ME 323: Mechanics of Materials **Homework 8**

Spring 2024 **Due: March 22, 2024**

Problem 1 (10 points):

The beam AD is fixed to a rigid wall at A and is supported by props at B and C as shown in *figure 1*. In sections AB and BC, the flexural rigidity is EI, but in section CD the flexural rigidity is 2EI. The beam supports a linearly distributed load over span BC.
Use Castigliano's Second Theorem (neglecting shear energy) to determine:

1. Reactions at end A.
2. Slope θ of the beam at the support C.

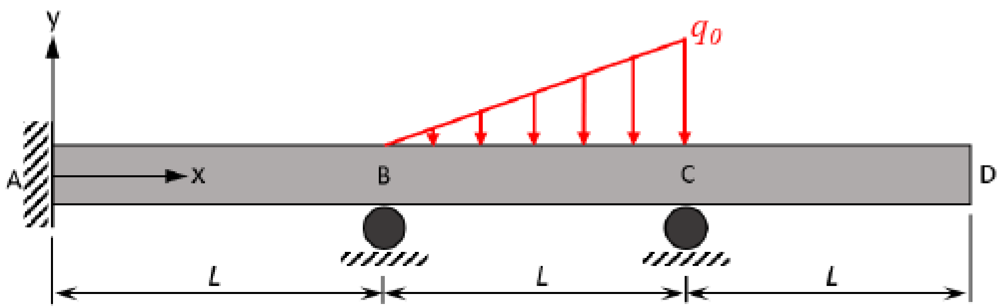
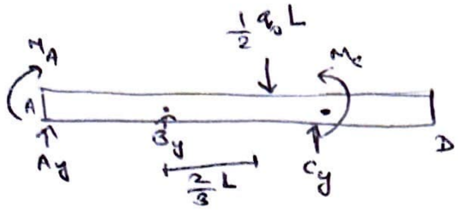


Figure 1: Beam loading for Problem 1

Problem 1

FBD:



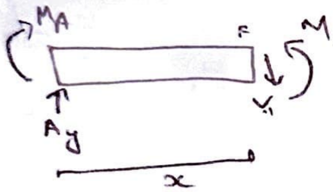
$$\Sigma F_y = A_y + B_y + C_y - \frac{q_0 L}{2} = 0 \Rightarrow A_y + B_y + C_y = \frac{q_0 L}{2}$$

$$\Sigma M_A = B_y L - M_A + C_y(2L) + M_c - \frac{q_0 L}{2} \times \frac{5L}{3} = 0$$

$$M_A = M_c + B_y L + C_y 2L - \frac{5 q_0 L^2}{6}$$

Indeterminate problem

Section AB $(0 \le x \le L)$



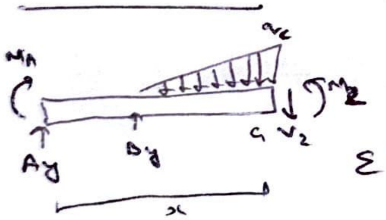
$$\Sigma M_F = M_1 - A_y x - M_A = 0$$

$$M_1 = M_A + A_y x$$

$$= M_c + B_y L + C_y 2L - \frac{5}{6} q_0 L^2 + \left(\frac{q_0 L}{2} - B_y - C_y\right) x$$

$$M_1 = M_c + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + B_y(L - x) + C_y(2L - x)$$

Section BC $(L < x \le 2L)$



$$q_c = \frac{q_0 (x - L)}{L}$$

$$\Sigma M_G = M_2 - M_A - A_y x - B_y(x - L) + \frac{q_c}{2}(x-L)\left(\frac{x-L}{3}\right) = 0$$

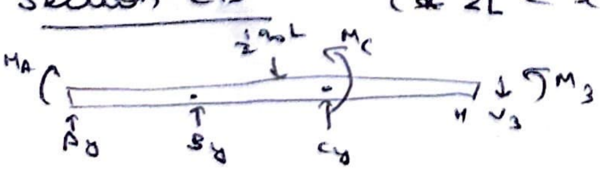
$$M_2 = M_A + A_y x + B_y(x - L) - \frac{q_0 (x-L)^3}{6L}$$

$$= M_c + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + B_y(L - x) + C_y(2L - x) + B_y(x - L) - \frac{q_0 (x-L)^3}{6L}$$

$$M_2 = M_c + C_y(2L - x) - \frac{q_0 (x-L)^3}{6L} + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right)$$

$$= M_c + C_y(2L - x) + \frac{q_0}{2}\left(xL - \frac{5L^2}{3} - \frac{(x-L)^3}{3L}\right)$$

Section CD $(2L < x \le 3L)$



$$\Sigma M_H = 0: M_3 - M_A - A_y x - B_y(x - L) + \frac{1}{2} q_0 L\left(x - \frac{5L}{3}\right) - C_y(x - 2L) + M_c = 0$$

$$M_3 = M_A - M_c + A_y x + B_y(x - L) - \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + C_y(x - 2L)$$

$$M_3 = M_c + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + B_y(L-x) - M_c + B_y(x-L) + C_y(2L-x) + C_y(x-2L) - \frac{q_0 L}{2}\left(x - \frac{5L}{3}\right)$$

$$M_3 = 0$$

$$U = U_1 + U_2 + U_3$$

$$= \frac{1}{2EI}\int_0^L M_1^2\,dx + \int_L^{2L} M_2^2\,dx + \int_{2L}^{3L} M_3^2\,dx \to 0$$

Now, B_y and C_y are redundant forces

$$\therefore \frac{\partial U}{\partial B_y} = 0 = \frac{1}{2EI}\int_0^L 2\left\{M_c + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + B_y(L-x) + C_y(2L-x)\right\}(L-x)\,dx + 0$$

$$\Rightarrow \int_0^L \left[\frac{q_0 L}{2}\left(x - \frac{5}{3}L\right)(L-x) + B_y(L^2 + x^2 - 2Lx) + C_y\underbrace{(2L-x)(L-x)}_{2L^2 - xL + x^2 - 2xL}\right] dx = 0$$

$$\int_0^L \left[\frac{q_0 L}{6}(8xL - 3x^2 - 5L^2) + x^2(B_y + C_y) - xL(2B_y + 3C_y) + L^2(B_y + 2C_y)\right] dx = 0$$

$$\int_0^L \left[x^2\left(B_y + C_y - \frac{q_0 L}{2}\right) - Lx\left(2B_y + 3C_y - \frac{4}{3}q_0 L\right) + L^2\left(B_y + 2C_y - \frac{5}{6}q_0 L\right)\right] dx = 0$$

$$\frac{L^3}{3}\left(B_y + C_y - \frac{q_0 L}{2}\right) - \frac{L^3}{2}\left(2B_y + 3C_y - \frac{4 q_0 L}{3}\right) + L^3\left(B_y + 2C_y - \frac{5 q_0 L}{6}\right) = 0$$

$$\frac{B_y}{3} + \frac{5}{6}C_y - \frac{q_0 L}{3} = 0$$

$$\frac{\partial U}{\partial C_y} = 0 = \frac{1}{2EI}\int_0^L 2\left(M_c + \frac{q_0 L}{2}\left(x - \frac{5}{3}L\right) + B_y(L-x) + C_y(2L-x)\right)(2L-x)\,dx$$

$$+ \frac{1}{2EI}\int_L^{2L} 2\left\{M_c + C_y(2L-x) + \frac{q_0}{2}\left(xL - \frac{5}{3}L^2 - \frac{(x-L)^3}{3L}\right)\right\}(2L-x)\,dx$$

$$= \int_0^L \left(\frac{q_0 L}{2}\left(2Lx - \frac{10L^2}{3} - x^2 + \frac{5}{3}Lx\right) + B_y(2L^2 - xL - xL + x^2) + C_y(4L^2 + x^2 - 4xL)\right) dx$$

$$+ \int_L^{2L} \left(C_y(4L^2 + x^2 - 4xL) + \frac{q_0}{2}\left(2xL - x^2 L - \frac{5L^3}{3} + \frac{5L^2 x}{3} - \frac{2}{3L}(x^3 - L^3 - 3x^2 L + 3xL^2)(2L - x)\right)\right) dx$$

$$= \left[\left(2B_y L^2 + 4C_y L^2 - \frac{5q_0 L^3}{2}\right)x - \left(\frac{3}{2}B_y L + 2C_y L - \frac{11}{12}q_0 L^2\right)x^2 + \frac{x^3}{3}\left(B_y + C_y - \frac{q_0 L}{2}\right)\right]\Bigg|_0^L$$

$$+ \left[\left(4C_y L^2 - \frac{4}{3}L^3 q_0\right)x - \left(2C_y L - \frac{q_0 L^2}{3}\right)x^2 + \left(\frac{C_y}{3} + \frac{q_0 L}{3}\right)x^3 - \frac{5q_0 x^4}{24} + \frac{q_0 x^5}{30L}\right]\Bigg|_L^{2L} = 0$$

$$\left(\frac{5B_yL^3}{6} + \frac{7C_yL^3}{3} - \frac{11q_0L^4}{12}\right) + \left(\frac{8C_yL^3}{3} - \frac{14q_0L^4}{15} - \frac{7C_yL^3}{3} + \frac{101}{120}q_0L^4\right) = 0$$

$$100B_y + 320C_y - 121q_0L = 0$$

$\Rightarrow$ Solving (4) equations and (4) unknowns,

$$A_y = -0.05q_0L \qquad B_y = 0.25q_0L$$

$$C_y = 0.3q_0L \qquad M_A = -0.2833q_0L^2$$

2) $$\theta_c = \left.\frac{\partial U}{\partial M_c}\right|_{M_c=0}$$

$$M_1 = M_c - q_0(0.2833L^2 + 0.05Lx)$$

$$M_2 = M_c - q_0\left(0.0667L^2 + 0.3Lx - 0.5x^2 + 0.1667\frac{x^3}{L}\right)$$

$$\frac{\partial U}{\partial M_c} = \frac{1}{2EI}\left[\int_0^L 2\left(M_c - q_0(0.2833L^2 + 0.05Lx)\right)dx + \int_{L}^{2L} 2\left(M_c - q_0\left(0.0667L^2 + 0.3Lx - 0.5x^2 + 0.1667\frac{x^3}{L}\right)\right)dx\right]$$

$$\theta_c = 0.0248\frac{q_0L^3}{EI}$$

Problem 2 (10 points):

A simply-supported beam *ABC* is subjected to a vertical load *P* at C. The cross section of the beam is shown in *figure 2*, and given that P = 1000 N and L = 2 m. The elastic and shear modulus of the material are 200 GPa and 75 GPa, respectively. The form factor for a square cross section is f_s= 6/5.

a) Compare the flexural energy and the shear energy due to bending.

b) Use Castigliano's second theorem to determine the vertical deflection of the end *C including the strain energy due to shear,.*

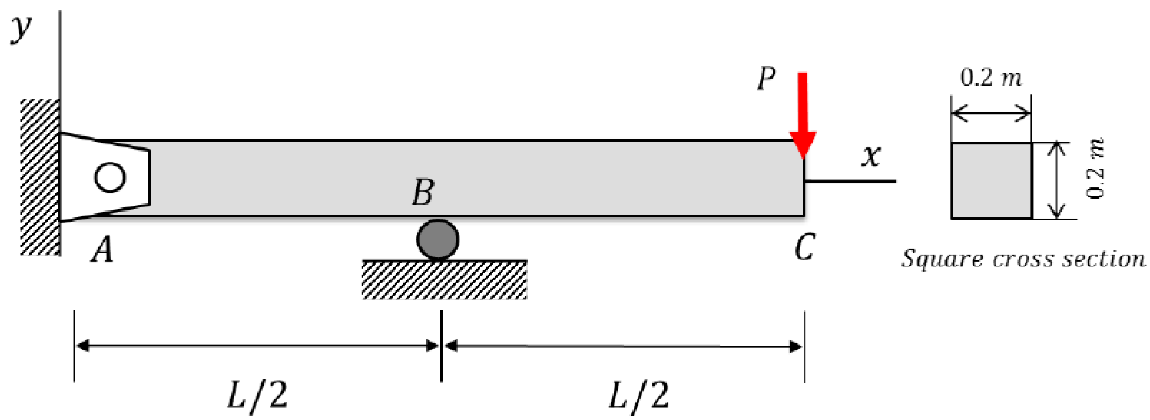
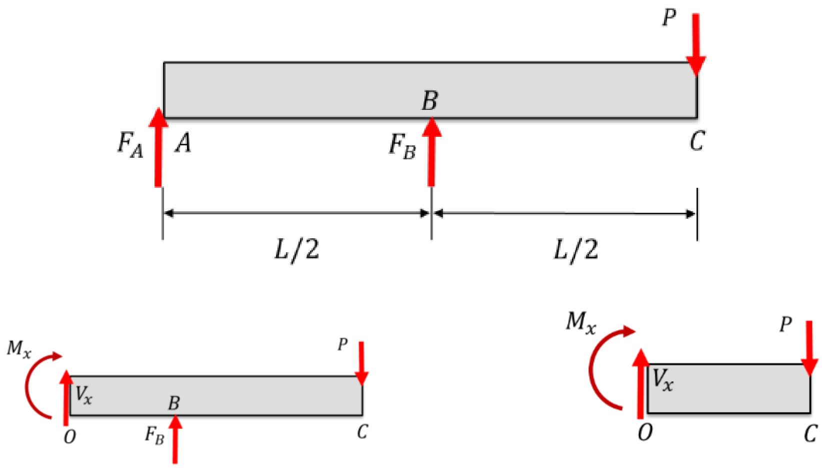


Figure 2: Beam ABC and its cross-section

SOLUTION

a)



$0 < x < 0.5L$ $\qquad$ $0.5L < x < L$

FBD analysis over the whole beam,

$$\Sigma M_A = F_B\left(\frac{L}{2}\right) - PL = 0 \Rightarrow F_B = 2P$$

$0 < x < 0.5L$:

$$\Sigma F_y = V_{1x} - P + F_B = 0 \Rightarrow V_{1x} = -P$$

$$\Sigma M_o = -M_{1x} - P(L-x) + F_B\left(\frac{L}{2} - x\right) = 0 \Rightarrow M_{1x} = -Px$$

$0.5L < x < L$:

$$\Sigma F_y = V_{2x} - P = 0 \Rightarrow V_{2x} = P$$

$$\Sigma M_o = -M_{2x} - P(L-x) = 0 \qquad \Rightarrow M_{2x} = P(x-L)$$

Flexural energy due to bending,

$$U_{flexural} = \int_0^{0.5L} \frac{{M_{1x}}^2 dx}{2EI} + \int_{0.5L}^{L} \frac{{M_{2x}}^2 dx}{2EI} = \frac{1}{2}\int_0^{0.5L} \frac{P^2x^2dx}{EI} + \frac{1}{2}\int_{0.5L}^{L} \frac{P^2(x-L)^2dx}{EI} = \frac{P^2L^3}{24EI}$$

Shear energy due to bending,

$$U_{shear} = \int_0^{0.5L} \frac{f_s {V_{1x}}^2 dx}{2GA} + \int_{0.5L}^{L} \frac{f_s {V_{2x}}^2 dx}{2GA} = \int_0^{L} \frac{f_s P^2 dx}{2GA} = \frac{f_s P^2 L}{2GA}$$

$$R = \frac{U_{flexural}}{U_{shear}} = \frac{\frac{P^2 L^3}{24EI}}{\frac{f_s P^2 L}{2GA}} = \frac{G}{f_s E}\left(\frac{L}{d}\right)^2 = \frac{75}{\frac{6}{5}(200)}\left(\frac{2}{0.2}\right)^2 = 31.25$$

b)

$$U_{total} = U_{flexural} + U_{shear} = 32.25 U_{shear} = 32.25 \frac{f_s P^2 L}{2GA} = 1.29 \times 10^{-2} J$$

Using work-energy principle,

$$U_{total} = \frac{1}{2} P v_c$$

$$v_c = \frac{2U_{total}}{P} = 2.58 \times 10^{-5}\ m$$

Using Castigliano's Theorem,

$$v_B = \frac{\partial U_{total}}{\partial P} = 32.25 \frac{f_s PL}{GA} = 2.58 \times 10^{-5}\ m$$

(Either work-energy principle or Castigliano's Theorem is OK)

Problem 3 (10 points):

Figure 3 shows a cantilevered beam (ABC) with modulus E and moment of inertia I. A distributed load of w_0 spans 1/3rd of its length. The beam is simply supported at point B. Use Castigliano's theorem to determine the reactions at A and B. You can neglect the shear energy due to bending.

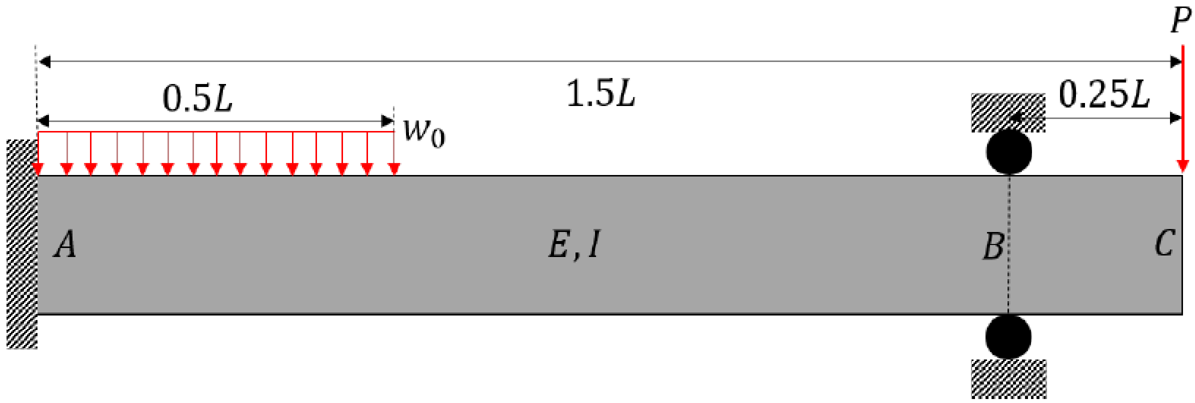
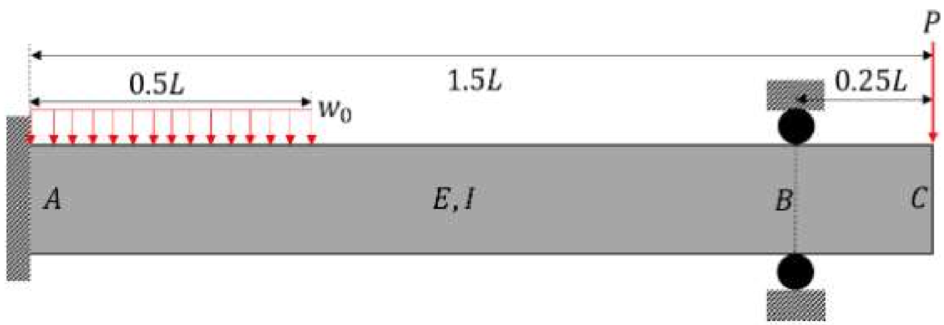
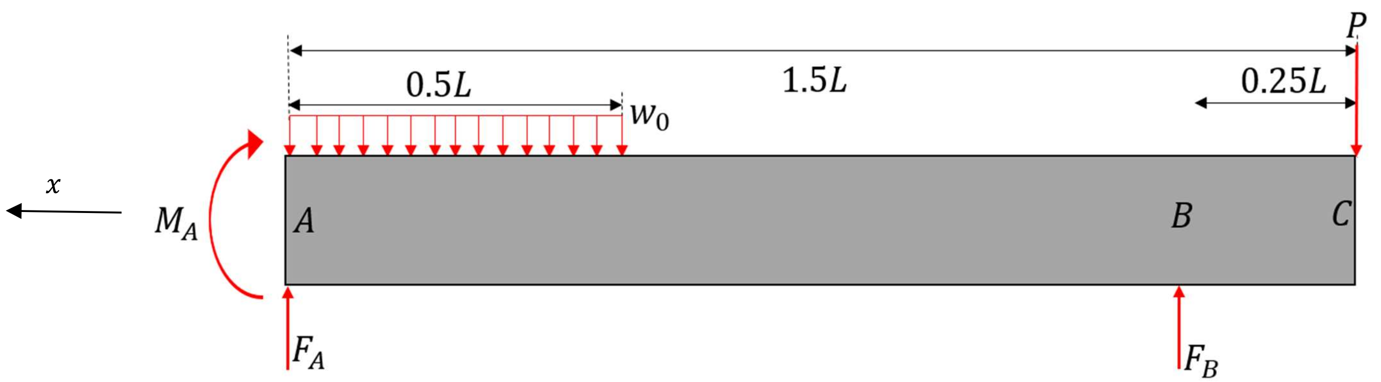


Figure 3: Cantilever beam loading for Problem 3



Solution:



2 pt.

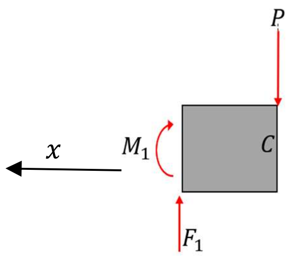
For all FBDs

Equilibrium: **Beam is statically indeterminate.** 1 pt.

$$\sum M_A = M_A - \frac{1}{2}w_o L\left(\frac{L}{2}\right) + \frac{5}{4}B_y L - \frac{3}{2}PL$$

$$M_A - \frac{1}{4}w_o L^2 + \frac{5}{4}B_y L - \frac{3}{2}PL = 0$$

$$\sum F_y = A_y + B_y - P - \frac{1}{2}w_o L = 0$$



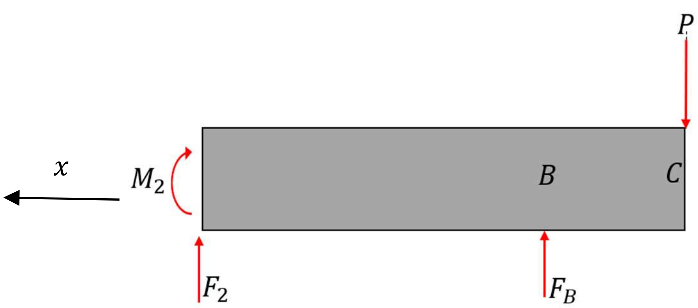
Making a cut at section $0 < x < \frac{L}{4}$ we get:

$$\sum M_{cut} = 0$$

$$-M_1 - Px = 0$$

1 pt.

$$M_1(x) = -Px, \qquad \frac{\delta M_1}{\delta_{B_y}} = 0$$



Making a cut at section $\frac{L}{4} < x < L$ we get:

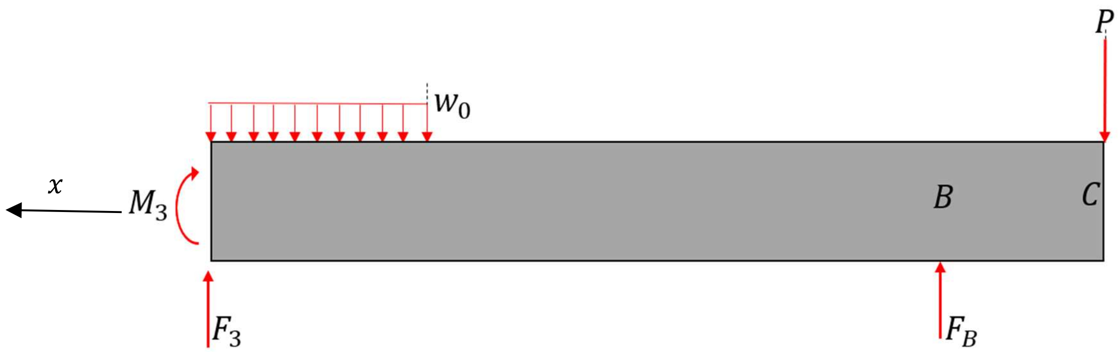
$$\sum M_{cut} = 0$$

$$-M_2 + B_y\left(x - \frac{L}{4}\right) - Px = 0$$

1 pt.

$$M_2(x) = B_y\left(x - \frac{L}{4}\right) - Px, \qquad \frac{\delta M_2}{\delta_{B_y}} = x - \frac{L}{4}$$

1 pt. for 3 correct cuts



Making a cut at section $L < x < \frac{3}{2}L$ we get:

$$\sum M_{cut} = 0$$

$$-M_3 - \frac{1}{2}w_o(x-L)^2 + B_y\left(x - \frac{L}{4}\right) - Px = 0$$

1 pt.

$$M_3(x) = -\frac{1}{2}w_o(x-L)^2 + B_y\left(x - \frac{L}{4}\right) - Px,$$

$$\frac{\delta M_3}{\delta B_y} = x - \frac{L}{4}$$

Total strain energy $= U = \int_0^{\frac{L}{4}} \frac{M_1^2}{2EI}dx + \int_{\frac{L}{4}}^{L} \frac{M_2^2}{2EI}dx + \int_L^{\frac{3}{2}L} \frac{M_3^2}{2EI}dx$ 1 pt.

$$\frac{\delta U}{\delta B_y} = 0 + \int_{\frac{L}{4}}^{L} \frac{M_2}{EI}\frac{\delta M_2}{\delta B_y}dx + \int_L^{\frac{3}{2}L} \frac{M_3}{EI}\frac{\delta M_3}{\delta B_y}dx = 0$$

1 pt.

$$\frac{1}{EI}\int_{\frac{L}{4}}^{L}\left(B_y\left(x-\frac{L}{4}\right) - Px\right)\left(x - \frac{L}{4}\right)dx + \frac{1}{EI}\int_L^{\frac{3}{2}L}\left(-\frac{1}{2}w_o(x-L)^2 + B_y\left(x-\frac{L}{4}\right) - Px\right)\left(x-\frac{L}{4}\right)dx = 0$$

$$\frac{9L^3}{128}(2B_y - 3P) + \frac{L^3}{384}(196B_y - 9Lw_o - 244P) = 0$$

$$\frac{128}{192}B_y = \frac{3}{128}w_oL + \frac{325}{384}P$$

$$\boxed{B_y = \frac{9}{256}w_oL + \frac{325}{256}P}$$

1 pt.

Optional solutions for A_y and M_a

$$A_y = P + w_o L - B_y = P + w_o L - \frac{9}{256} w_o L - \frac{325}{256} P$$

$$\boxed{A_y = \frac{247}{256} w_o L - \frac{69}{256} P}$$

$$M_A = \frac{1}{4} w_o L^2 - \frac{5}{4} B_y L + \frac{3}{2} PL = \frac{1}{4} w_o L^2 - \frac{45}{1024} w_o L^2 - \frac{1625}{1024} PL + \frac{3}{2} PL$$

$$\boxed{M_A = \frac{211}{1024} w_o L^2 - \frac{89}{1024} PL}$$

Problem 4 (5 points):

Considering all the contributions to strain energy, the total strain energy for the figure 4.1 will have how many non-zero terms?

a. 3
b. 5
c. 6
d. 7

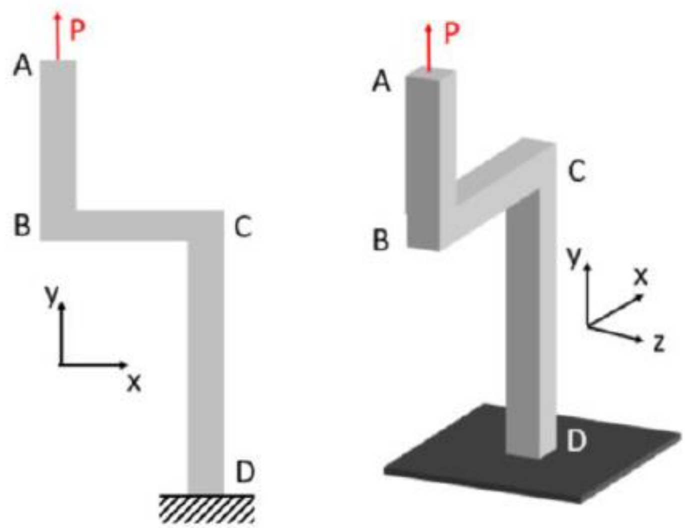
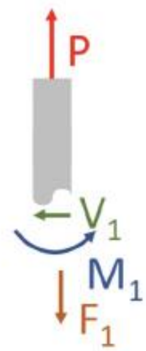
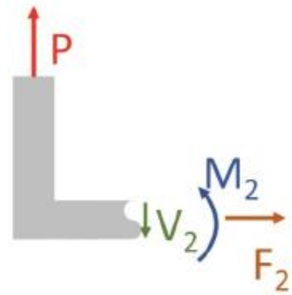


Figure 4: Loading of 3D structure for Problem 4

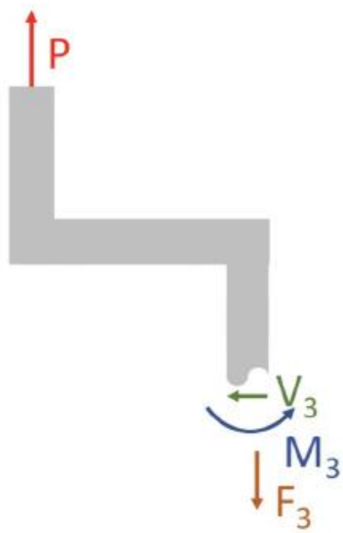
Solution:



$F_1 = P$
$V_1 = 0$
$M_1 = 0$



$F_2 = 0$
$V_2 = P$
$M_2 = Px$



$F_3 = P$
$V_3 = 0$
$M_3 = PL_2$

$$U = \frac{1}{2EA}\int_0^{L_1} F_1^2\,dy + \frac{f_s}{2GA}\int_0^{L_2} V_2^2\,dx + \frac{1}{2EI}\int_0^{L_2} M_2^2\,dx + \frac{1}{2EA}\int_0^{L_3} F_3^2\,dy + \frac{1}{2EI}\int_0^{L_3} M_3^2\,dy$$

→ 5 non-zero terms