_{M,2} = \sigma_y = 0$$

4. $300\vec{j}$ Nm (Torsion)

$$\tau_{M,2} = \tau_{yz} = -\left|\frac{M_y R}{J}\right| = \frac{-300(0.1)}{1.571\times10^{-4}}$$

$$= -1.91\times10^{5} \text{ Pa}$$

$$\tau_{N,2} = \tau_{yx} = \left|\frac{M_y R}{J}\right| = \frac{300(0.1)}{1.571\times10^{-4}} = 1.91\times10^{5} \text{ Pa}$$

5. $150\vec{k}$ Nm (Bending moment)

$$\sigma_{N,3} = \sigma_y = 0$$

$$\sigma_{M,3} = \sigma_y = \left|\frac{M_z R}{I}\right| = \frac{(150)(0.1)}{7.854\times10^{-5}}$$

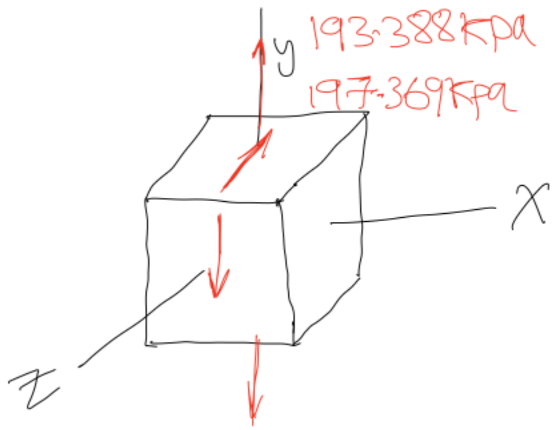
$= 1.91 \times 10^5$ pa

In summary,

The state of stress at M:

$\sigma_{M,y} = \sigma_{M,1} + \sigma_{M,2} + \sigma_{M,3}$
$= 193.388$ Kpa

$\tau_{M,yz} = \tau_{M,1} + \tau_{M,2} = -197.369$ Kpa



The state of stress at N:

$\sigma_{N,y} = \sigma_{N,1} + \sigma_{N,2} + \sigma_{N,3} = 193.388$ Kpa

$\tau_{N,yx} = \tau_{N,1} + \tau_{N,2} = 191$ Kpa

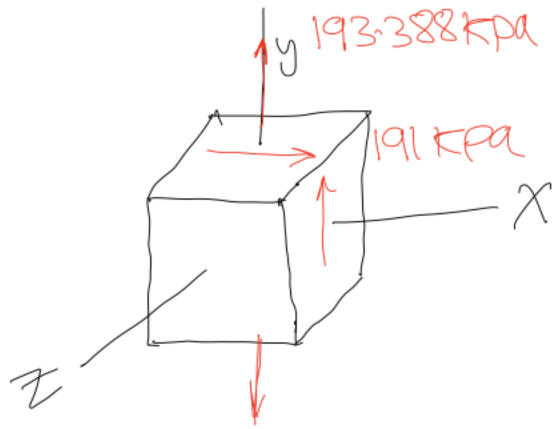


Table method :

	Reaction at A	stress comp. @ "M"	stress comp. @ "N"
1	$F_y = 75N$	$\sigma_y = \frac{F_y}{A} = 2388.53$ Pa	$\sigma_y = \frac{F_A}{A} = 2388.53$ Pa
2	$F_z = -150N$	$\tau_{yz} = \frac{4F_z}{3A} = -636942$ Pa	—
3	$M_x = -150Nm$	—	$\sigma_y = \left\|\frac{M_x R}{I}\right\| = 1.91\times10^5$ Pa
4	$M_y = 300N.m$	$\tau_{yz} = -\left\|\frac{M_y R}{J}\right\| = -1.91\times10^5$ Pa	$\tau_{yx} = \left\|\frac{M_y R}{J}\right\| = 1.91\times10^5$ Pa
5	$M_z = 150N.m$	$\sigma_y = \left\|\frac{M_z R}{I}\right\| = 1.91\times10^5$ Pa	—

Problem 11.2

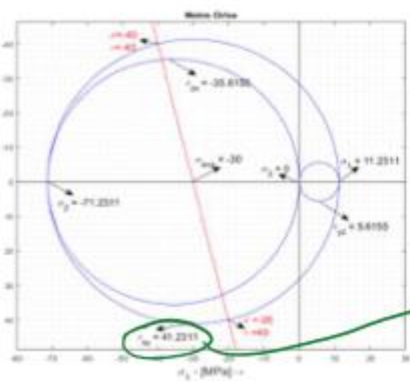
a) For fig. 11.1(a)

$\sigma_x = -40\,MPa$; $\sigma_y = -20\,MPa$; $\tau_{xy} = -40\,MPa$

For fig. 11.1(b)

$\sigma_z = 5\,ksi$; $\tau_{xz} = 10\,ksi$

$\sigma_x = 0\,ksi$



11.a

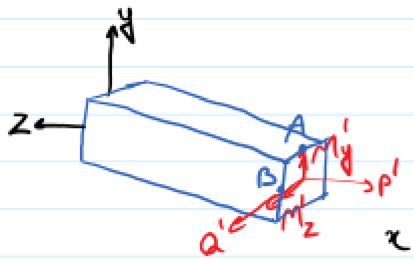
$\tau_{max,abs.} = 41.231\,MPa$

11.b

$\tau_{max,abs} = 10.31\,ksi$

Problem 11.3

a) FBD



Reactions at face AB:

1) $P' = P = -10$ kips (Axial force)

2) $Q' = Q = 5$ kips (Shear force)

3) $M_y' = M_y - Q(5) = 5 - 25 = -20$ kips·in (Bending moment)

4) $M_z' = M_z = 2$ kips·in (Bending moment)

Geometrical properties of face AB:

$A = bh = (1)(3) = 3 \text{ in}^2$

$I_z = \frac{1}{12}(b)(h)^3 = 2.25 \text{ in}^4$

$I_y = \frac{1}{12}(h)(b)^3 = 0.25 \text{ in}^4$

Stresses caused due to each of the above four reactions:

1) P':

$$\sigma_{x,A} = \sigma_{x,B} = \frac{P'}{A} = -3.33 \text{ ksi}$$

2) Q':

$$\tau_{xz,A} = \frac{Q' A^* z^*}{I_y t} \; ; \quad A^* = (3)(0.5) = 1.5 \text{ in}^2$$

$z^* = 0.25$ in

$t = 3$ in

$\Rightarrow \tau_{xz,A} = 2.5$ ksi ; $\tau_{xz,B} = 0$ ($\because$ it's at neutral axis)

3) M_y' :

$$\sigma_{x,B} = -\left|\frac{M_y' z}{I_y}\right|$$

$= -40$ ksi ; $z = 0.5$ in

$\sigma_{x,A} = 0$ ($\because z = 0$)

4) M_z' :

$$\sigma_{x,A} = -\left|\frac{M_z' y}{I_z}\right| \; ; \; y = 1.5 \text{ in}$$

$= -1.33$ ksi

$\sigma_{x,B} = 0$ ($\because y = 0$)

$\therefore$ Total Stresses at A :

$\sigma_{x,A} = -3.33 + 0 - 1.33 = -4.67$ ksi

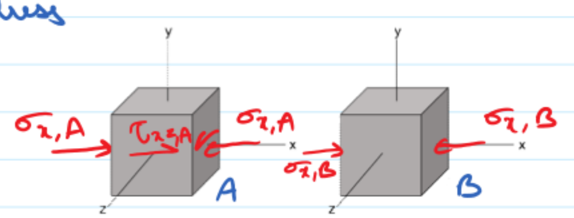
$\tau_{xz,A} = 2.5$ ksi

Total stresses at B :

$\sigma_{x,B} = -3.33 - 40 + 0 = -43.33$ ksi

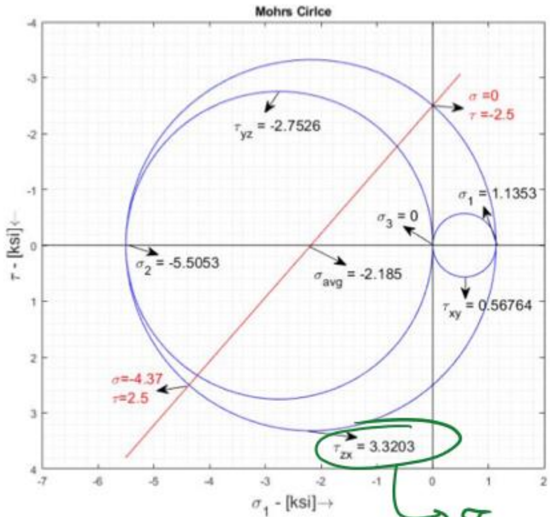
$\tau_{xz,B} = 0$

States of stress



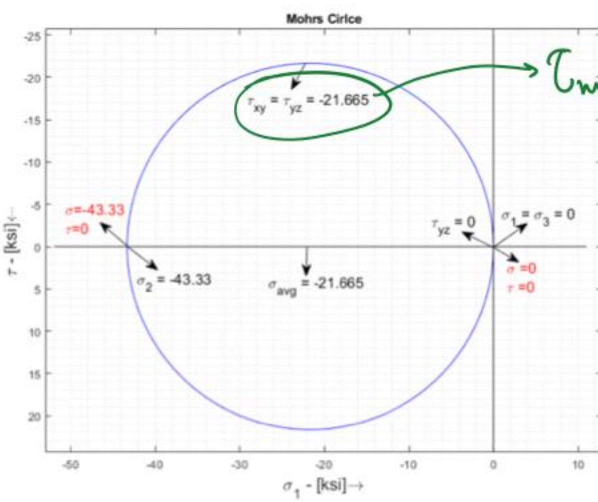
b)

A



$\tau_{max, abs} = 3.32$ ksi

B



$\tau_{max, abs} = 21.665$ ksi

Problem 4

Critical load :

$$P_{cr} = \frac{\pi^2 E I}{(KL)^2} \quad , \quad I = \frac{\pi R^4}{4}$$

(a) $K = 1$ (pinned - pinned)

$$P_{cr}^{(a)} = \frac{\pi^2 \times E \times \pi R^4}{4 \times (1 \times L)^2} = \frac{1}{4}\left(\frac{\pi^3 E R^4}{L^2}\right)$$

(b) $K = 2$ (fixed - free)

$$P_{cr}^{(b)} = \frac{\pi^2 \times 4E \times \pi \times (R/2)^4}{4 \times (2L)^2} = \frac{1}{64}\left(\frac{\pi^3 E R^4}{L^2}\right)$$

(c) $K = 0.7$ (fixed - pinned)

$$P_{cr}^{(c)} = \frac{\pi \times 3E \times \pi R^4}{4 \times \left(\frac{0.7L}{3}\right)^2} = 13.78\left(\frac{\pi^3 E R^4}{L^2}\right)$$

(d) $K = 0.5$ (fixed - fixed)

$$P_{cr}^{(d)} = \frac{\pi \times 4E \times \pi (2R)^4}{4 \times (0.5L)^2} = 64\left(\frac{\pi^3 E R^4}{L^2}\right)$$

$$\therefore \boxed{P_{cr}^{(d)} > P_{cr}^{(c)} > P_{cr}^{(a)} > P_{cr}^{(b)}}$$