-Course reviews.

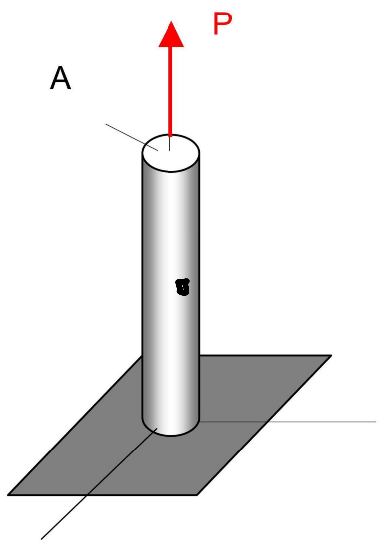
18. Column buckling

Objectives:

To study the possible failure mode of a column under a compressive axial load as a result of a "*buckling instability*".

Background:

- A column under an axial loading P will experience a material failure due to yielding when the magnitude of the normal stress $|\sigma| = \frac{|P|}{A}$ exceeds the yield strength of the material σ_Y.



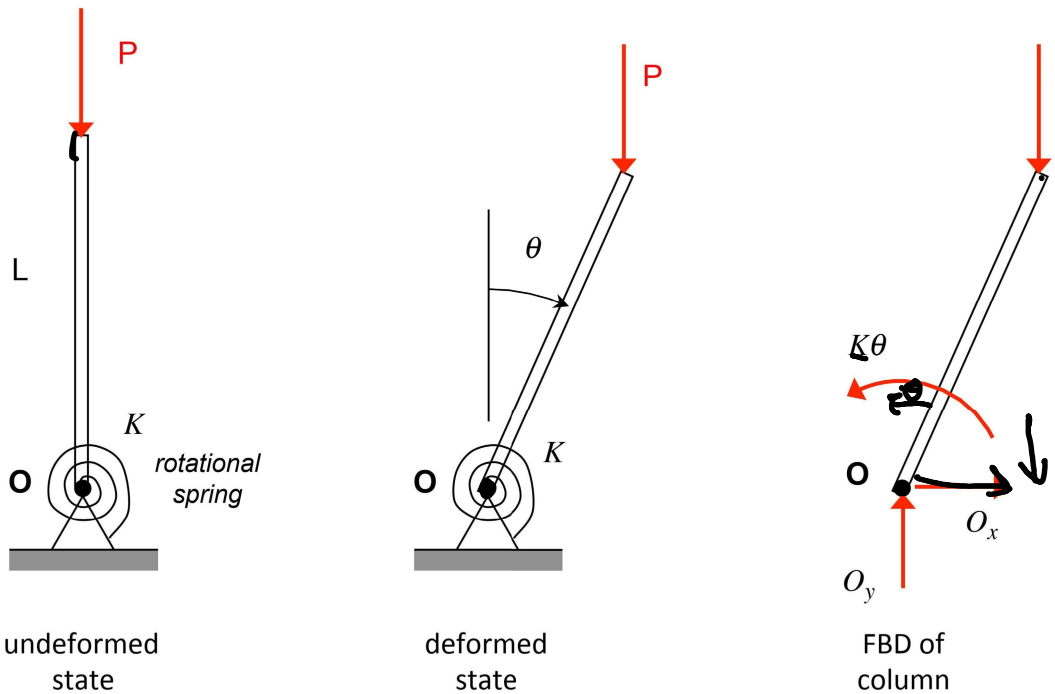
- For thin columns experiencing *compressive* axial loading, failure is possible for axial loads that are considerable less than that which $\frac{|P|}{A} = \sigma_Y$ due to a "buckling instability".
- The *moment-curvature equation* for a thin bending beam is given by:

$$M(x) = EI\frac{d^2v}{dx^2}$$

P
V
M
θ
v

Simple model for buckling instability – rigid bar

Consider a rigid bar of length L is pinned to ground at O. A rotational spring (spring which exerts a couple that is proportional to its angular deformation) is attached between the bar and ground at O. A downward vertical force P acts at the free end of the bar. Ignore the influence of gravity.



For equilibrium of the column:

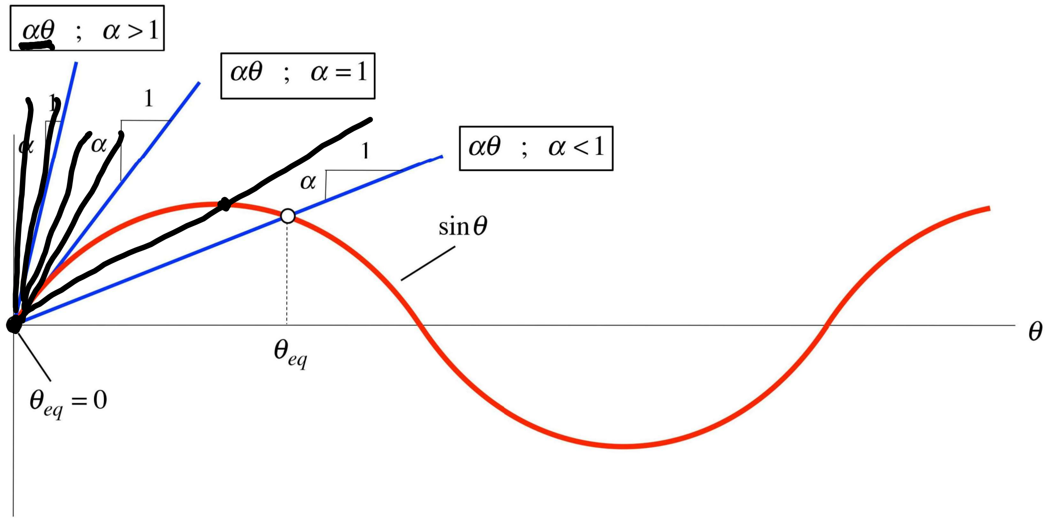
$$\sum M_O = -K\theta + PL\sin\theta = 0 \quad \Rightarrow \quad \sin\theta_{eq} = \frac{K}{PL}\theta_{eq} = \alpha\theta_{eq} \tag{1}$$

where $\alpha = K/PL$.

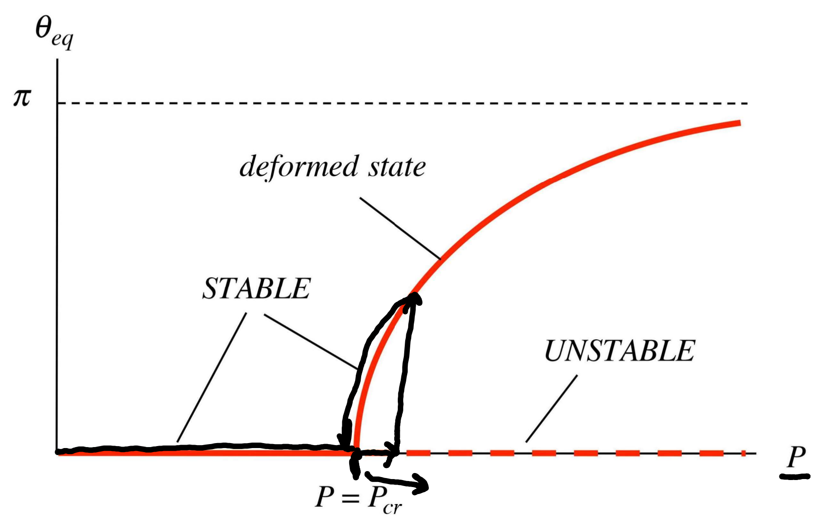
The above equation tells us the relationship among the loading P, the column length L, the stiffness of the rotational spring and the equilibrium angle θ_{eq} of the column. A couple observations:

- An undeformed equilibrium state is possible for all values of loading P. To see this, $\theta_{eq} = 0$ satisfies equation (1) regardless of the value of P. This result agrees with one's intuition since the loading P does not cause of moment about O when $\theta = 0$.
- For $\alpha < 1$ ($P > K/L$) there exists another nonzero solution of θ_{eq} for equation (1). To see this, consider the plots of $\alpha\theta_{eq}$ and $\sin\theta_{eq}$ vs. θ_{eq} in the following graph. Stated in words, for large loadings P, the column can take on deformed configuration. It can be shown that when this non-zero equilibrium state exists, the zero solution $\theta_{eq} = 0$ is *UNSTABLE*. That is, even though the column can be

in equilibrium, any disturbance away from that state will have the column take on the stable, "buckled", deformed state.



A sketch of the solution of the above equilibrium equation (1) is shown below:



In summary:

- For $P < K/L$ ($\alpha > 1$), the undeformed state of the column is *STABLE*.
- For $P > K/L$ ($\alpha < 1$) the undeformed state of the column is *UNSTABLE* and the column will take on the "buckled" deformed state.
- $P_{cr} = \dfrac{K}{L}$ is known as the critical load for buckling of the bar. This is the largest magnitude of a compressive axial force for which the undeformed state is not unstable.

Euler buckling model for columns

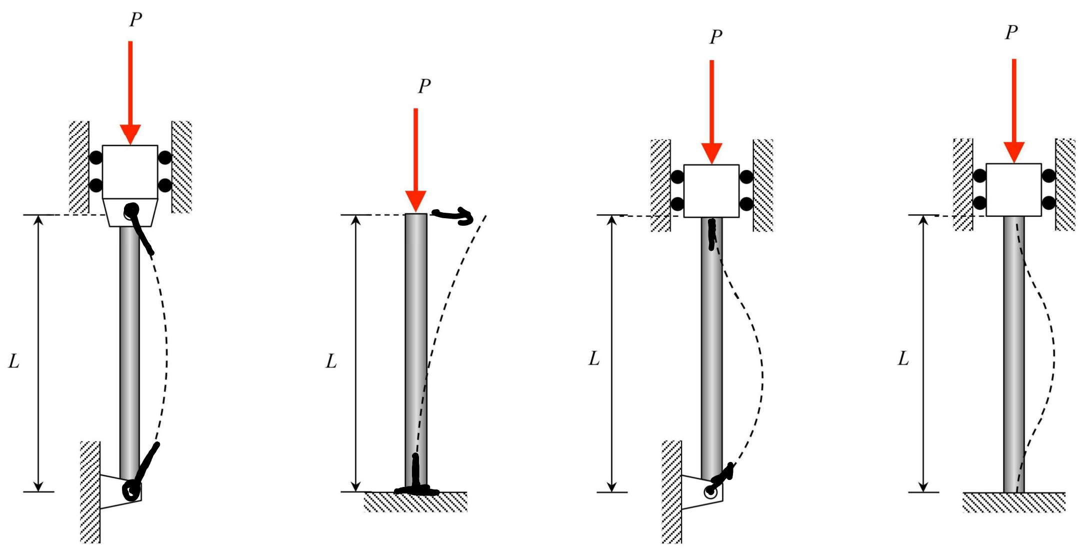
Here we will consider a thin flexible column that is acted upon by a compressive axial loading P. The Euler-Bernoulli bending "continuous" model will be used in our buckling analysis. We will study the buckling load *P* for four different types of boundary conditions for the column (shown in the figures below):

A) pinned-pinned column

B) fixed-free column

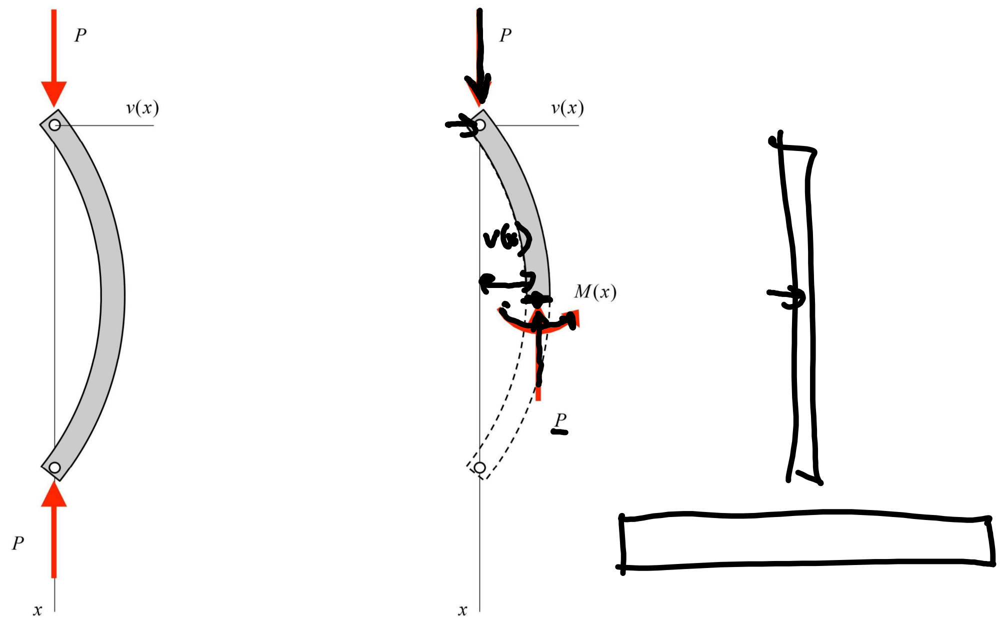
C) fixed-pinned column

D) fixed-fixed column



pinned-pinned fixed-free fixed-pinned fixed-fixed

A) Pinned-pinned column



FBD of column ***FBD of top section of column***

Here we have the axial load P and the beam moment (using Euler-Bernoulli theory for thin beams):

$$M(x) = EI\frac{d^2v}{dx^2}$$

acting on the top section of the column at the cut. For equilibrium:

$$\sum M_B = M(x) + Pv(x) = 0 \quad \Rightarrow$$

$$EI\frac{d^2v}{dx^2} + Pv = 0 \tag{2}$$

Equation (2) is the equation of equilibrium for a deflected column acted upon by an axial load P. The general form of the solution for this differential equation is:

$$v(x) = A\cos\left(x\sqrt{P/EI}\right) + B\sin\left(x\sqrt{P/EI}\right)$$

Enforcing the following boundary conditions for the pinned-pinned column:

$$v(0) = v(L) = 0 \tag{3}$$

gives:

$$v(0) = 0 = A\cos(0) + B\sin(0) = A \quad \Rightarrow \quad A = 0$$

and:

$$v(L) = 0 = A\cos\left(L\sqrt{P/EI}\right) + B\sin\left(L\sqrt{P/EI}\right) = B\sin\left(L\sqrt{P/EI}\right) \quad \Rightarrow$$

$$B\sin\left(L\sqrt{P/EI}\right) = 0 \tag{4}$$

Equation (4) says that *EITHER*:

- $B = 0$, *OR*
- $\sin\left(L\sqrt{P/EI}\right) = 0$

If $B = 0$, then $v(x) = 0$. That is, the undeformed shape of the column is always possible, just as it was for the rigid-bar model. For a deformed shape of the column ($B \neq 0$), then we must have:

$$\left[\sin\left(L\sqrt{P/EI}\right) = 0\right] \Rightarrow \left[L\sqrt{\frac{P}{EI}} = n\pi\right]; \quad n = 1,2,3,... \tag{5}$$

The values of P satisfying equation (5) are known as the critical loads for buckling of the column:

$$\boxed{P_{cr} = n^2\pi^2\frac{EI}{L^2}} \; * \tag{6}$$

For loads above the lowest critical value of P,

$$\left[P > P_{cr} = \pi^2\frac{EI}{L^2} = \pi^2\frac{EA}{(L/r)^2}\right] \tag{7}$$

the column is assumed to be in a buckled state, where A is the cross-sectional area of the column and:

$$\left[r = \sqrt{\frac{I}{A}}\right]$$

is the radius of gyration of the beam's cross-section.

The axial stress in the column corresponding to the critical buckling load P_{cr} for a pinned-end column is given by:

$$\left[\sigma_{cr} = \frac{P_{cr}}{A} = \pi^2\frac{E}{(L/r)^2}\right]. \tag{8}$$

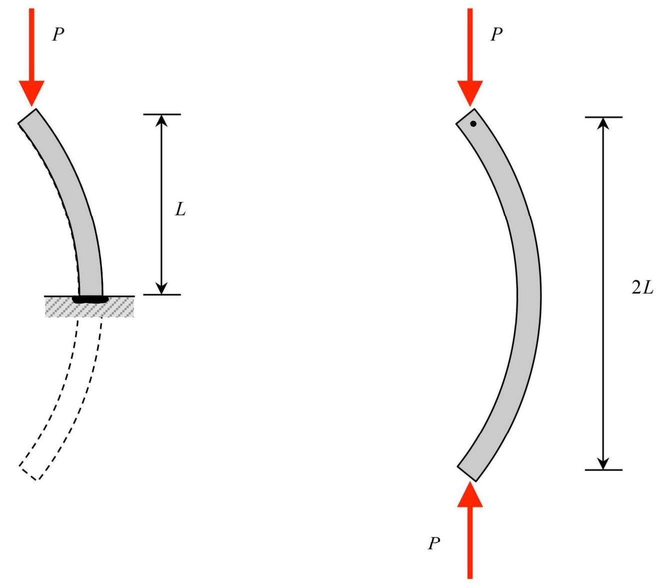
For the other boundary conditions, we will write the critical load and the critical stress as:

$$P_{cr} = \pi^2\frac{EI}{L_{eff}^2} = \pi^2\frac{EA}{\left(L_{eff}/r\right)^2} \tag{9}$$

$$\sigma_{cr} = \pi^2\frac{E}{\left(L_{eff}/r\right)^2} \tag{10}$$

where L_{eff} is the "effective length" of the column for that particular set of boundary conditions. From equation (9) we see that if the effective length of the column L_{eff} is greater than the physical length L for a given set of boundary conditions, then the critical buckling load is lower than that of the pinned-pinned column.

B) Fixed-free column



fixed-free column of length L

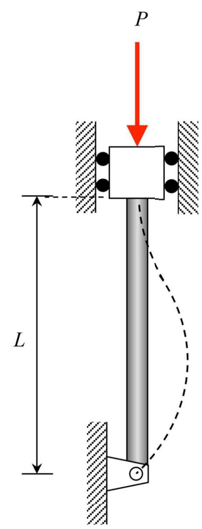
pinned-pinned column of length 2L

From the above figure, we see that a *fixed-free* column under an axial load P is structurally the same as a pinned-pinned column of half the length. Based on this, we can directly write down that the critical load for the buckling of a fixed-free column as:

$$P_{cr} = \pi^2 \frac{EI}{L_{eff}^2}$$

where $L_{eff} = 2L$ = the effective length of the column for buckling.

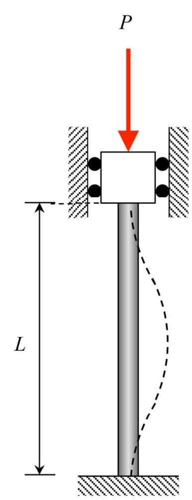
C) Fixed-pinned column



It can be shown that the effective length of a *fixed-pinned* column of length L for buckling is:

$$L_{eff} = 0.7L$$

D) Fixed-fixed column

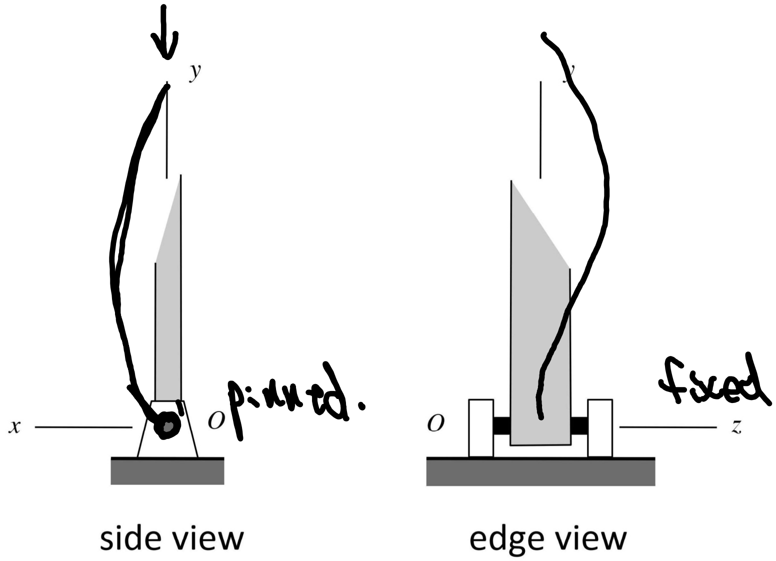


It can be shown that the effective length of a *fixed-fixed* column of length L for buckling is:

$$L_{eff} = 0.5L$$

Comments on boundary conditions and buckling analysis

Recall that a pinned-end condition corresponds to boundary conditions of zero bending moment at the pin. This is apparent since a smooth pin does not resist rotation about the pin axis (the z-axis, perpendicular to the page in the side view below). However, the physical makeup of a pin joint does resist rotation about the x-axis (perpendicular to the page as seen from the edge view shown below). For a rigid pin, the rotation of the pin about the x-axis is completely restrained, producing a "fixed" (or, clamped) boundary condition for rotation about the x-axis.



There are also other combinations of end conditions that may exist in two directions for other boundaries. One needs to look at these conditions carefully to determine which conditions to use in the buckling analysis.

In summary, we need to consider buckling about two axes of bending, generally in the direction perpendicular to the page and in the direction as seen by the edge view of the structure. Columns will typically have different second area moments, I, for these two directions. In addition, the pinned end condition places different boundary conditions on rotations for these two different directions of rotations. Generally, one solves for buckling critical loads in each direction, and chooses smallest critical load when assessing a design.

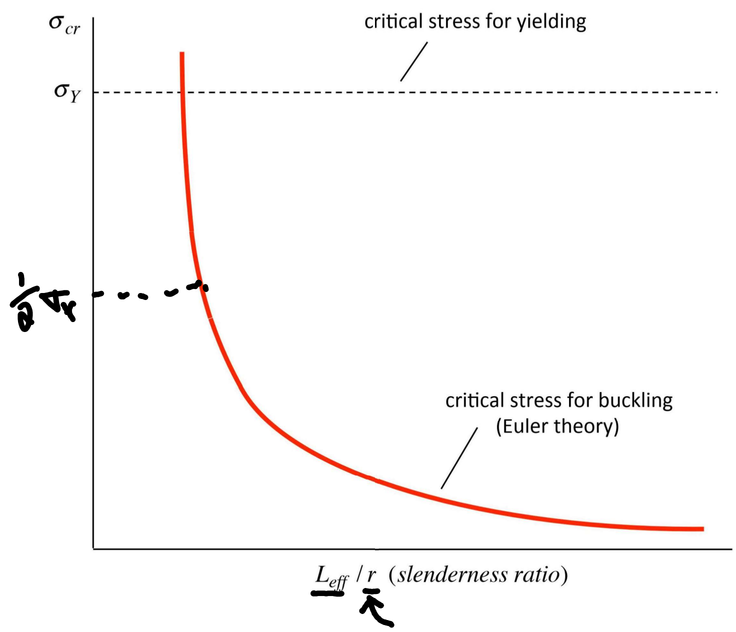
Limitations on the Euler buckling theory

As we have seen, the Euler theory provides us with the following expression for the critical buckling stress of a column of length L and radius of gyration:

$$\sigma_{cr} = \pi^2 \frac{E}{\left(L_{eff} / r\right)^2} \tag{11}$$

As discussed earlier in the course, the Euler theory for beams is valid for slender beams (the length dimension is large compared to a characteristic beam thickness dimension). Stated differently, the Euler theory is valid for a large "slenderness ratio", L_{eff} / r.

A plot of the critical buckling stress given in equation (11) is shown in the following figure.



Some comments on this plot:

- For large slenderness ratios, the critical stress for buckling is quite low, as one would expect for long, slender beams. For small slenderness ratios, the Euler theory predicts unreasonably high critical stresses required for buckling. This is a shortcoming of the Euler theory.
- In addition to the critical stress for buckling, we need to keep in mind that the induced axial stress must not exceed the yield strength σ_Y of the material. In short, the induced stress in the column must lie below both the yield strength and the critical stress for buckling.
- Experiments have shown that the Euler theory is actually only valid for induced stresses up to $0.5\sigma_Y$. From equation (11), we see that this says we should not use the Euler buckling theory for slenderness ratios less than $\left(L_{eff} / r\right)_c$, where:

$$\left(L_{eff} / r\right)_c = \sqrt{\frac{\pi^2 E}{0.5\sigma_Y}}$$

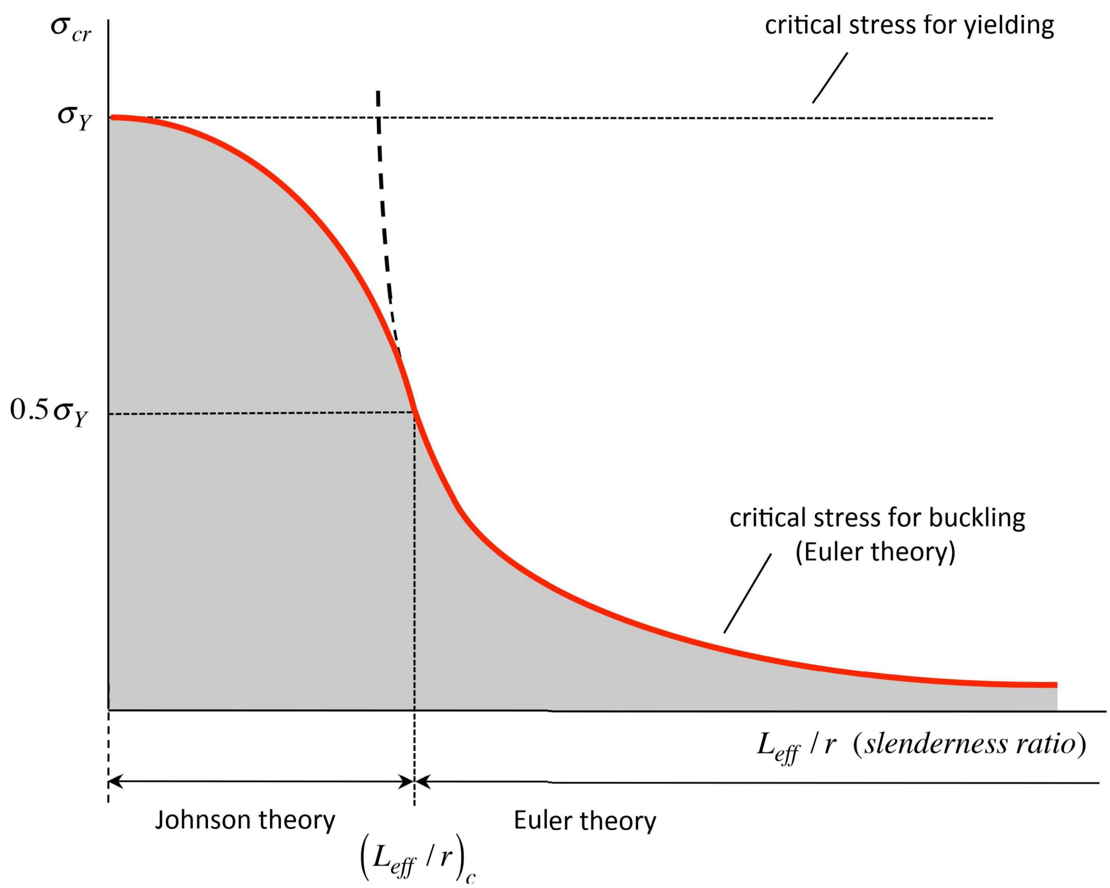
Not on exam.

The following quadratic transition curve for slenderness ratio values of $0 < L_{eff}/r < \left(L_{eff}/r\right)_c$ has been proposed and used extensively:

$$\sigma_{cr} = \left[1 - \frac{\left(L_{eff}/r\right)^2}{2\left(L_{eff}/r\right)_c^2}\right]\sigma_Y$$

The above is known as the "Johnson column formula".

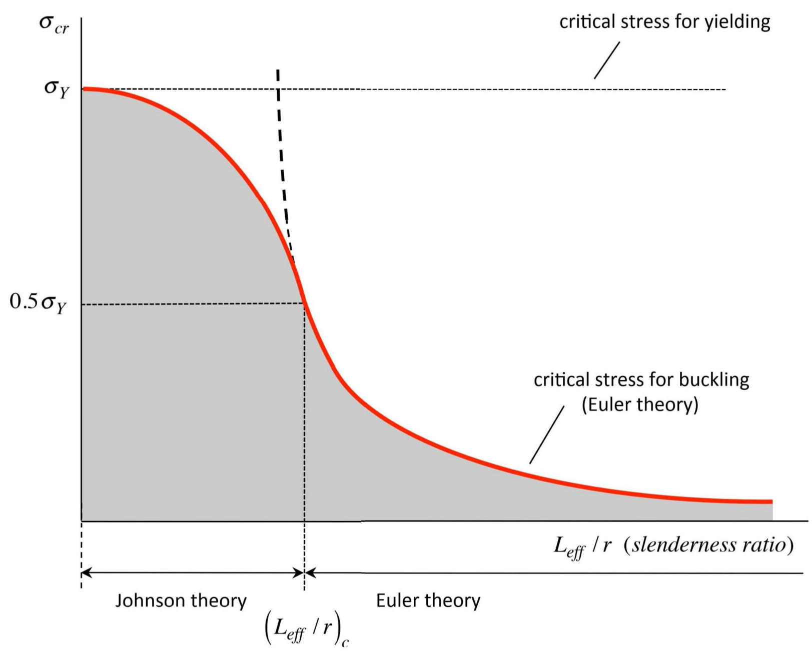
The modified critical stress for buckling plot is shown below. For $0 < L_{eff}/r < \left(L_{eff}/r\right)_c$, one is to use the Johnson formula, whereas for $L_{eff}/r > \left(L_{eff}/r\right)_c$ the Euler formula is to be used.



Solving buckling problems using the Euler/Johnson formulas

1. ***Calculate*** – Leading up to choosing the correct formula for determining critical loads, you need to determine the following:
   - Determine the compressive axial load acting on the member.
   - Determine the appropriate boundary conditions for the column. From this, determine the effective length L_{eff}.
   - Calculate the *radius of gyration* for the column cross section: $r = \sqrt{I / A}$.
   - Calculate the *slenderness ratio* for the column, L_{eff} / r. (This number depends on only the *geometric properties* of the column.)
   - Calculate the *critical slenderness ratio*: $\left(L_{eff} / r\right)_c = \sqrt{2\pi^2 E / \sigma_Y}$. (This number depends on only the *material properties* of the column.)
2. ***Compare*** – Compare the slenderness ratio L_{eff} / r with the critical slenderness ratio value $\left(L_{eff} / r\right)_c$.
3. ***Choose*** – Choose the appropriate formula for the critical load value:
   - If $L_{eff} / r > \left(L_{eff} / r\right)_c$, then use the Euler buckling formula: $P_{cr} = \left[\dfrac{\pi^2}{\left(L_{eff} / r\right)^2}\right] EA$.
   - If $L_{eff} / r < \left(L_{eff} / r\right)_c$, then use the Johnson buckling formula:

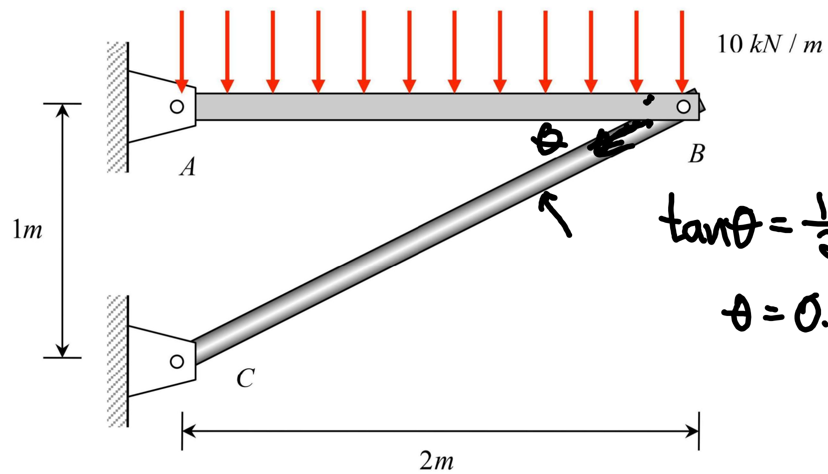
$$P_{cr} = \sigma_{cr} A = \left[1 - \frac{\left(L_{eff} / r\right)^2}{2\left(L_{eff} / r\right)_c^2}\right] \sigma_Y A .$$



$$\left[P_{cr} = \pi^2 \frac{EI}{L_{eff}^2} \right.$$

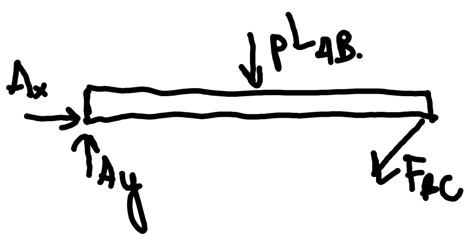
Example 18.2

The steel compression strut BC of the frame ABC is a steel tube with an outer diameter of d = 48 mm and a wall thickness of t = 5 mm. Determine the factor of safety against elastic buckling if a distributed load of 10 kN/m is applied to the horizontal frame member AB as shown. Let E = 210 GPa and σ_Y = 340 MPa.



$\tan\theta = \frac{1}{2}$

$\theta = 0.464$ rad



$$(\Sigma M)_A = -F_{BC}\sin\theta(L_{AB}) - (pL_{AB})\left(\frac{L_{AB}}{2}\right) = 0$$

$$F_{BC} = -\frac{pL_{AB}}{2\sin\theta}$$

$$F_{BC} = 22\ 361\ N.$$

$$L_{eff} = L_{BC} = \sqrt{1^2+2^2} = \sqrt{5}$$

$$I = \frac{\pi}{4}\left(\left(\frac{d}{2}\right)^4 - \left(\frac{d}{2}-t\right)^4\right) = 1.58\times10^{-7}\ m^4$$

$$P_{cr} = \pi^2 \frac{EI}{L_{eff}^2}$$

$$P_{cr} = \frac{\pi^2(210\times10^9)(1.58\times10^{-7})}{5}$$

$$P_{cr} = 65\ 587\ N$$

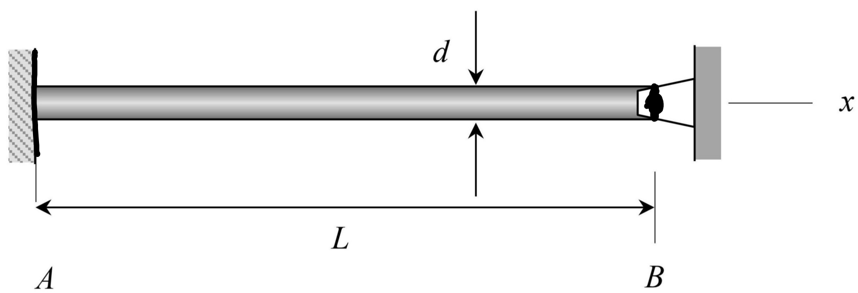
$$F.S = \frac{P_{cr}}{F_{BC}} = \frac{65\,587}{22\,361} = 2.94$$

Example 18.4

A straight, slender rod is fixed to a rigid support at end A and pinned to a rigid support at end B. At the reference temperature, the rod is perfectly stress-free.

a) Derive a formula that expresses the uniform increase in temperature ΔT_{cr} required to cause elastic buckling of the compression member.

b) Determine the value of ΔT_{cr} required to cause elastic buckling of an aluminum rod with a diameter of $d = 20\ mm$ and a length of $L = 1m$. The coefficient of thermal expansion, Young's modulus and yield strength for aluminum are $\alpha = 23\times10^{-6}\ /\ {}^{\circ}C$, $E = 10.6\times10^{3} ksi$ and $\sigma_Y = 60\ ksi$, respectively.



a) $e = 0 = \frac{PL}{EA} + \alpha \Delta T L = 0$

$-P = \alpha \Delta T E A.$

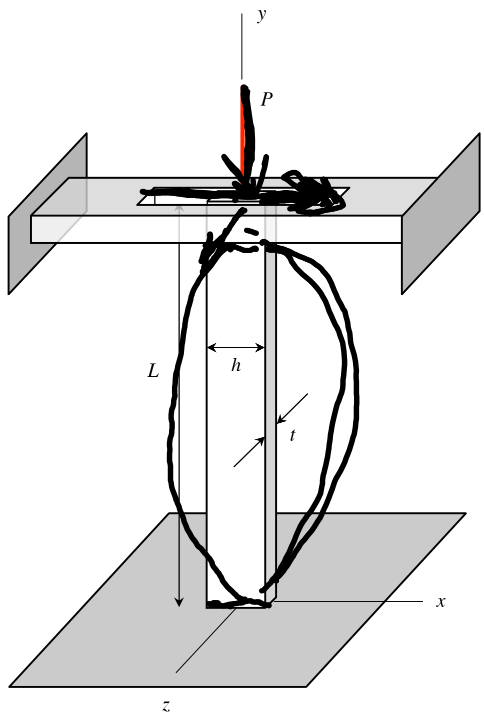
$P_{cr} = \alpha \Delta T E A = \frac{\pi^2 E I}{L_{eff}^2}$

$L_{eff} = 0.7L = 0.7\ m.$

$\Delta T_{cr} = \frac{\pi^2 I}{L_{eff}^2 \alpha A}$

Example 18.5

The column shown below is clamped onto to ground at its bottom, with the top of the beam able to slide within a slot. The column carries an axial load of P. What is the largest load P that the column can withstand without buckling? Use $h = 3t$ and $L = 10h$.



y-z

$L_{eff} = 0.5L = 5h = 15t$

$P_{cr} = \frac{\pi^2 E I}{L_{eff}^2}$

$I_{zz} = \frac{ht^3}{12} = \frac{3t^4}{12} = \frac{t^4}{4}$

$P_{cr} = \frac{\pi^2 E t^4}{4(15t)^2}$

$P_{cr} = \pi^2 E t^2 (0.0011)$

x-y

$L_{eff} = 2L = 20h = 60t$

$P_{cr} = \frac{\pi^2 E I}{L_{eff}^2}$

$I_{xx} = \frac{th^3}{12} = \frac{t(3t)^3}{12} = \frac{27t^4}{12}$

$P_{cr} = \frac{\pi^2 E (27t^4)}{12(60t)^2}$

$P_{cr} = \pi^2 E t^2 (0.000625)$