Course Eval: 40 (0.7) = 28.

Course Roadmap

Ch 13: Mohr's Circles

- Given the loading conditions at a point, what are the stress states at different angles?
- At what angle does the max normal stress and max shear stress occur?

Ch 14: Combined Loading

- What are the normal and shear stresses at points on a cross section due to combined axial, torsion, and bending loading?
- Determine the principal stresses and max shear stress at these points – use Mohr's circles.

Ch 15: Failure Analysis

- Given the stress states at a point, under what condition will a 3D structure fail?

15. Failure analysis

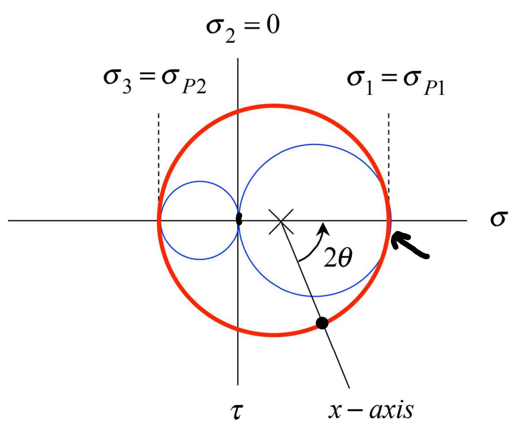
Objectives:
To study and apply theories of material failure to stress states existing in loaded structural components to determine design criteria to prevent component failure.

Background:

- For a *uni-axial loading* in which a normal component of stress σ exists along the axis of loading, the maximum shear stress at a point is exactly half of σ:

$$\tau_{\max} = \sigma / 2$$

- A *general state of plane stress* is completely characterized by its principal components of stress, σ_{P1} and σ_{P2}, along with the angle of the plane for one of these principal stresses.



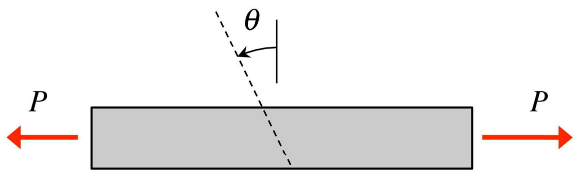
Lecture topics:

a) Failure theories for *ductile* materials:
   - maximum shear stress theory
   - maximum distortion energy theory

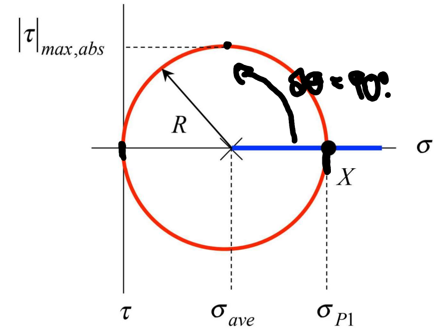
b) Failure theories for *brittle* materials:
   - maximum normal stress theory
   - Mohr's failure criteria

Failure in members loaded with an axial force

A rod (having a cross-sectional area of A) is acted upon by a *force* P along its axis.



$\sigma_{P1} = P/A$



- What is the maximum normal stress in the rod? On which plane does this stress exist?

  σ_{P1} at 0°.

- What is the maximum magnitude shear stress in the rod? On which plane does this stress exist?

  $\tau_{max,abs} = \frac{\sigma_{P1}}{2}$ at 45°.

Shown below are photos of the failure planes for aluminum (ductile material) and chalk (brittle material) experiencing applied tensile axial forces.

Aluminum – failure due to normal or shear stress?

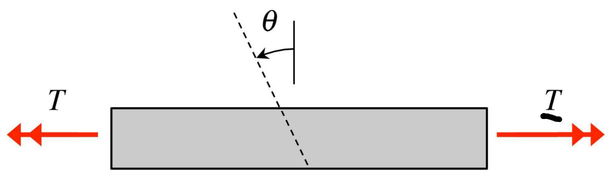
Ductile.
steel

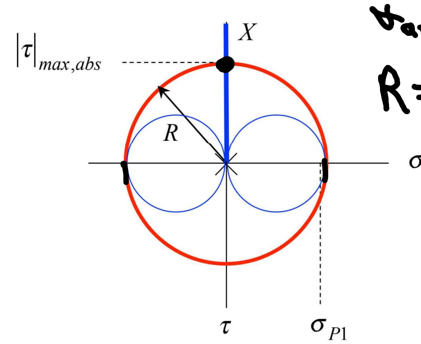
Chalk – failure due to normal or shear stress?

Brittle.
chalk.

Failure in members loaded with an axial torque

A shaft (having a polar area moment of I_P) is acted upon by a *torque* T along its axis.





$\sigma_{avg} = 0$

$R = \tau$

- What is the maximum normal stress in the shaft? On which plane does this stress exist?

$|\sigma_{P1}| = |\sigma_{P2}| = \frac{TR}{I_P}$ at $-45°$ or $135°$. $45°$ CW.

- What is the maximum magnitude shear stress in the shaft? On which plane does this stress exist?

$\tau_{max} = \frac{TR}{J_P}$ at $0°$.

Shown below are photos of the failure planes for mild steel (ductile material) and chalk (brittle material) experiencing applied axial torques.

Mild steel – failure due to normal or shear stress?

Shear stress ⇒ ductile.

Chalk – failure due to normal or shear stress?

Normal stress ⇒ Brittle.

Failure theories

As a design engineer you are more likely to use the simple theories of torsion, bending and axial extension of thin beams that we have developed to make back-of-the-envelope calculations on expected stresses in a product. As a stress analyst engineer, you will more likely use more sophisticated finite element methods that can be used to calculate stresses based on the 3D equations of elasticity directly.

In either case, the state of stress at a point in the product is likely to consist of many components, i.e. *multi-axial state of stress*. In such a setting it is not clear if the failure would occur if the maximum normal stress exceeded a certain value or if the absolute maximum shear stress exceeded a threshold, or if the strain energy density exceeded a certain critical value.

To answer these questions, several researchers form the past have proposed a host of failure theories, some very simple and some very advanced. We will discuss four basic theories, two that are more applicable to ductile materials and two others that are better suited for brittle materials.

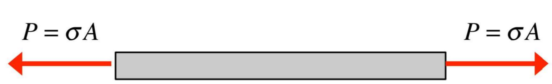
Ductile: - Max shear stress
- von Mises stress.

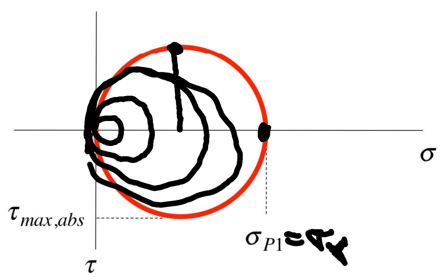
Brittle: - Mohr's criterion.

a) Failure theory for ductile materials – maximum shear stress (Tresca, Coulomb, Guest)

When ductile material like structural steel is tested in tension, it has been observed that the mechanism actually responsible for yielding is slip, i.e., shearing along planes of maximum shear stress on a plane oriented 45° with respect to the axis of the member. Slip planes correspond to planes of maximum atomic density in the crystal. Initial yielding is associated with the appearance the first slip line, and as the strain increases more slip lines appear until the entire specimen has yielded. Failure occurs when $\tau_{max,abs}$ exceeds a certain value. But at what value? We will now discuss that.

Uni-axial material testing:
Suppose that uni-axial tension test is conducted on a specimen. When the axial stress reaches $\sigma = \sigma_Y$, a *ductile* material is known to yield with the maximum shear stress occurring at $45°$ to the member's axis.



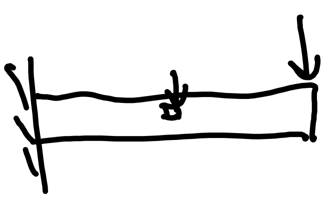


Therefore, from the Mohr's circle for this test, we see that $\tau_{max} = \dfrac{\sigma_Y}{2}$ at failure.

For a general state of stress:
According to the "maximum shear stress" (MSS) theory, a ductile material in a general state of stress will fail when the shear stress reaches the same value as the shear stress when it fails under a uni-axial test; that is, when:

$$\tau_{max,abs} = \frac{\sigma_Y}{2}$$

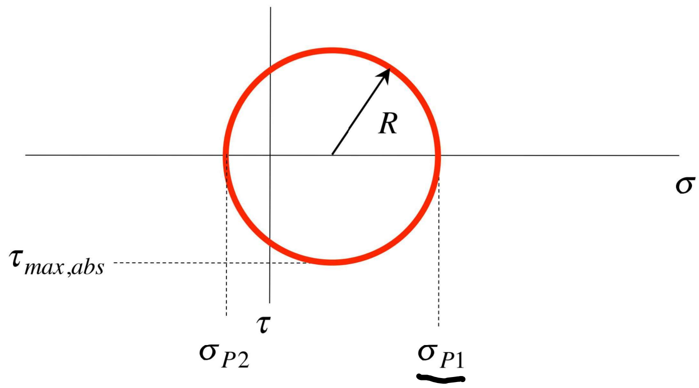
That's it. Nothing else. Failure analysis for plane stress using the maximum shear stress theory consists of calculating the absolute maximum shear from a known state of stress.



Graphical representation of the Maximum Shear Stress theory

For later use, let's take a look at the implications of this failure theory in terms of the principal components of stress of a given state of stress. Recall that the maximum shear stress for a general state of stress depends on the signs of the principal stresses, σ_{P1} and σ_{P2}. Specifically:

- For the case where σ_{P1} and σ_{P2} have OPPOSITE signs:



we have:

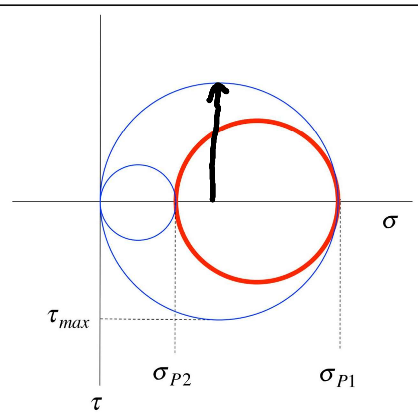
$$|\tau_{max}| = R = \frac{|\sigma_{P1} - \sigma_{P2}|}{2}$$

Therefore, for FAILURE:

$$|\tau_{max}| = \frac{|\sigma_{P1} - \sigma_{P2}|}{2} > \frac{\sigma_Y}{2} \quad \Rightarrow \quad |\sigma_{P1} - \sigma_{P2}| > \sigma_Y$$

- For the case where σ_{P1} and σ_{P2} have the SAME sign:

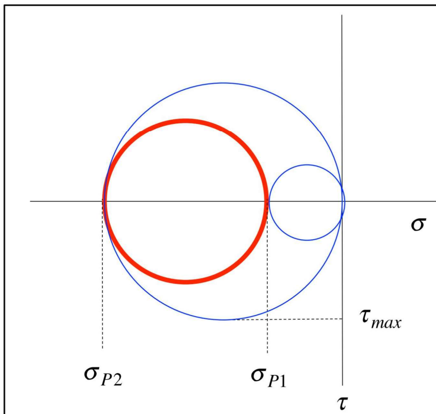
σ_{P1} and σ_{P2} both POSITIVE



In this case: $|\tau_{max}| = \frac{\sigma_{P1}}{2}$

Therefore, for FAILURE: $|\tau_{max}| = \frac{\sigma_{P1}}{2} > \frac{\sigma_Y}{2} \quad \Rightarrow \quad \sigma_{P1} > \sigma_Y$

σ_{P1} and σ_{P2} both NEGATIVE



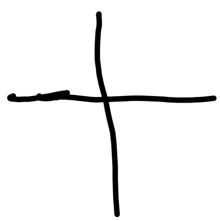
In this case: $|\tau_{max}| = \frac{|\sigma_{P2}|}{2}$

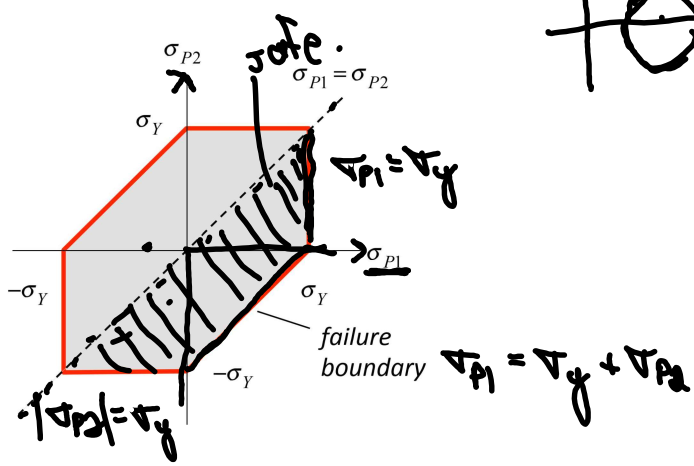
Therefore, for FAILURE: $|\tau_{max}| = \frac{|\sigma_{P2}|}{2} > \frac{\sigma_Y}{2} \quad \Rightarrow \quad |\sigma_{P2}| > \sigma_Y$

In summary, the maximum shear stress theory for ductile materials says that failure will occur when $|\tau_{max}| > \frac{\sigma_Y}{2}$ where σ_Y is the yield strength of the material. Our goal is to identify the region in the σ_{P1} vs. σ_{P2} plane for safe operation. Such a region is bounded by principal stress values for which $|\tau_{max}| = \frac{\sigma_Y}{2}$. From above we can say:

- In the *first* quadrant (both $\sigma_{P1} > 0$ and $\sigma_{P2} > 0$), this boundary is given by $\sigma_{P1} = \sigma_Y$ and $\sigma_{P2} = \sigma_Y$.
- In the *third* quadrant (both $\sigma_{P1} < 0$ and $\sigma_{P2} < 0$), this boundary is given by $\sigma_{P1} = -\sigma_Y$ and $\sigma_{P2} = -\sigma_Y$.
- In the *second* quadrant ($\sigma_{P1} < 0$ and $\sigma_{P2} > 0$), this boundary is given by $\sigma_{P1} - \sigma_{P2} = -\sigma_Y$.
- In the *fourth* quadrant ($\sigma_{P1} < 0$ and $\sigma_{P2} > 0$), this boundary is given by $\sigma_{P1} - \sigma_{P2} = \sigma_Y$.

The failure boundary for the maximum shear stress theory is shown in the following figure. A stress state having principal stress components inside this hexagonal region corresponds to safe operation; outside of this region corresponds to failure as predicted by the maximum shear stress theory.



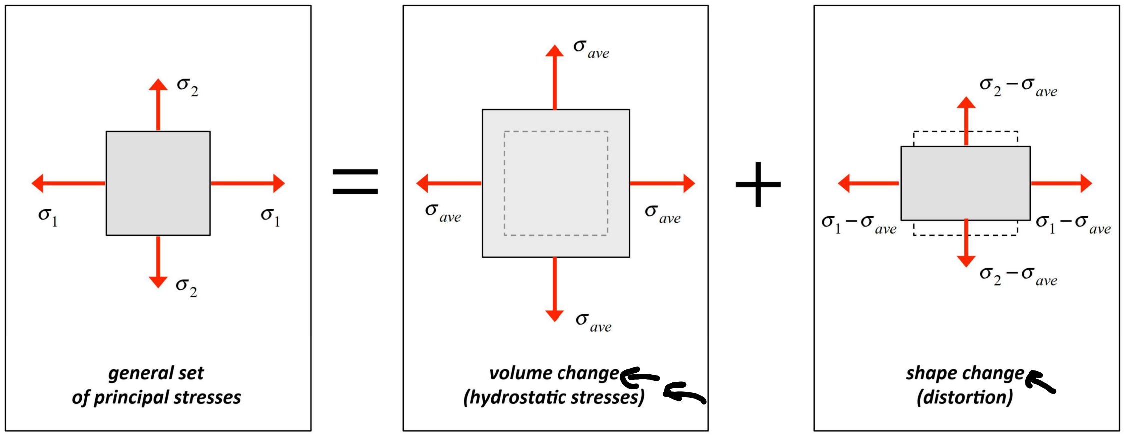


Question: We need to consider only the region below the dashed line for $\sigma_{P1} = \sigma_{P2}$ shown in the figure above – why is that?

b) Failure theory for ductile materials – maximum distortional energy (MDE).

The strain energy density in a body experiencing plane stress can be written in terms of the principal stresses as:

$$u = \frac{1}{2E}\left(\sigma_{P1}^2 + \sigma_{P2}^2 - 2\nu\sigma_{P1}\sigma_{P2}\right)$$

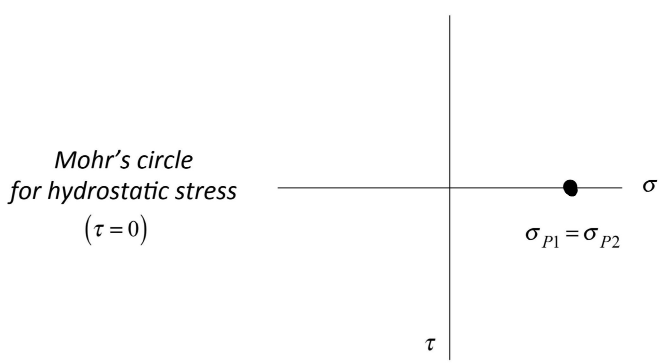


A portion of this energy goes into changing the volume of a stress element, with the remainder going into the change in shape (or "distortion"). The contributions to changes in volume and to distortion are demonstrated in the following figures.

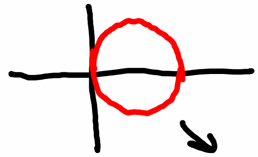
It can be shown that the contribution to the strain energy density above due to distortion effects is given by:

$$u_{distort} = \frac{1}{6G}\left(\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2\right) \quad (1)$$

Why is distortional energy relevant to failure? Recall from our earlier discussions of stress transformations and Mohr's circle, we discovered that having "hydrostatic stresses" (i.e., identical principal stresses) does not produce in-plane shear stresses. Since ductile materials fail in shear, the volume changing effects do not lead to failure of ductile materials. This suggests that distortional effects lead to failure of ductile materials. Consider the following failure theory for ductile materials:



$\tau_{avg} = \tau_0$
$R = 0$



$u_{dist} = \frac{1}{6G}\left(\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2\right)$

<u>*Maximum distortional energy failure criterion*</u>
Yielding of a *ductile* material occurs when the *distortion energy density* equals or exceeds the *distortional energy density* when the same material *yields* in a uni-axial test.

To implement this theory, note that a tensile test specimen at the yield stress σ_Y produces the following distortional energy density (set $\sigma_{P2} = 0$ and $\sigma_{P1} = \sigma_Y$ in the above equation):

$$u_{yield} = \frac{1}{6G}\sigma_Y^2 = \frac{1}{6G}\left(\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2\right) \qquad (2)$$

Through the above theory, we equate (1) and (2) producing the following failure boundary in the $\sigma_{P1} - \sigma_{P2}$ plane:

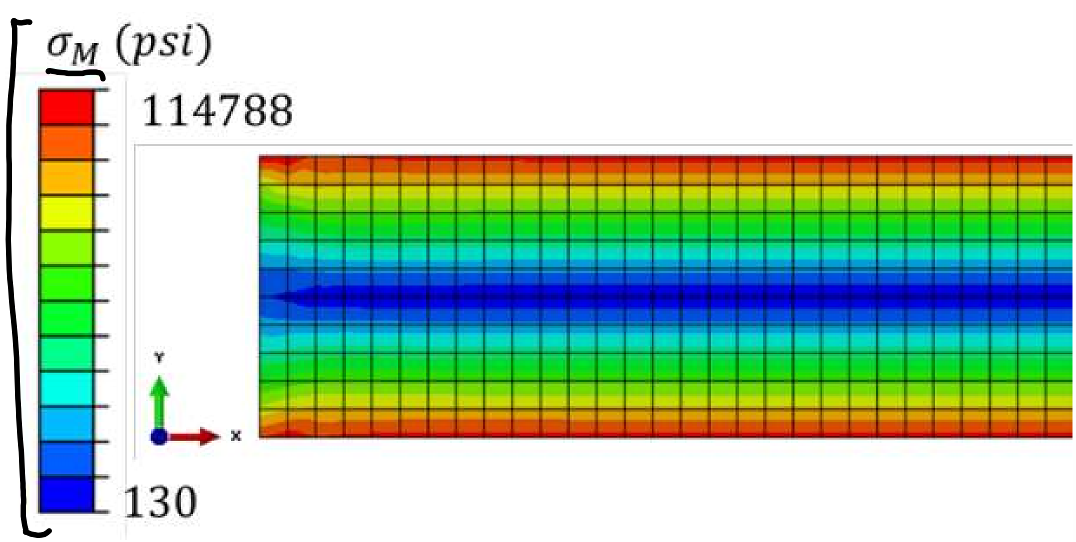
$$\sigma_Y = \sigma_M \qquad (3)$$

where:

$$\sigma_M = \sqrt{\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2} = \text{"von Mises" stress}$$

In words, the *Maximum Distortional Energy* (MDE) failure criterion says that if the von Mises stress calculated from the principal components of plane stress exceeds the yield strength of the material, failure of the material is predicted.

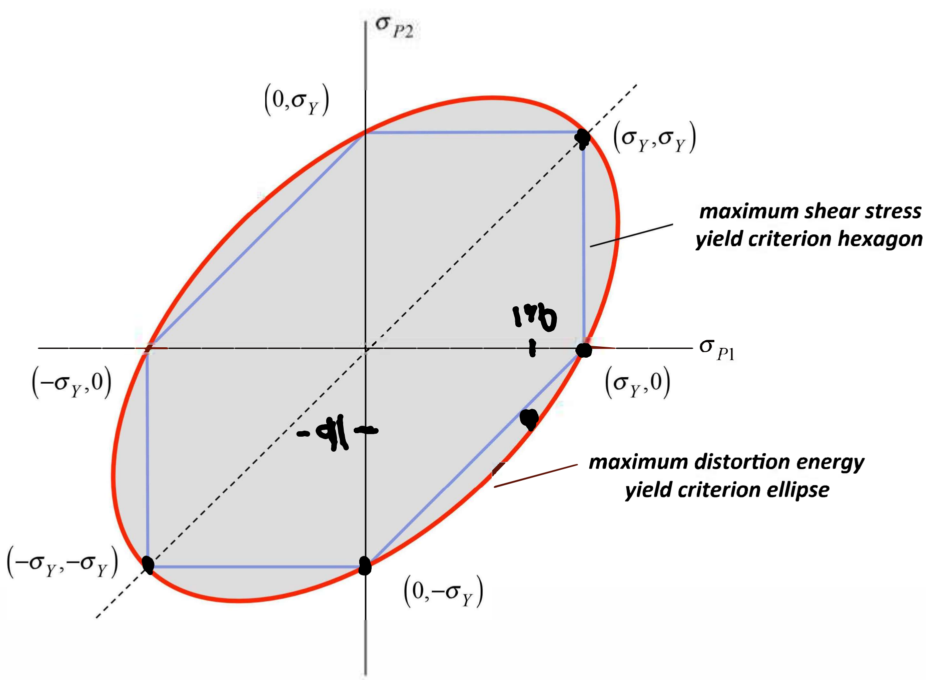
d) von Mises equivalent stress σ_M



Comparison of the Maximum Shear Stress theory with the Maximum Distortional Energy theory for failure of a ductile material

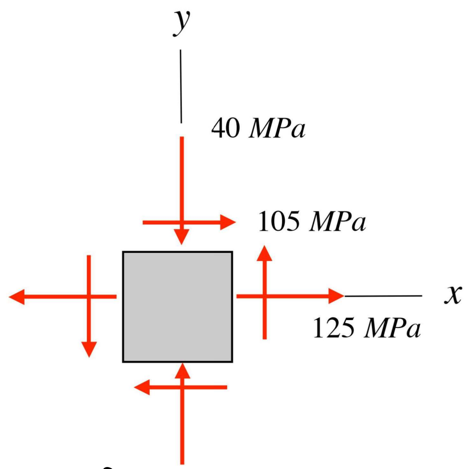
From our above analysis we see that:

- In the $\sigma_{P1} - \sigma_{P2}$ plane, the maximum distortional energy criterion of equation (3) represents a rotated ellipse.
- Note that the six corners of the hexagonal region defined by the maximum shear theory lie on this ellipse. That is, the following six sets of points in the $\sigma_{P1} - \sigma_{P2}$ plane satisfy equation (3): $(\sigma_{P1}, \sigma_{P2}) = (\pm\sigma_Y, \pm\sigma_Y)$, $(\sigma_{P1}, \sigma_{P2}) = (\pm\sigma_Y, 0)$, $(\sigma_{P1}, \sigma_{P2}) = (0, \pm\sigma_Y)$.
- In conclusion, the ellipse of (3) circumscribes the hexagonal region defined by the maximum shear stress theory. *Since this hexagonal region is completely contained within the ellipse, the Maximum Shear Stress (MSS) theory is more conservative than the Maximum Distortional Energy (MDE) criterion.*



Example 15.1

Consider the state of stress shown below in a component made up of a ductile material with a shear strength of $\sigma_Y = 250 MPa$. Does the maximum shear stress theory predict failure for the material? Does the maximum distortional energy theory predict failure of the material?



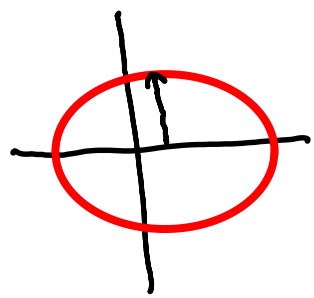
a) MSS $\tau_{max,abs} > \frac{\sigma_Y}{2} \Rightarrow$ fail.

$\sigma_{avg} = \frac{125-40}{2} = 42.5 MPa$

$R = 133.5 MPa$.

$\sigma_{P1} = \sigma_{avg} + R = 176 MPa$

$\sigma_{P2} = \sigma_{avg} - R = -91 MPa$.



$\tau_{max,max} = 133.5 MPa > \frac{250}{2} = 125 \Rightarrow$ fails.

b) $\sigma_M > \sigma_Y \Rightarrow$ fail.

$\sigma_M = \sqrt{\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2}$

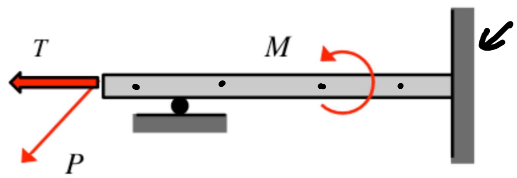
$\sigma_M = \sqrt{176^2 + 176(91) + 91^2} = 235.2 < \sigma_Y = 250 MPa$.

$\Rightarrow$ no failure.

Summary: failure analysis *(what the whole course of ME 323 leads up to…)*

EQUILIBRIUM ANALYSIS

COMBINED loading : axial, torsion and bending

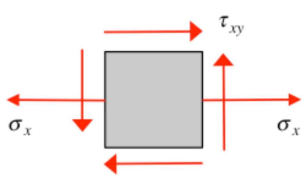


Includes: determining reactions, determining internal force/moment analysis and shear force/bending moment diagrams (to determine critical locations). For INDETERMINATE components, this also includes deflection analysis.

UNIAXIAL tensile loading

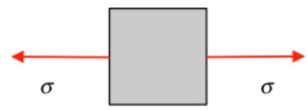
Perform uniaxial loading experiment to determine ultimate and yield strength of material

STRESS ANALYSIS



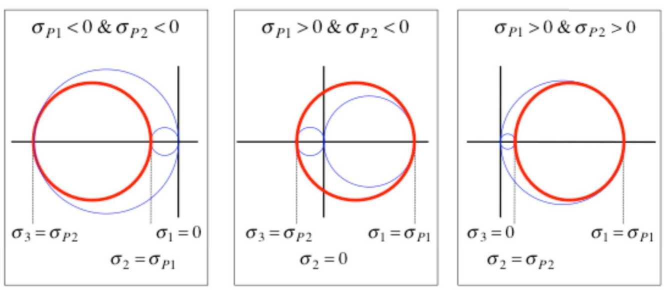
$\sigma_{axial} = \frac{F_n}{A}$ $\quad \sigma_{bending} = -\frac{My}{I}$

$\tau_{torsion} = \frac{T\rho}{J}$ $\quad \tau_{bending} = \frac{VQ}{It}$

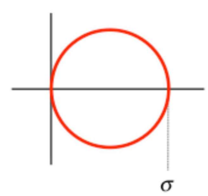


$\sigma = \frac{F}{A}$

STRESS TRANSFORMATION AND MOHR'S CIRCLE



$\sigma_{min} = \sigma_3 \quad \sigma_{max} = \sigma_1 \quad \tau_{max,abs} = (\sigma_1 - \sigma_3)/2$

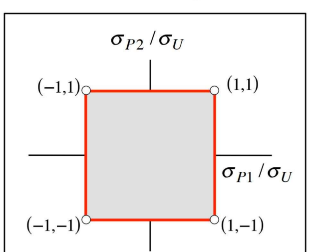


$\sigma_U =$ *ULTIMATE strength*

$\sigma_Y =$ *YIELD strength*

$\tau_{max} = \frac{\sigma}{2}$

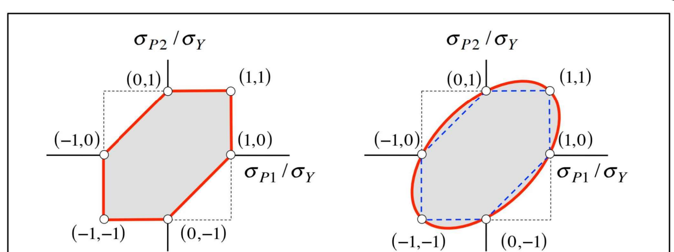
FAILURE THEORIES



maximum NORMAL STRESS

$\sigma_U = |\sigma|_{max} = \max(|\sigma_{P1}|, |\sigma_{P2}|)$

BRITTLE material



maximum SHEAR STRESS

$\sigma_Y / 2 = |\tau|_{max,abs}$

maximum DISTORTIONAL ENERGY

$\sigma_Y = \sigma_M = \sqrt{\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2}$

DUCTILE material

c) Failure theory for <u>brittle</u> materials – maximum normal stress

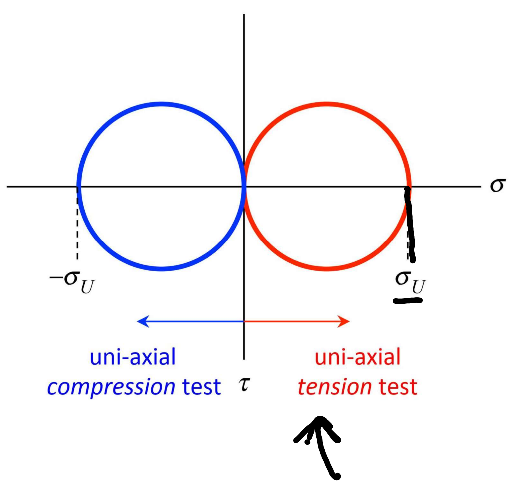
Brittle materials fail suddenly by fracture without prior yielding. Experiments have shown that the value of the normal stress on the failure plane for a biaxial (general) state of stress is not significantly different from the fracture stress σ_U in a uni-axial stress test.

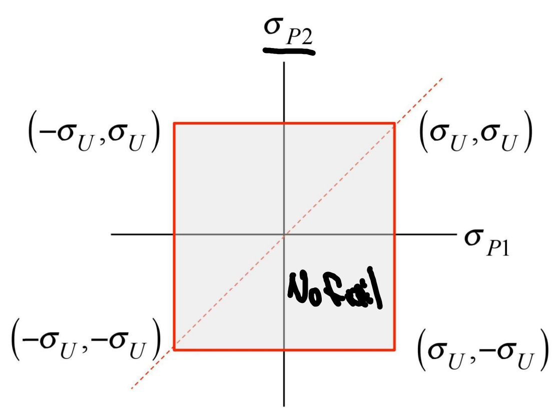
<u>Maximum normal stress failure criterion</u>

Fracture of a *brittle* material occurs when the *maximum principal stress* equals or exceeds the *ultimate normal stress* when the same material *fractures* in a uni-axial test. [Note that this criterion assumes that compressive failure occurs at the same ultimate stress value as do tension failures.]

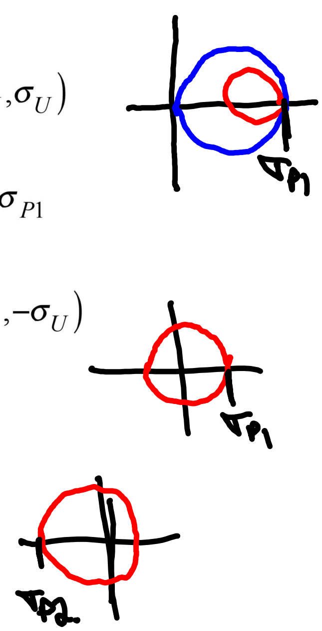
In terms of principal stresses, the failure boundary for a brittle material by the Maximum Normal Stress failure criterion is defined by:

$$\left|\sigma_{P1}\right| = \sigma_U \quad \text{OR} \quad \left|\sigma_{P2}\right| = \sigma_U$$





general state of plane stress

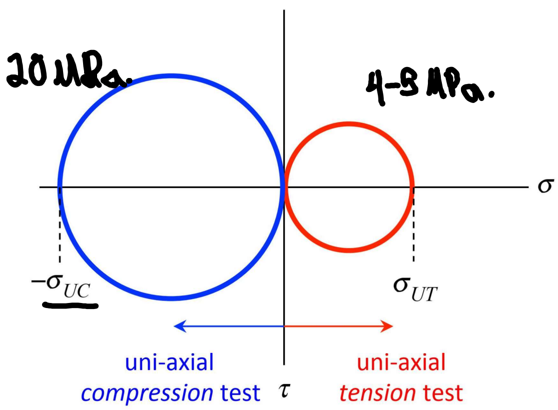


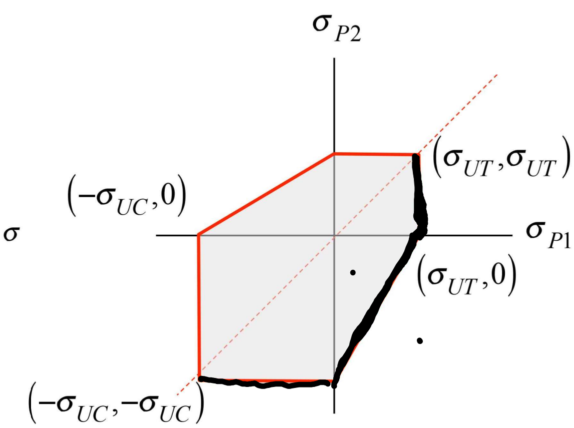
d) Failure theory for brittle materials – Mohr's criterion

Many brittle materials have a lower ultimate strength in tension σ_{UT} than in compression σ_{UC}; i.e., $\sigma_{UT} < \sigma_{UC}$. In this case, the maximum normal stress criterion cannot be used. An alternative theory, Mohr's criterion, states that:

- when σ_{P1}, and σ_{P2} have the same sign, failure occurs if either of the following stress limits is reached: $\sigma_{\max} = \sigma_{TU}$ *OR* $\sigma_{\min} = -\sigma_{CU}$.
- when σ_{P1} and σ_{P2} have opposite signs, Mohr's failure criterion states that failure occurs when: $\dfrac{\sigma_{P1}}{\sigma_{UT}} = \dfrac{\sigma_{P2}}{\sigma_{UC}} + 1$. Mohr's failure criterion.

The above combination constitutes Mohr's failure criterion.





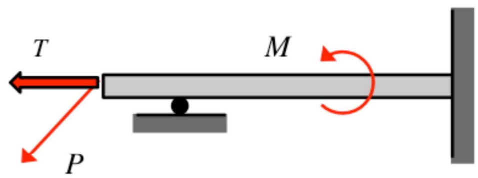
general state of plane stress

$$\sigma_{P2} = \frac{\sigma_{UC}}{\sigma_{UT}}\sigma_{P1} - \sigma_{UC}$$

Summary: failure analysis *(what the whole course of ME 323 leads up to…)*

EQUILIBRIUM ANALYSIS

COMBINED loading : axial, torsion and bending

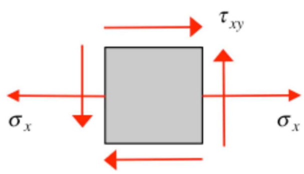


Includes: determining reactions, determining internal force/moment analysis and shear force/bending moment diagrams (to determine critical locations). For INDETERMINATE components, this also includes deflection analysis.

UNIAXIAL tensile loading

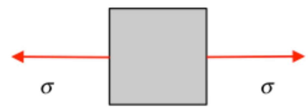
Perform uniaxial loading experiment to determine ultimate and yield strength of material

STRESS ANALYSIS



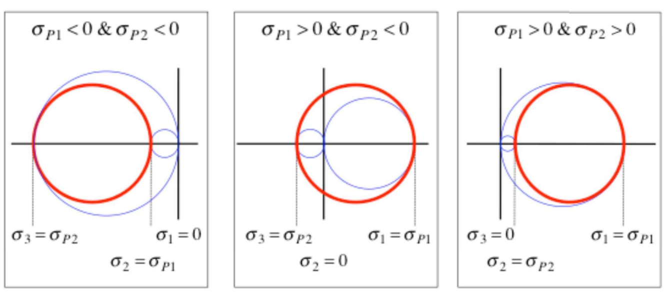
$\sigma_{axial} = \frac{F_n}{A}$ $\quad \sigma_{bending} = -\frac{My}{I}$

$\tau_{torsion} = \frac{T\rho}{J}$ $\quad \tau_{bending} = \frac{VQ}{It}$

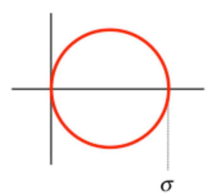


$\sigma = \frac{F}{A}$

STRESS TRANSFORMATION AND MOHR'S CIRCLE



$\sigma_{min} = \sigma_3 \quad \sigma_{max} = \sigma_1 \quad \tau_{max,abs} = (\sigma_1 - \sigma_3)/2$

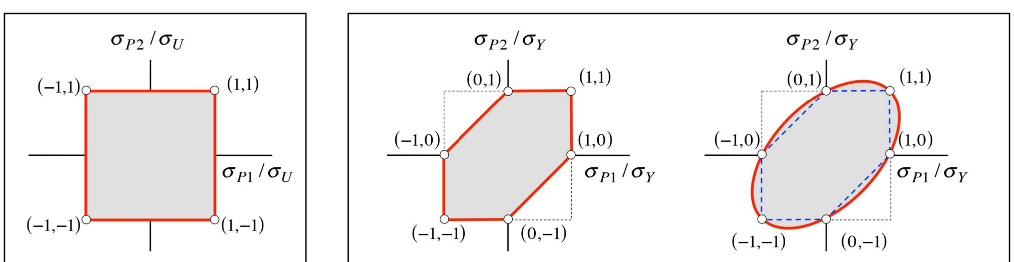


σ_U = *ULTIMATE strength*

σ_Y = *YIELD strength*

$\tau_{max} = \frac{\sigma}{2}$

FAILURE THEORIES



maximum NORMAL STRESS

$\sigma_U = |\sigma|_{max} = \max(|\sigma_{P1}|, |\sigma_{P2}|)$

BRITTLE material

maximum SHEAR STRESS

$\sigma_Y / 2 = |\tau|_{max,abs}$

maximum DISTORTIONAL ENERGY

$\sigma_Y = \sigma_M = \sqrt{\sigma_{P1}^2 - \sigma_{P1}\sigma_{P2} + \sigma_{P2}^2}$

DUCTILE material