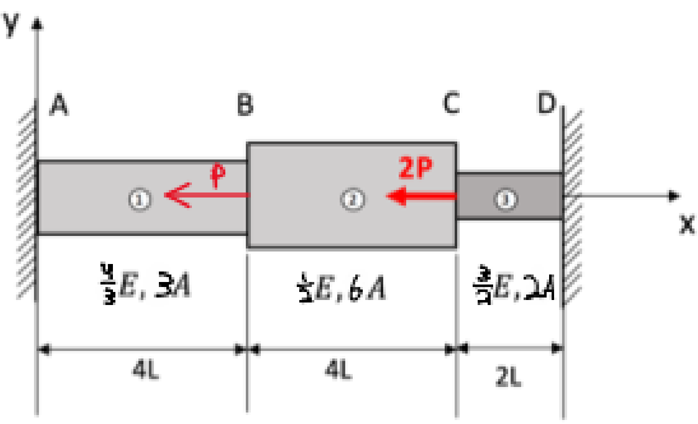
Problem 9.1 (10 points)

A three-segment rod AD is fixed to walls at A and D. An external load $2P$ is applied at C. The properties are shown in the figure.

(1) Use three finite elements (one element per segment), write down the stiffness matrix $[K]$ and the forcing vector $[F]$.
(2) Enforcing the boundary conditions, write the reduced system of equations and solve for the displacements at B and C.
(3) Determine the reactions at A and D.



Solution:

$$k_1 = \frac{\frac{4}{3}E * 3A}{4L} = \frac{EA}{L}$$

$$k_2 = \frac{\frac{1}{2}E * 6A}{4L} = \frac{3EA}{4L}$$

$$k_3 = \frac{\frac{3}{2}E * 2A}{2L} = \frac{3EA}{2L}$$

Stiffness matrix:

$$[K] = \begin{bmatrix} k_1 & -k_1 & & \\ k_1 & k_1 + k_2 & -k_2 & \\ & -k_2 & k_2 + k_3 & -k_3 \\ & & -k_3 & k_3 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & & \\ -1 & 1.75 & -0.75 & \\ & -0.75 & 2.25 & -1.5 \\ & & -1.5 & 1.5 \end{bmatrix}$$

Forcing vector:

$$[F] = \begin{bmatrix} F_A \\ F_B \\ F_C \\ F_D \end{bmatrix} = \begin{bmatrix} F_A \\ -P \\ -2P \\ F_D \end{bmatrix}$$

Displacements (using boundary conditions $u_A = u_D = 0$)

$$[u] = \begin{bmatrix} u_A \\ u_B \\ u_C \\ u_D \end{bmatrix} = \begin{bmatrix} 0 \\ u_B \\ u_C \\ 0 \end{bmatrix}$$

$$[F] = [K][u]$$

$$\begin{bmatrix} F_A \\ -P \\ -2P \\ F_D \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & & \\ -1 & 1.75 & -0.75 & \\ & -0.75 & 2.25 & -1.5 \\ & & -1.5 & 1.5 \end{bmatrix}\begin{bmatrix} 0 \\ u_B \\ u_C \\ 0 \end{bmatrix}$$

Reduced system of equations:

$$\begin{bmatrix} -P \\ -2P \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1.75 & -0.75 \\ -0.75 & 2.25 \end{bmatrix}\begin{bmatrix} u_B \\ u_C \end{bmatrix}$$

solving for displacements at B and C:

$$u_B = -\frac{10}{9}\frac{PL}{EA}$$

$$u_c = -\frac{34}{27}\frac{PL}{EA}$$

Reactions at A and D:

$$F_A = \left(-\frac{EA}{L}\right) * \left(-\frac{10}{9}\frac{PL}{EA}\right) = \frac{10}{9}P$$

$$F_D = \left(-\frac{3EA}{2L}\right) * \left(-\frac{34}{27}\frac{PL}{EA}\right) = \frac{17}{9}P$$

$$\left(-\frac{PL}{EA} + .75C\right)\frac{1}{1.75} = B$$

$$-P = \frac{EA}{L}\left(1.75B - 0.75C\right)$$

$$\frac{3}{4}\cdot\frac{4}{7} = \frac{3}{7}\cdot\frac{3}{4} = \frac{9}{28}$$

$$-2P = \frac{EA}{L}\left(-.75B + 2.25C\right)$$

$$-2P = \frac{EA}{L}\left(-\frac{0.75}{1.75}\left(-\frac{PL}{EA} + .75C\right) + 2.25C\right)$$

$$-2P = \frac{EA}{L}\left(+\frac{3}{7}\frac{PL}{EA} - \frac{9}{28}C + \frac{9}{4}C\right)$$

$$-2P = \frac{3}{7}P + \frac{EA}{L}C\left(-\frac{9}{28} + \frac{63}{28}\right)$$

$$-\frac{14P}{7} - \frac{3}{7}P = \frac{EA}{L}C\left(\frac{54}{28}\right)$$

$$-\frac{17}{7}P = \frac{54}{28}\frac{EA}{L}u_c$$

$$u_c = -\frac{PL}{EA}\left(\frac{476}{378}\right) = -\frac{PL}{EA}\left(\frac{34}{27}\right)$$

$$u_B = \frac{4}{7}\left(-\frac{PL}{EA} + \frac{3}{4}\left(-\frac{34}{27}\right)\frac{PL}{EA}\right)$$

$$= \frac{4}{7}\left(-\frac{18}{18}\frac{PL}{EA} - \frac{17}{18}PL/EA\right)$$

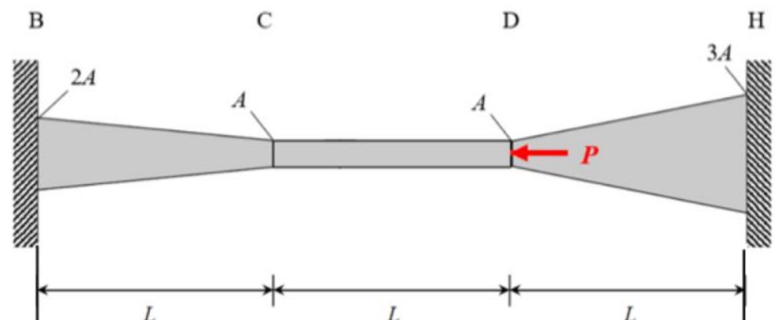
$$= -\frac{10}{9}\frac{PL}{EA}$$

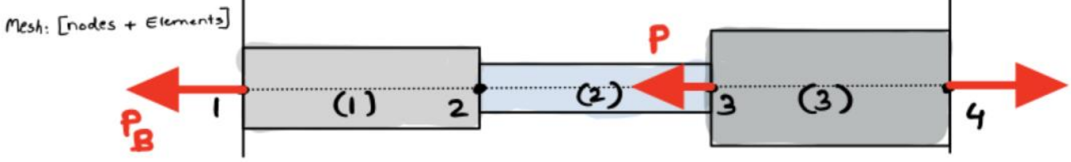
Problem 9.2 *(10 points)*

A rod is made of three segments: BC, CD, and DH. All segments have length L and are made of a material with Young's modulus E. The cross-sectional area of segment BC decreases linearly from 2A at B to A at C. The cross-sectional area of segment CD is constant. The cross-sectional area of segment DH increases linearly from A at D to 3A at H. A force P acts to the left at point D.

Using a three-element (four-node) finite element model, do the following:

(a) Construct the global stiffness matrix [K] in terms of E, A and L.
(b) Construct the force vector [F] in terms of P.
(c) Enforce the displacement boundary conditions.
(d) Solve for the nodal displacements in terms of PL/EA.
(e) Determine the reactions at walls B and H in terms of P.





(A) $[K] = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix}$

$$K_1 = \frac{(E A_{avg})_1}{L_1} = \frac{3EA}{2L}$$

$$K_2 = \frac{(E A_{avg})_2}{L_2} = \frac{EA}{L}$$

$$K_3 = \frac{(E A_{avg})_3}{L_3} = \frac{2EA}{L}$$

$$\therefore [K] = \frac{EA}{L}\begin{bmatrix} 3/2 & -3/2 & 0 & 0 \\ -3/2 & 5/2 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$$

(b) $\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} -P_B \\ 0 \\ -P \\ P_H \end{Bmatrix}$

(c) $u_1 = 0$, $u_4 = 0$, thus reduce the eq $\rightarrow [K][U] = [F]$

$$\frac{EA}{L}\begin{bmatrix} 3/2 & -3/2 & 0 & 0 \\ -3/2 & 5/2 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -P_B \\ 0 \\ -P \\ P_H \end{bmatrix}$$

$u_1 \rightarrow 0$, $u_4 \rightarrow 0$

(d) Solving the reduced system of equations results in $u_3 = \frac{5}{2}u_2$

or $u_2 \Rightarrow \frac{-PL}{13AE}$ $u_3 \Rightarrow \frac{-5PL}{13AE}$

$u_2, u_3 < 0$ thus both move to the left

(e) Now that $[u] = \begin{bmatrix} 0 \\ -PL/13AE \\ -5PL/13AE \\ 0 \end{bmatrix}$

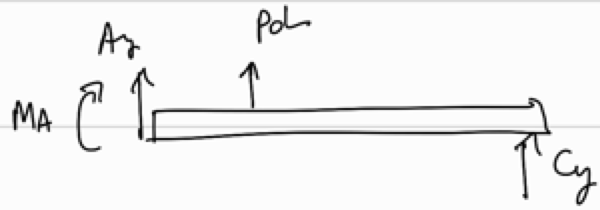
We can solve the crossed out rows to get the reactions:

$$\Rightarrow \begin{bmatrix} -P_B \\ -P \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 3/2 & -3/2 & 0 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -PL/13AE \\ -5PL/13AE \\ 0 \end{bmatrix}$$

$\therefore$ $P_B = \frac{-3}{13}P$ and $P = \frac{10}{13}P$

[It is in the opposite direction to what was assumed initially]

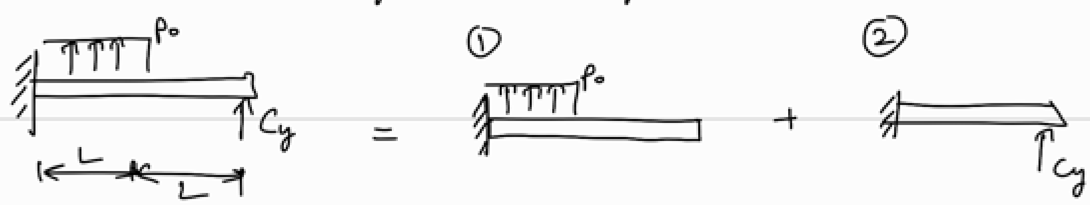
9.3 (a)



(b) ΣF_y : $A_y + p_0 L + C_y = 0$

ΣM_A : $-M_A + \frac{p_0 L^2}{2} + 2 C_y L = 0$

(c) Given loading is split as



Problem ① :

$$v_1(x) = \begin{cases} \frac{p_0 x^2}{24EI}(6L^2 - 4Lx + x^2) & 0 \le x \le L \\ \frac{p_0 L^3}{24EI}(4x - L) & L \le x \le 2L \end{cases}$$

Problem ② :

$$v_2(x) = \frac{C_y x^3}{6EI}(6L - x)$$

Total displacement :

$$v(x) = \begin{cases} \frac{p_0 x^2}{24EI}(6L^2 - 4Lx + x^2) + \frac{C_y x^3}{6EI}(6L - x), & 0 \le x \le L \\ \frac{p_0 L^3}{24EI}(4x - L) + \frac{C_y x^3}{6EI}(6L - x), & L \le x \le 2L \end{cases}$$

$$V(2L) = 0 = \frac{p_0 \cancel{L^3}}{24\cancel{EI}} \cdot 7L + \frac{8 C_y \cancel{L^3}}{3\cancel{EI}}$$

$$\boxed{C_y = -\frac{7}{64} p_0 L}$$

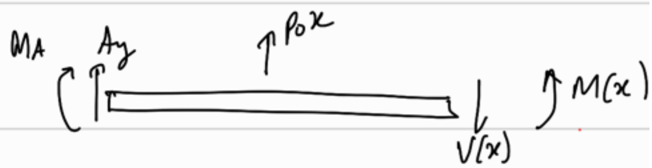
Substituting into equilibrium,

$$\boxed{A_y = -\frac{57}{64} p_0 L}$$

$$M_A = \frac{p_0 L^2}{2} - \frac{7}{32} p_0 L^2$$

$$\boxed{M_A = \frac{9}{32} p_0 L^2}$$

(d,e) $0 \leq x \leq L$:

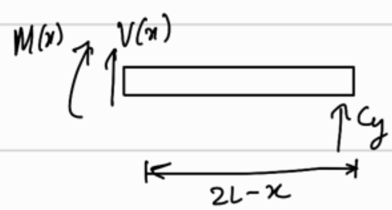


$$V(x) = p_0 x + A_y = p_0 x - \frac{57}{64} p_0 L$$

$$M(x) = M_A + A_y \cdot x + \frac{p_0 x^2}{2}$$

$$M(x) = \frac{9}{32} p_0 L^2 - \frac{57}{64} p_0 L x + \frac{p_0 x^2}{2}$$

$L \leq x \leq 2L$:



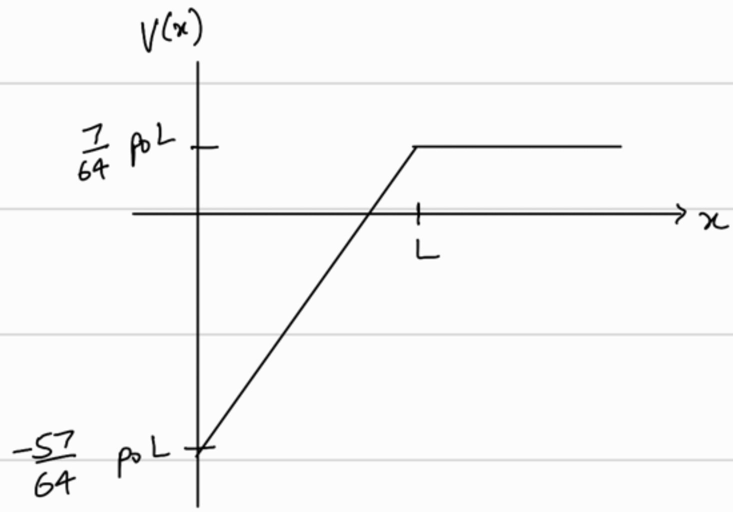
$$V(x) = -C_y = \frac{7}{64} p_0 L$$

$$M(x) = C_y(2L-x) = +\frac{7}{64} p_0 x L - \frac{7}{32} p_0 L^2$$

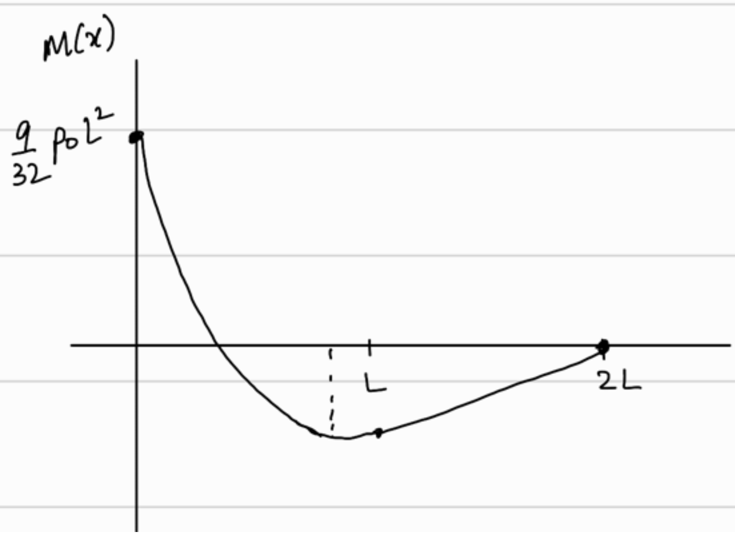
$$V(x) = \begin{cases} p_0 x - \frac{57}{64} p_0 L & 0 \le x \le L \\ \frac{7}{64} p_0 L & L \le x \le 2L \end{cases}$$

$$M(x) = \begin{cases} \frac{9}{32} p_0 L^2 - \frac{57}{64} p_0 L x + \frac{p_0 x^2}{2} & 0 \le x \le L \\ + \frac{7}{64} p_0 x L - \frac{7}{32} p_0 L^2 & L \le x \le 2L \end{cases}$$

(f)



(g)



Problem 9.4 ***(2.5 +2.5 points)***

1) A shaft has a stiffness matrix $[K] = \begin{bmatrix} K1 & -K1 & 0 \\ -K1 & K1+K2 & -K2 \\ 0 & -K2 & K2 \end{bmatrix}$. The right end of the shaft is free.

   Simplify the matrix.

   a) $\begin{bmatrix} K1 & -K1 \\ -K1 & K1+K2 \end{bmatrix}$

   b) $\begin{bmatrix} K1+K2 & -K2 \\ -K2 & K2 \end{bmatrix}$ row 1 and column1 can be removed as the left side only is fixed

   c) $\begin{bmatrix} K1 & -K1 \\ -K1 & K1+K2 \\ 0 & -K2 \end{bmatrix}$

2) What is the order of the reduced stiffness matrix for a shaft composed of 4 elements and with both ends fixed.

   a) 4
   b) 3 order of a stiffness matrix is (N+1) where N is the number of elements. Then each fixed end will reduce the order by one. (4+1)-2= 3
   c) 5
   d) 6