ME 323 – Mechanics of Materials
Exam 2
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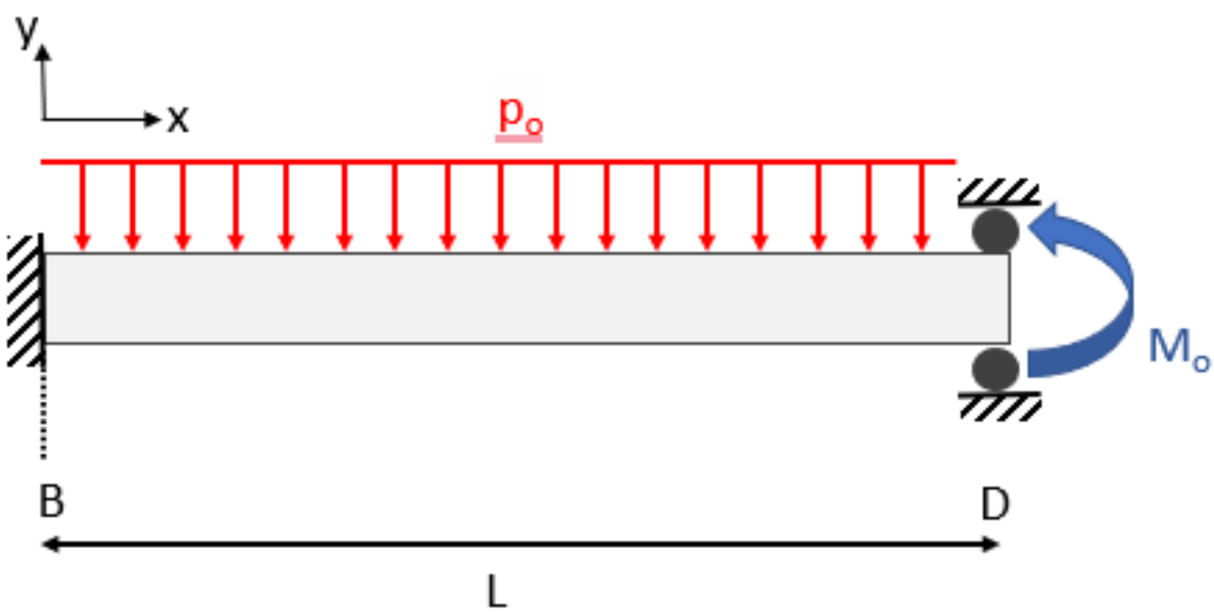
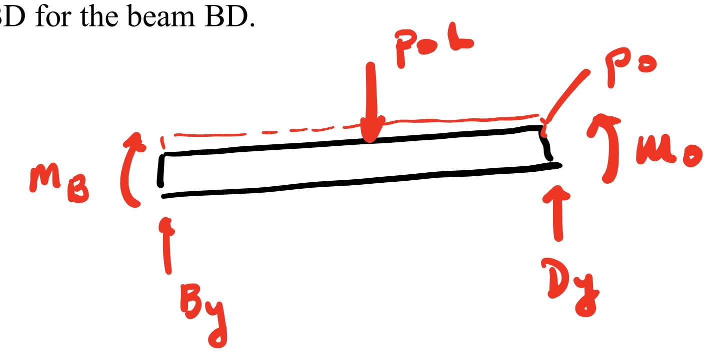
PROBLEM #1 (25 points)

The beam BD has a distributed load p_o acting between B and D and a concentrated moment M_o applied at D as shown. The beam is fixed at B and is supported by a roller at D. *E* and *I* are constant along the beam. p_o has a value of 16P/L and M_o has a value of 2PL. Use **the second-order integration method** to answer the following:

$p_o = 16P/L$ $M_o = 2PL$



a) Draw the FBD for the beam BD.



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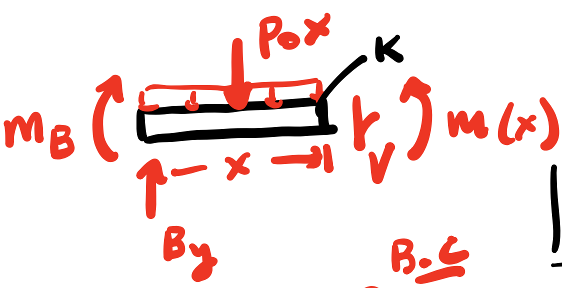
b) Write the equilibrium equations for the beam BD.

$$\Sigma M_D = M_0 + P_0 L\left(\frac{L}{2}\right) - B_y L - M_B = 0 \quad (1)$$

$$\Sigma F_y = B_y - P_0 L + D_y = 0 \quad (2)$$

c) Calculate the reactions on the beam at B and D in terms of *P and L*.

Use 2nd order integration:



$$\Sigma M_K = M(x) + P_0 x\left(\frac{x}{2}\right) - B_y x - M_B = 0$$

$$\boxed{M(x) = M_B + B_y x - P_0 \frac{x^2}{2}} = EI\frac{d\theta}{dx}$$

B.C

$$\theta(x) = \cancel{\theta(0)}^{\,0} + \frac{1}{EI}\int_0^x \left(M_B + B_y x - P_0 \frac{x^2}{2}\right) dx$$

$$\boxed{\theta(x) = \frac{1}{EI}\left[M_B x + B_y \frac{x^2}{2} - P_0 \frac{x^3}{6}\right]} = \frac{dv}{dx}$$

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0 B.C

$$v(x) = \cancel{v(0)} + \frac{1}{EI}\int_0^x \left[M_B x + B_y \frac{x^2}{2} - p_0 \frac{x^3}{6} \right] dx$$

$$v(x) = \frac{1}{EI}\left[M_B \frac{x^2}{2} + B_y \frac{x^3}{6} - p_0 \frac{x^4}{24} \right]$$

B.C: $v(L) = 0$

$$v(L) = M_B \frac{\cancel{L^2}}{\cancel{2}} + B_y \frac{\cancel{L^3}}{\cancel{6}_{3}} - p_0 \frac{\cancel{L^4}^{2}}{\cancel{24}_{12}} = 0$$

$$\boxed{M_B + B_y \frac{L}{3} - p_0 \frac{L^2}{12} = 0} \quad (3)$$

Solve (1) & (3) to calculate B_y:

$$B_y = \frac{3}{2L}\left[5 p_0 L + M_0 \right]$$

Replace $p_0 = \frac{16P}{L}$ $\quad M_0 = 2PL$

d) Find the equation for the vertical displacement, *v(x)* throughout the beam in terms of *P*, *L*, *E*, and *I*.

$\boxed{B_y = 13P}$ Use (2): $\boxed{D_y = 3P}$

Solve M_B from (3): $\boxed{M_B = -3PL}$

Replace B_y, D_y & M_B in $v(x)$

$$v(x) = \frac{1}{EI}\left[-\frac{3}{2}PLx^2 + \frac{13}{6}Px^3 - \left(\frac{16P}{L}\right)\frac{x^4}{24} \right]$$

$$\boxed{v(x) = \frac{P}{EI}\left[-\frac{3}{2}Lx^2 + \frac{13}{6}x^3 - \frac{2}{3}\frac{x^4}{L} \right]}$$

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e) Find the slope at point D in terms of *P, L, E, and I*.

$$\theta(L) = \frac{1}{EI}\left[-3PL(L) + \frac{13}{2}PL^2 - \left(\frac{16P}{L}\right)\frac{L^3}{6}\right]$$

$$= \frac{PL^2}{EI}\left[-3 + \frac{13}{2} - \frac{8}{3}\right]$$

$$\boxed{\theta(L) = \frac{5PL^2}{6EI}}$$

Using indefinite integrals, or other correct solutions was acceptable.

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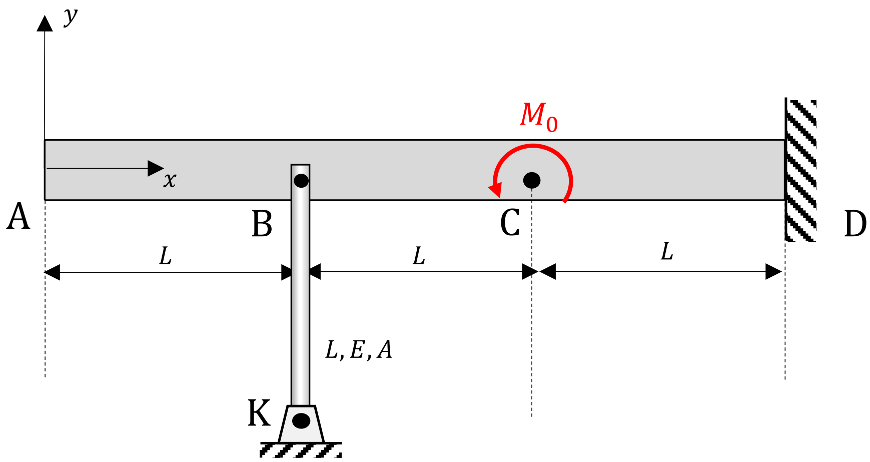
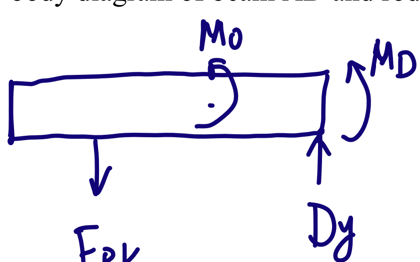
<u>PROBLEM #2</u> (25 points)

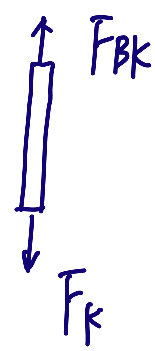
Cantilever beam AD of the bending stiffness *EI* is subjected to a concentrated moment M_0 at C. The beam is also pin-supported at B by a rod BK of the length *L*, Young's modulus *E*, and cross-section area *A*.



(a) Draw the free body diagram of beam AD and rod BK. List the equilibrium equations.

FBD:





$F_{BK} = F_K$

$\Sigma F_y = D_y + F_{BK} = 0$

$\Sigma M_D = F_{BD} \cdot 2L + M_0 + M_D = 0$

(b) Is AD a statically determined or indeterminate structure? If it is statically indeterminate, state your choice of redundant support.

statically indeterminate

Choose F_K as the redudant support

(Alternatively, we can also chose F_{BK} or M_D as redundant support)

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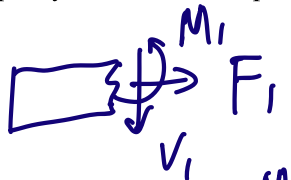
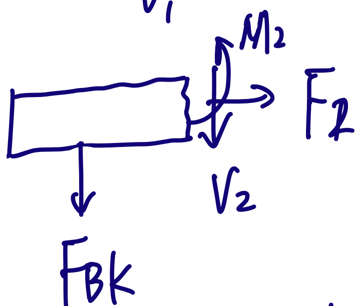
(c) Determine the internal reactions of the sections AB, BC, and CD, as well as rod BK in terms of the redundant support you identified in part (b).

For AB:



$F_1 = V_1 = M_1 = 0$

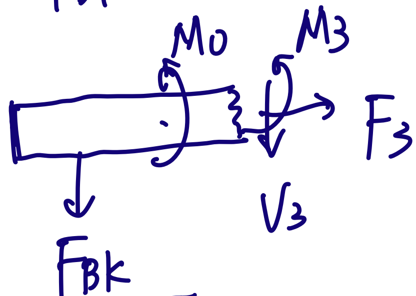
For BC:



$F_2 = 0, \quad V_2 = F_{BK} = F_K$

$M_2(x) = -F_{BK}(x-L) = -F_K(x-L)$

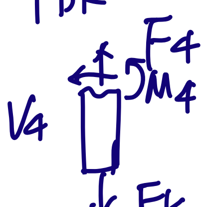
For CD:



$F_3 = 0, \quad V_3 = F_{BK} = F_K$

$M_3(x) = -F_K(x-k) - M_0$

For BK



$F_4 = F_K, \quad V_4 = M_4 = 0$

(d) Show the strain energies of the sections AB, BC, CD and BK. The strain energies may include the uniaxial strain energy U_A, shear strain energy due to torsion U_T, flexural strain energy due to bending U_σ. Neglect the shear strain energy due to bending U_τ.

strain energies: $U_{AB} = 0,$

$$U_{BC} = \frac{1}{2}\int_L^{2L} \frac{M_2^2(x)}{EI}\,dx = \frac{1}{2}\int_L^{2L} \frac{F_K^2 (x-L)^2}{EI}\,dx$$

$$U_{CD} = \frac{1}{2}\int_{2L}^{3L} \frac{M_3^2(x)}{EI}\,dx = \frac{1}{2}\int_{2L}^{3L} \frac{\left(F_K(x-L)+M_0\right)^2}{EI}\,dx$$

$$U_{BK} = \frac{1}{2}\int_0^{L} \frac{F_4^2}{EA}\,dx = \frac{1}{2}\int_0^{L} \frac{F_K^2}{EA}\,dx$$

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(e) Using Castigliano's 2nd theorem, determine the reaction force of the rod BK ~~and the wall reactions.~~

Total strain energy of the system:

$$U = U_{BC} + U_{CD} + U_{BK} = \frac{1}{2}\int_L^{2L} \frac{F_k^2 (x-L)^2}{EI}\,dx$$

$$+ \frac{1}{2}\int_{2L}^{3L} \frac{(F_k(x-L)+M_0)^2}{EI}\,dx + \frac{1}{2}\int_0^L \frac{F_k^2}{EA}\,dx$$

$$\Delta_k = \frac{\partial U}{\partial F_k} = 0 \Rightarrow \int_L^{2L} \frac{F_k(x-L)^2}{EI}\,dx + \int_{2L}^{3L} \frac{(F_k(x-L)+M_0)(x-L)}{EI}\,dx$$

$$+ \int_0^L \frac{F_k}{EA}\,dx$$

(f) Using Castigliano's 2nd theorem, determine the rotation angle at C.

Continue (e):

$$= \frac{1}{3}F_k \frac{(x-L)^3}{EI}\Big|_L^{2L} + \frac{1}{3}F_k \frac{(x-L)^3}{EI}\Big|_{2L}^{3L} + \frac{1}{2}M_0 \frac{(x-L)^2}{EI}\Big|_{2L}^{3L}$$

$$+ \frac{F_k L}{EA}$$

$$= \frac{1}{3}\frac{F_k \cdot L^3}{EI} + \frac{1}{3}F_k \cdot \frac{7L^3}{EI} + \frac{1}{2}M_0 \frac{3L^2}{EI} + \frac{F_k L}{EA} = 0$$

$$\Rightarrow F_K = -\frac{3}{2}\frac{M_0 L}{I} \Big/ \left(\frac{8}{3}\frac{L^2}{I} + \frac{1}{A}\right)$$

Alternatively: If we choose F_{BK} as the redundant support, then system energy

$$U = U_{BC} + U_{CD} = \frac{1}{2}\int_L^{2L} \frac{F_K^2 (x-L)^2}{EI} dx + \frac{1}{2}\int_{2L}^{3L} \frac{(F_K(x-L)+M_0)^2}{EI} dx$$

The compatibility condition:

$$\Delta_B = \frac{\partial U}{\partial F_{BK}} = -\frac{F_{BK} L}{EA}.$$ It gives the same conclusion.

part (f): Rotation angle at C:

$$\theta_C = \frac{\partial U}{\partial M_0} = \int_{2L}^{3L} \frac{F_K(x-L)+M_0}{EI} dx$$

$$= \frac{M_0 L}{EI} + \frac{\frac{3}{2}F_K L^2}{EI}$$

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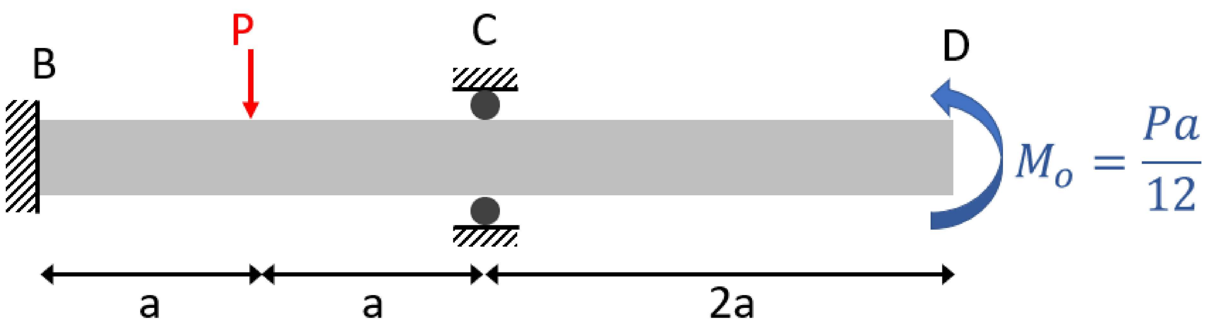
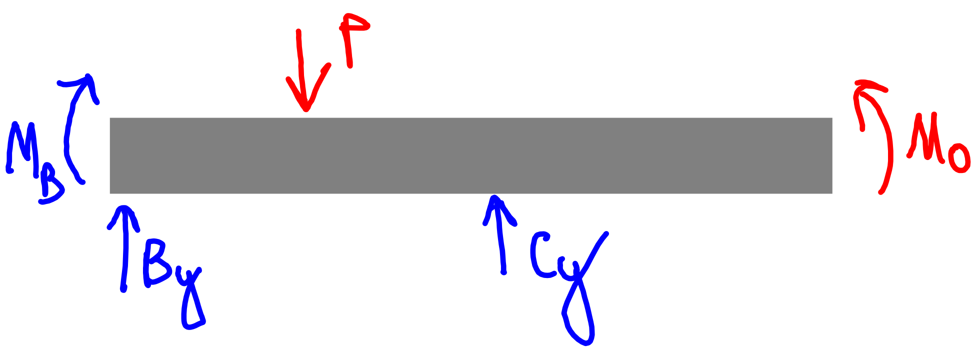
<u>PROBLEM #3</u> (25 points):

Beam BCD has a length of 4a and is supported by a fixed support at B and a roller at C. A concentrated force and a concentrated moment are applied as shown in the diagram below. The beam has a circular cross-section with a radius (r) of a/10. i.e. a = 10r



(a) Draw the free body diagram of the beam.



(b) Write the equilibrium equations for the beam.

(1) $\Sigma F_y = B_y + C_y - P = 0$

(2) $(\Sigma M)_B = -M_B - Pa + C_y(2a) + M_o = 0$

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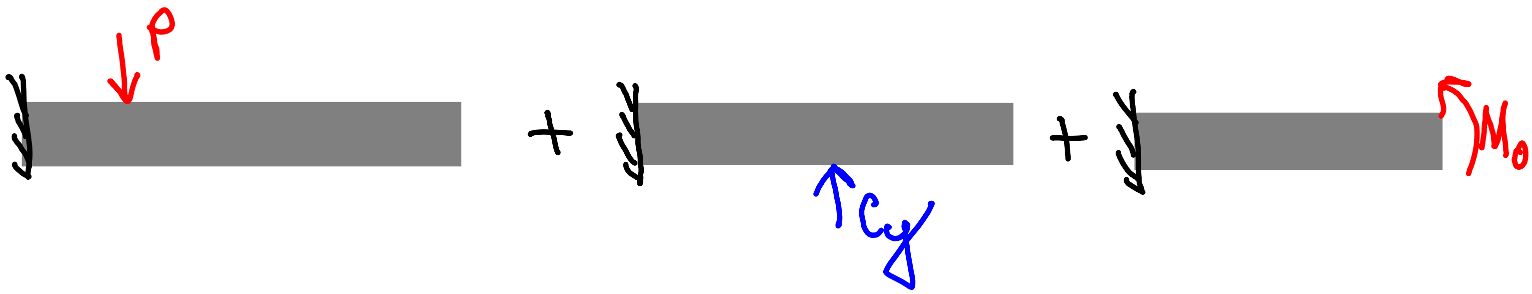
(c) Use the superposition tables to determine the reactions at B and C.



$v_P = -\frac{Px^2}{6EI}(3a-x)$ $\quad 0<x<a \Rightarrow$ not valid at $x=2a$.

$v_P = -\frac{Pa^2}{6EI}(3x-a)$ $\quad a<x<4a \Rightarrow$ need to use this one

$v_{C_y} = \frac{C_y x^2}{6EI}(6a-x)$ $\quad 0<x<2a$

$v_{C_y} = \frac{C_y (2a)^2}{6EI}(3x-2a)$ $\quad 2a<x<4a$

both valid at $x=2a$.

$v_{M_0} = \frac{M_0 x^2}{2EI}$ $\quad 0<x<4a$

$$v(2a)=0=-\frac{Pa^2}{6EI}(6a-a)+\frac{C_y(2a)^2}{6EI}(4a)+\left(\frac{Pa}{12}\right)\frac{(2a)^2}{2EI}$$

$$0=-\frac{5}{6}P+\frac{16}{6}C_y+\frac{1}{6}P$$

$\Rightarrow C_y=\frac{P}{4}$

(2) $M_B=-Pa+2(2a)\left(\frac{P}{4}\right)+\frac{Pa}{12}$

$\Rightarrow M_B=-\frac{5}{12}Pa$

(1) $B_y=P-C_y$

$B_y=\frac{3}{4}P$

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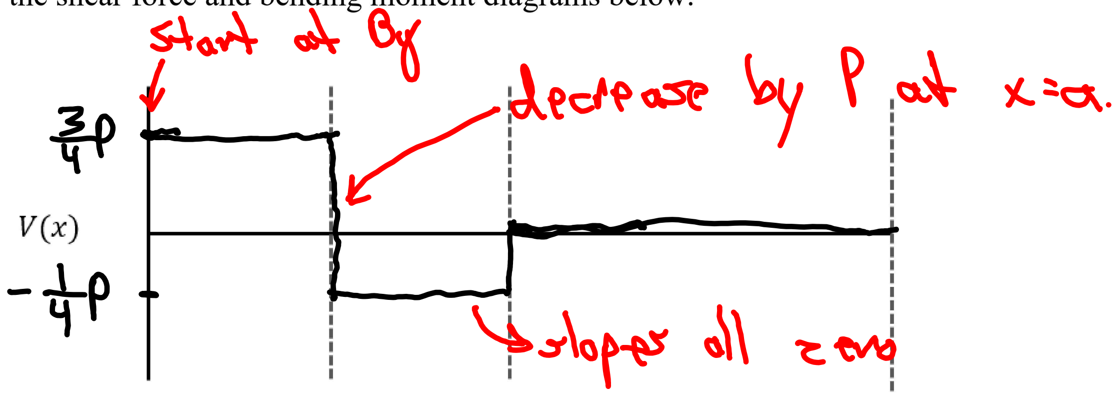
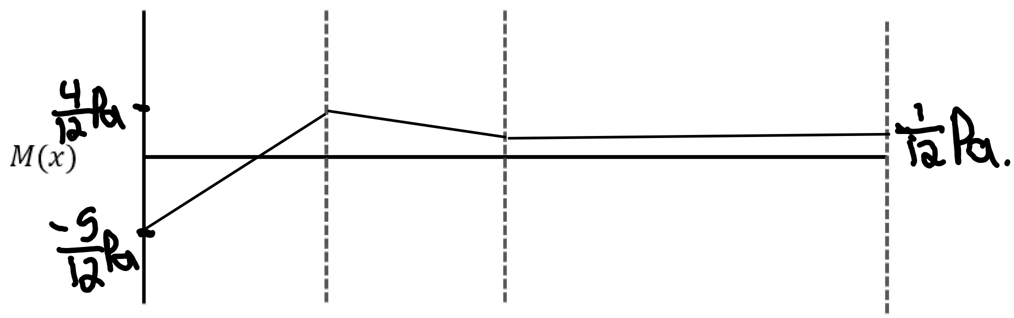
(d) Draw the shear force and bending moment diagrams below.





(e) Determine the location (along x) of the maximum flexural stress in the beam.

$x = 0$ (at B)

(f) Determine the magnitude of the maximum flexural stress in the beam.

$$|\sigma| = \frac{My}{I} = \frac{\left(\frac{5}{12}Pa\right)\left(\frac{a}{10}\right)}{\frac{\pi}{4}\left(\frac{a}{10}\right)^4} = \frac{5000P}{3\pi a^2}$$

(g) Determine the location (along x) of the maximum shear stress in the beam.

$0 < x < a$

(h) Determine the magnitude of the maximum shear stress in the beam.

$$\tau = \frac{4}{3}\frac{V}{A} = \frac{4}{3}\left(\frac{3}{4}P\right)\frac{1}{\pi\left(\frac{a}{10}\right)^2} = \frac{100P}{\pi a^2}$$

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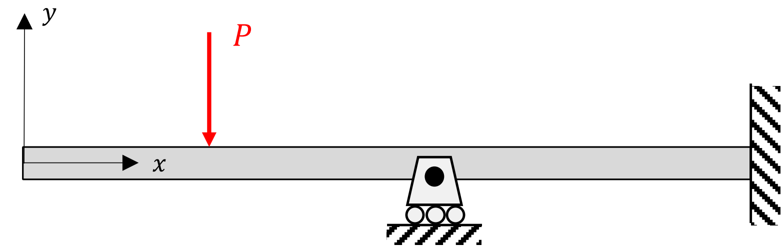
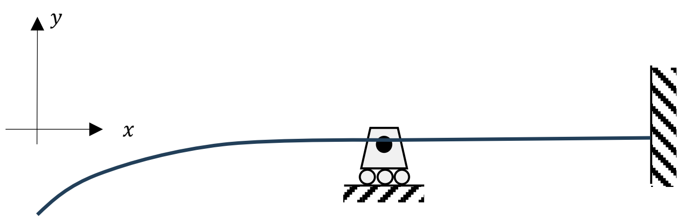
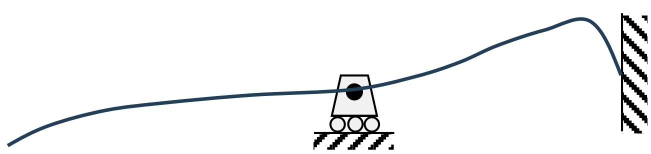
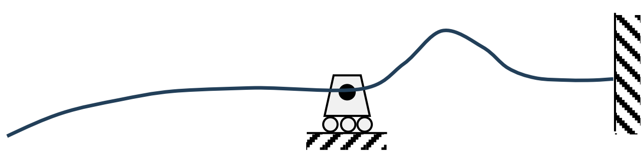
PROBLEM #4 (25 points)

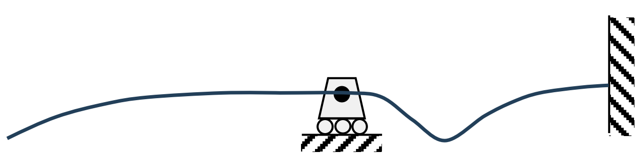
Part (A): (5 pts)
Circle which of the schematics presented below depicts the deflection curve of the following beam:



(a)


(b)


(c)


(d)


(e) None of the above

Note: both (C) and (e) are accepted as a correct answer. The beam slope at the roller appears to be zero which was not intended to be.

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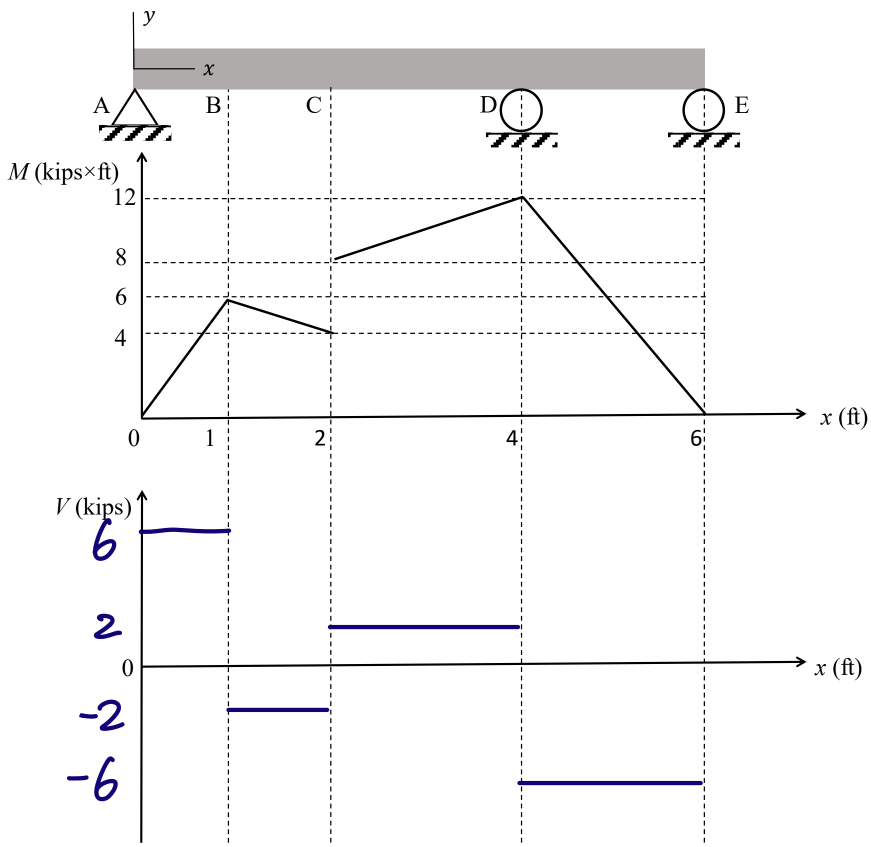
Part (B): (5 pts)
The following beam is loaded with concentrated moments and concentrated forces. Loading is unknown, however, the bending moment diagram for the beam is provided below:



(i) Draw the shear force diagram in the above figure.
(ii) What is the reaction force at the roller support D? Assuming no loading is applied at D.

$F_D = -8$ kips

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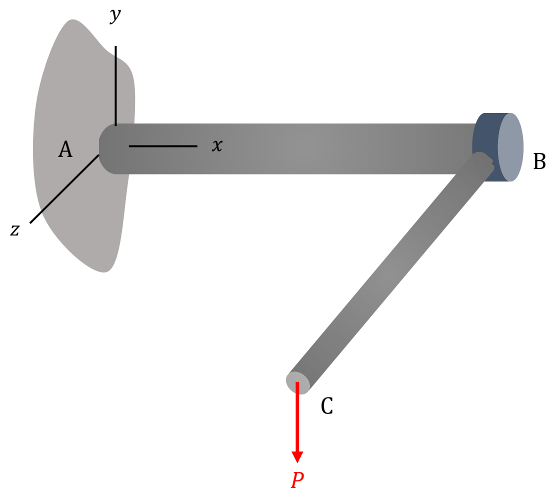
Part (C): (5 pts)
The following rigid structure is composed of arms AB and BC and is fixed to the wall at A. It is subject to a force P at C in the negative y direction. How many strain energy terms contribute to the total strain energy of the system? Note that the strain energy may include the uniaxial strain energy U_A, shear strain energy due to torsion U_T, flexural strain energy due to bending U_σ, and shear strain energy due to bending U_τ.



(a) 3

(b) 4

(c) 5

(d) 6

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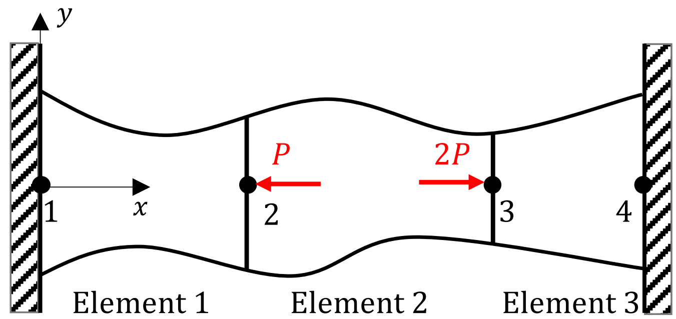
Part (D): (10 pts)

The double-cantilever beam AD is of an arbitrary shape. Two concentrated forces are applied where $P = 100\ lbs$. In solving this mechanics problem using the finite element method, beam AD is discretized into 3 elements and 4 nodes which are marked as 1, 2, 3, and 4 in the following figure, where nodes 1 and 4 are on the walls.



(i) The stiff matrix is given below, however, some components are missing. Fill up the missing components in the matrix.

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \times 10^5\ lb/in$$

$k_1 = 2\times10^5\ lb/in$

$k_2 = 1\times10^5\ lb/in$

$k_3 = 3\times10^5\ lb/in$

(ii) Determine the displacements at nodes 2 and 3.

Enforce the fixed boundary condition

$$\begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} \times10^5 \cdot \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -100 \\ 200 \end{Bmatrix}$$

$$u_2 = -\frac{2\times10^{-3}}{11}\ in, \qquad u_3 = \frac{5\times10^{-3}}{11}\ in$$

(iii) Determine the internal forces in the three elements.

$$F_1 = k_1 \cdot (u_2 - u_1) = 2\times10^5 \cdot \left(-\frac{2}{11}\times10^{-3}\right) = -\frac{400}{11}\ lb$$

$$F_2 = k_2\ (u_3 - u_2) = 1\times10^5 \cdot \left(\frac{7}{11}\times10^{-3}\right) = \frac{700}{11}\ lb$$

$$\ldots \left(\ldots 5\times10^{-3}\right) \quad 1500 \ldots\ lb$$

$F_3 = k_3 (u_4 - u_3) = 3\times10^3 \left(-\frac{\quad}{11}\right) = -\frac{\quad}{11} w$

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