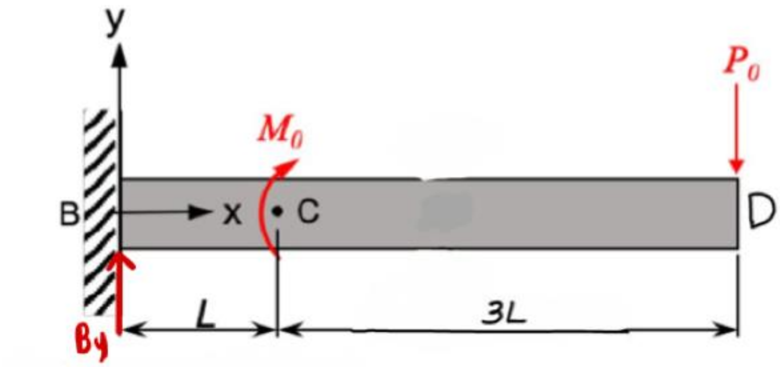
Problem 1 ***(10 points)*** The beam shown below has a Young's modulus E and second area moment of inertia I. Determine the following using the second-order integration method:

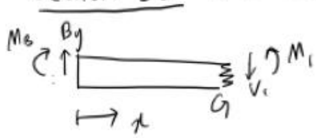
(a) Bending moment M(x) along the length of the beam.
(b) The slope of the beam $\theta(x)$.
(c) The deflection of the beam v(x).



$$\Sigma F_y = 0 \; : \; B_y - P_0 = 0 \quad \Rightarrow B_y = P_0 \quad ①$$

$$+\circlearrowleft \Sigma M_B = 0 \; : \; -M_B - M_0 - P_0(4L) = 0 \quad \Rightarrow M_B = -M_0 - 4P_0L$$

Section BC $\quad 0 \le x \le L$



$$\Rightarrow M_G : M_1 - M_B - B_y x = 0$$

$$\Rightarrow M_1 = M_B + B_y x$$

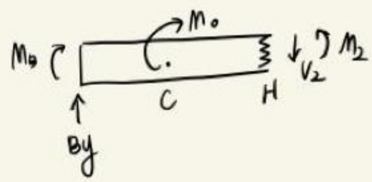
$$\theta_1(x) = \cancel{\theta(0)} + \frac{1}{EI}\int_0^x M_1 \, dx$$

$$= \frac{1}{EI}\left(M_B x + \frac{B_y}{2} x^2\right)$$

$$v_1(x) = \cancel{v(0)} + \int_0^x \theta_1(x)\,dx$$

$$= \frac{1}{EI}\left(\frac{M_B}{2}x^2 + \frac{B_y}{6}x^3\right)$$

<u>Section CD</u> $\quad L \le x \le 4L$



$$\Rightarrow M_H : \; M_2 - M_0 - M_B - B_y x = 0$$

$$\Rightarrow M_2 = M_B + M_0 + B_y x$$

$$\theta_2(x) = \theta_1(L) + \frac{1}{EI}\int_L^x M_2\,dx$$

$$= \frac{1}{EI}\left(M_B L + \frac{B_y}{2}L^2\right) + \frac{1}{EI}\Big[(M_B + M_0)x + \frac{B_y}{2}x^2\Big]_L^x$$

$$= \frac{1}{EI}\left(\cancel{M_B L} + \cancel{\frac{B_y}{2}L^2}\right) + \frac{1}{EI}\left\{(M_B + M_0)x + \frac{B_y}{2}x^2\right\}$$

$$- \frac{1}{EI}\left\{(\cancel{M_B} + M_0)L + \cancel{\frac{B_y}{2}L^2}\right\}$$

$$= \frac{1}{EI}\left\{(M_B + M_0)x + \frac{B_y}{2}x^2 - M_0 L\right\}$$

$$v_2(x) = v_1(L) + \int_L^x \theta_2(x)\,dx$$

$$= \frac{1}{EI}\left(\cancel{\frac{M_B}{2}L^2} + \cancel{\frac{B_y}{6}L^3}\right) + \frac{1}{EI}\left\{\frac{1}{2}(M_B + M_0)x^2 + \frac{B_y}{6}x^3 - M_0 L x\right\}$$

$$- \frac{1}{EI}\left\{\frac{1}{2}(\cancel{M_B} + M_0)L^2 + \cancel{\frac{B_y}{6}L^3} - M_0 L^2\right\}$$

$$= \frac{1}{EI}\left\{(M_B + M_0)\frac{x^2}{2} + \frac{B_y}{6}x^3 - M_0 L x + \frac{M_0}{2}L^2\right\}$$

Eqn ① & ②

$\Rightarrow$

(a) $$M(x) = \begin{cases} -M_0 - 4P_0L + P_0 x & 0 \le x \le L \\ -4P_0L + P_0 x & L \le x \le 4L \end{cases}$$

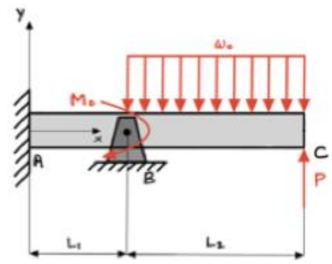
(b) $$\theta(x) = \begin{cases} \frac{1}{EI}\left\{(-M_0 - 4P_0L)x + \frac{P_0}{2}x^2\right\} & 0 \le x \le L \\ \frac{1}{EI}\left\{-4P_0Lx + \frac{P_0}{2}x^2 - P_0L\right\} & L \le x \le 4L \end{cases}$$

(c) $$v(x) = \begin{cases} \frac{1}{EI}\left\{(-M_0 - 4P_0L)\frac{x^2}{2} + \frac{P_0}{6}x^3\right\} & 0 \le x \le L \\ \frac{1}{EI}\left\{-2P_0Lx^2 + \frac{P_0}{6}x^3 - M_0Lx + \frac{M_0}{2}L^2\right\} & L \le x \le 4L \end{cases}$$

Problem 2 *(10 points)*

A cantilever beam ABC with a circular cross section has a roller support at B and is subjected to a uniformly distributed load w_0 between B and C, a concentrated load P at C, and an external couple M_0 at B. Using the **superposition principles,** determine:

(a) The reaction at point B.
(b) The analytical expression for the deflection of the beam v_x
(c) The deflection at point C.

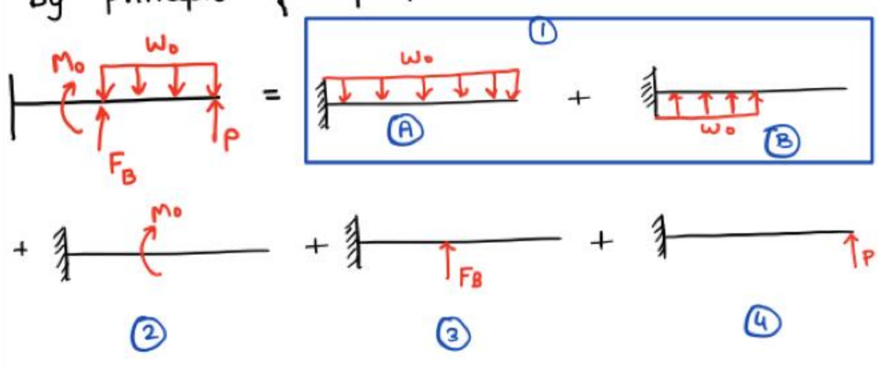


Data: $L_1 = 1$ m, $L_2 = 2$ m, $d = 40$ mm, $E = 200$ GPa, P = 150 N, $M_0 = 100$ N.m, and $w_0 = 80$ N/m.

Note: Please refer to the attached table for the necessary beam deflection equations.

By inspection, we can state that beam is Statically, indeterminate.

By principle of superposition:



Their individual deflections can be estimated from the charts:

① :

→ Ⓐ : $V_A(x) = -\frac{80x^2}{24EI}(54 - 12x + x^2) \qquad 0 \le x \le 3$

Ⓑ : $V_B(x) = \begin{cases} \frac{80x^2}{24EI}(6 - 4x + x^2) & 0 \le x \le 1 \\ \frac{80}{24EI}(4x - 1) & 1 \le x \le 3 \end{cases}$

② : $V_2(x) = \begin{cases} -\frac{50x^2}{EI} & 0 \le x \le 1 \\ -\frac{50}{EI}(2x-1) & 1 \le x \le 3 \end{cases}$

③ : $V_3(x) = \begin{cases} \frac{F_B x^2}{6EI}(3-x) & 0 \le x \le 1 \\ \frac{F_B}{6EI}(3x-1) & 1 \le x \le 3 \end{cases}$

④ : $V_4(x) = \frac{150x^2}{6EI}(9-x) \qquad 0 \le x \le 3$

To get the overall deflection of the beam, we will have to add 1, 2, 3 & 4 together [in their respective domains], and the value will add upto zero at $x = 1$.

Now,

$$V_{ABC}(x) = \begin{cases} \overbrace{-\frac{80x^2}{24EI}(54 - 12x + x^2) + \frac{80x^2}{24EI}(6 - 4x + x^2) - \frac{50x^2}{EI} + \frac{F_B x^2}{6EI}(3 - x) + \frac{150x^2}{6EI}(9 - x)}^{0 \le x \le 1m} \\ \overbrace{-\frac{80x^2}{24EI}(54 - 12x + x^2) + \frac{80}{24EI}(4x - 1) - \frac{50}{EI}(2x - 1) + \frac{F_B}{6EI}(3x - 1) + \frac{150x^2}{6EI}(9 - x)}^{1 \le x \le 3m} \end{cases}$$

@

$V_{ABC}(1m) = 0$

$$\therefore -\frac{80}{24\cancel{EI}}(54 - 12 + 1) + \frac{80}{24\cancel{EI}}(6 - 4 + 1) - \frac{50}{\cancel{EI}}$$

$$+ \frac{F_B}{6\cancel{EI}}(2) + \frac{150}{6\cancel{EI}}(8) = 0$$

$$\Rightarrow \quad -\frac{80}{24}(43) + 10 - 50 + \frac{F_B}{3} + 200 = 0$$

$$F_B = -50N$$

(b)

The analytical expression $V_{ABC}(x)$ can now be modified as follows:

$$V_{ABC}(x) = \begin{cases} 0 \le x \le 1m \\ \frac{1}{EI}\left(10x^3 - 10x^2\right) \\ 1 \le x \le 3m \\ \frac{1}{EI}\left(-10x^4 + 15x^3 + 45x^2 - \frac{335}{3}x + 55\right) \end{cases}$$

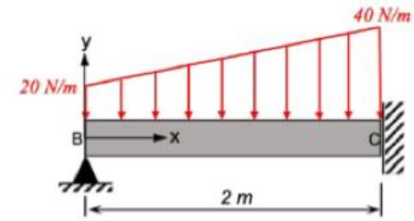
(c) Deflection at C:

$$V_{ABC}(3m) = \frac{\left(-10(81) + 15(27) + 45(9) - 335 + 55\right)}{(200\,GPa)\left(\frac{\pi}{4}(0.07)^4\right)}$$

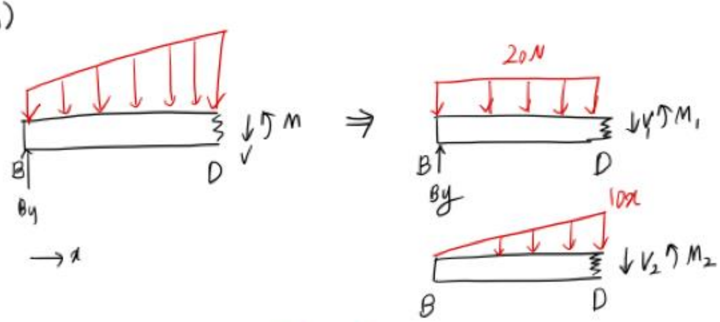
$$= 0.0103\ m$$

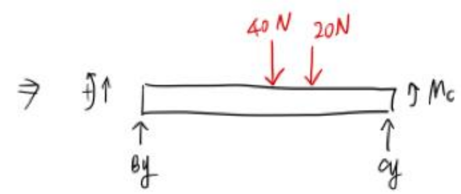
Problem 3. ***(10 points)*** The beam shown below has a Young's modulus of 50 GPa and has a square cross-section of side 1 cm. Determine the following using the second-order integration method:

(a) Bending moment M(x) along the length of the beam.
(b) The slope of the beam θ(x).
(c) The deflection of the beam v(x).
(d) Slope of the beam at support θ_B



(a)





$\Sigma F_y = B_y + C_y - 60 = 0$

$B_y + C_y = 60$ — ③

$\Sigma M_c = M_c - B_y(2) + 40(1) + 20\left(\frac{2}{3}\right) = 0$

$\Rightarrow M_c - 2B_y = -\frac{160}{3}$ — ②

$$M_D = M_1 + M_2 = \{B_y x - 20x(\tfrac{x}{2})\} + \{-5x^2(\tfrac{x}{3})\}$$

$$M(x) = B_y x - 10x^2 - \frac{5}{3}x^3$$ (indeterminate beam)

(b) $$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x M(x)\,dx$$

$$= \theta(0) + \frac{1}{EI}\left[\frac{B_y}{2}x^2 - \frac{10}{3}x^3 - \frac{5}{12}x^4\right]$$

(c) $$v(x) = \cancel{v(0)} + \int_0^x \theta(x)\,dx$$

$$= \theta(0)\,x + \frac{1}{EI}\left[\frac{B_y}{6}x^3 - \frac{5}{6}x^4 - \frac{1}{12}x^5\right]$$

B.C's

$v(2) = 0$

$$\Rightarrow v(2) = 2\theta(0) + \frac{1}{EI}\left[\frac{4}{3}B_y - \frac{40}{3} - \frac{8}{3}\right]$$

$$\Rightarrow \theta(0) = -\frac{1}{EI}\left(\frac{2}{3}B_y - 8\right)$$

$\theta(2) = 0$

$$\Rightarrow \theta(2) = \theta(0) + \frac{1}{EI}\left(2B_y - \frac{100}{3}\right)$$

$$= -\frac{1}{EI}\left(\frac{2}{3}B_y - 8\right) + \frac{1}{EI}\left(2B_y - \frac{100}{3}\right)$$

$\Rightarrow B_y = 19N \quad \Rightarrow C_y = 41N, \; M_c = -15.33 N\cdot m$

$$I = \frac{b^4}{12} = \frac{(0.01)^4}{12} \Rightarrow EI = 41.67\ N \cdot m^2$$

$$\theta(0) = -\frac{1}{41.67}\left(\frac{14}{3}\right) = -0.112\,\text{rad} \approx -6.42^\circ$$

$$\Rightarrow \quad M(x) = 19x - 10x^2 - \frac{5}{3}x^3 \quad (N \cdot m)$$

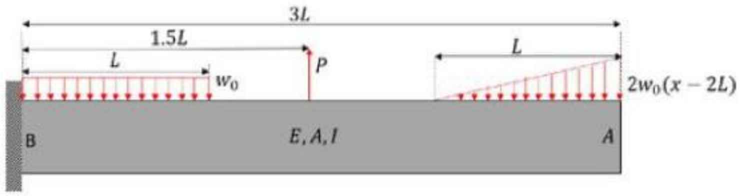
$$\theta(x) = -0.112 + \frac{1}{41.67}\left\{\frac{19}{2}x^2 - \frac{10}{3}x^3 - \frac{5}{12}x^4\right\} \text{rad}$$

$$V(x) = -0.112x + \frac{1}{41.67}\left\{\frac{19}{6}x^3 - \frac{5}{6}x^4 - \frac{1}{12}x^5\right\} m$$

(d) $\theta_B = \theta(0) = -0.112\ \text{rad} = -6.42^\circ$

Problem 4 *(5 points)*

1. Consider the cantilevered beam with distributed loads as shown below



Select the general form that the equation for flexual energy due to bending will take (note that $f_i(x)$ represents some continuous function $M(x)/(2EI)$ for the length of integration):

(a) $U = \int_0^L f_1(x)dx + \int_L^{2L} f_2(x)dx + \int_{2L}^{3L} f_3(x)dx$

(b) $U = \int_0^L f_1(x)dx + \int_L^{1.5L} f_2(x)dx + \int_{1.5L}^{3L} f_3(x)dx$

(c) $U = \int_0^L f_1(x)dx + \int_L^{1.5L} f_2(x)dx + \int_{1.5L}^{2L} f_3(x)dx + \int_{2L}^{3L} f_4(x)dx$

(d) $U = \int_0^L f_1(x)dx + \int_0^{1.5L} f_2(x)dx + \int_0^{2L} f_3(x)dx + \int_0^{3L} f_4(x)dx$

$\{0 < x < L\}$, $\{L < x < 1.5L\}$, $\{1.5L < x < 2L\}$, and $\{2L < x < 3L\}$ each have unique equations for bending moment. Integration is performed section by section so that the sum of integrals spans the region of interest.