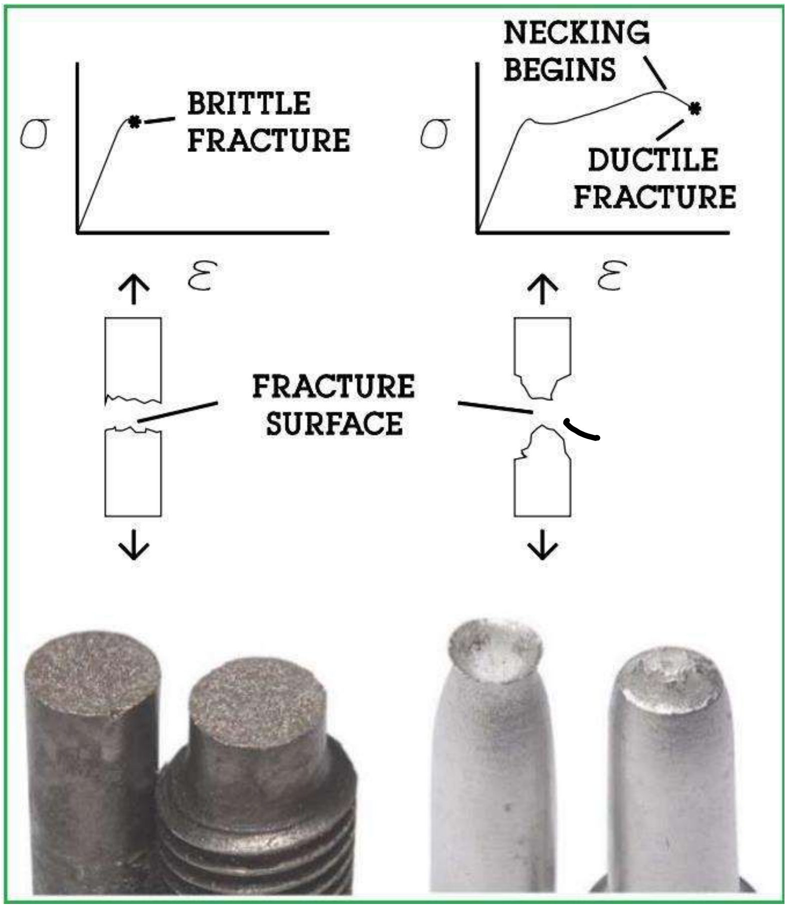
NECKING
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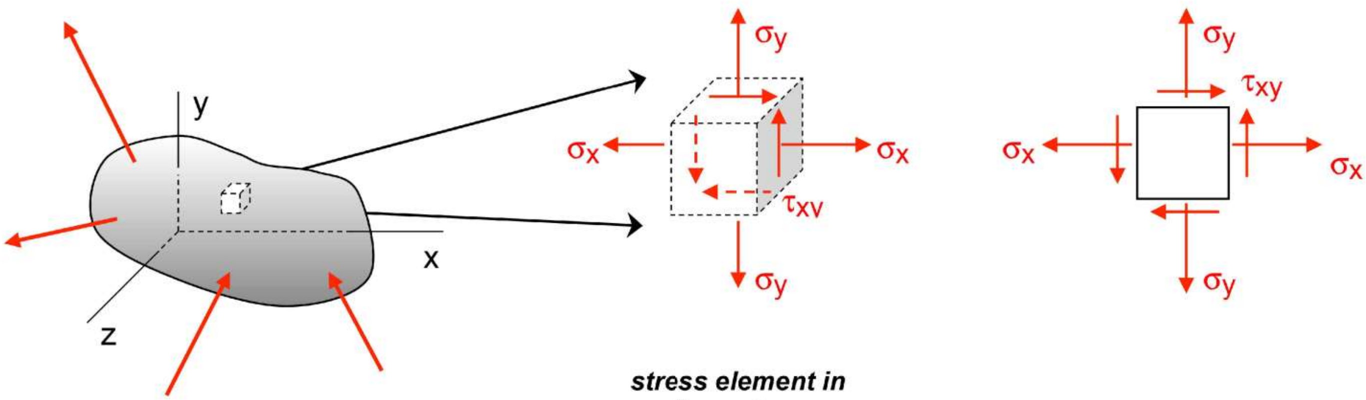
13. Transformation of stresses and Mohr's Circle

Objectives:
To study the projection of a given state of plane stress at a point onto an arbitrarily-aligned stress element.

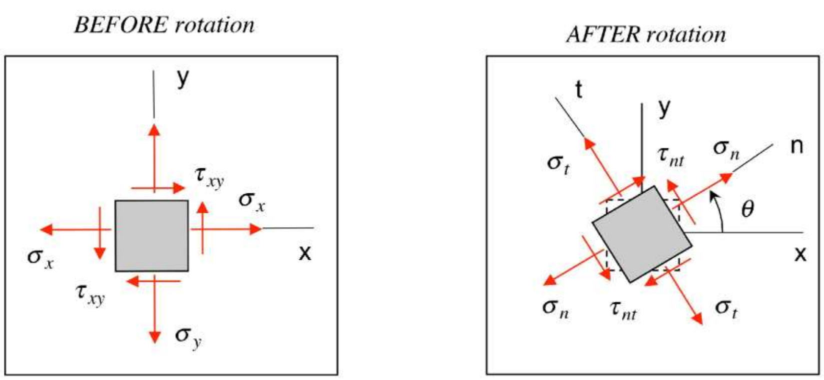
Background:

- If the stresses acting on a stress element all act in a single plane in the structural component, then the component is said to be in a state of "*plane stress*". For example, if $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, then the only non-zero stresses lie in the *xy*-plane and the component is in a state of plane stress in the *xy*-plane.



stress element in plane stress

- *Sign conventions:* Suppose we have an arbitrary face of a stress element that is originally oriented in such a way that its normal "*n*" is aligned with the positive *x*-axis. If this element is rotated CCW through an angle θ from the positive *x*-axis, the positive *t*-axis for this face is then defined by the right-hand-rule: "*z* crossed into *n* gives *t*". Positive normal stresses on face n, σ_n, points *outward* from face n. Positive shear stress, τ_{nt}, acts in the +t direction on the n-axis face.



Lecture topics:

a) Simple examples of plane stress.

b) Stress transformation equations for plane stress.

c) Principal normal stresses and maximum shear stress.

d) Mohr's circle.

e) Absolute maximum shear stress for plane stress.

Lecture notes

a) Simple examples of plane stress

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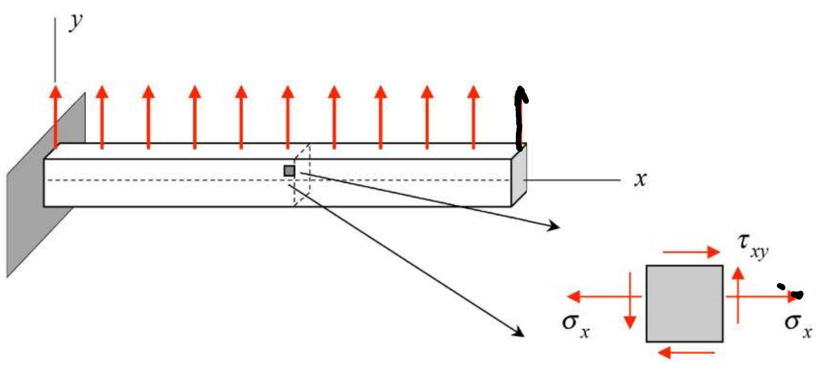
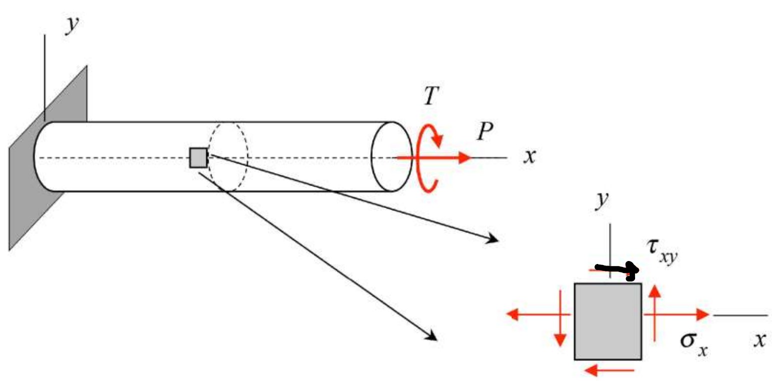
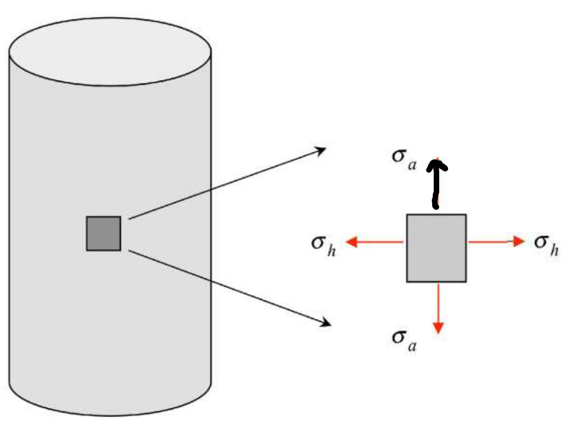
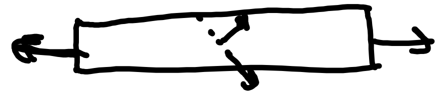


TORSIONALLY AND AXIALLY LOADED SHAFT



THIN-WALLED PRESSURE VESSEL

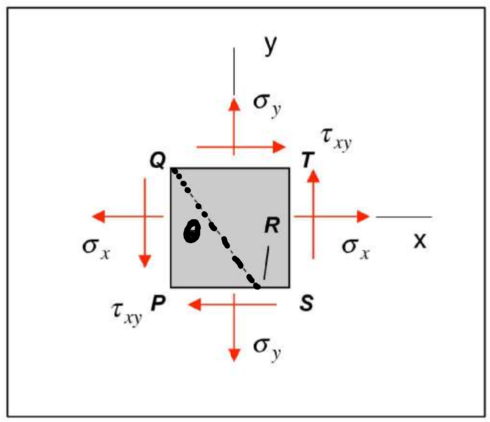




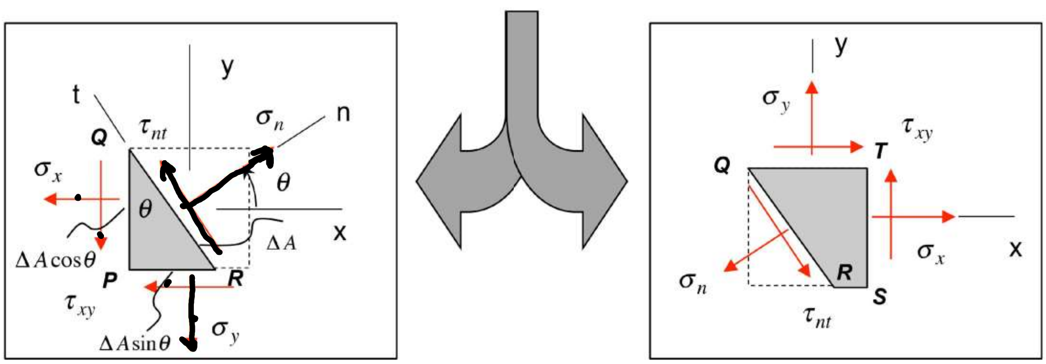
b) Stress transformation for plane stress

Here we start with a state of plane stress with normal stresses σ_x and σ_y acting on faces perpendicular to the *x*- and *y*-axes, respectively, and shear stress τ_{xy} acting on the four faces. Our goal here is to determine the normal and shear components of stress acting on a face, QR, whose normal "*n*" is at an angle of θ measured CCW from the *x*-axis, as indicated in the figure below.

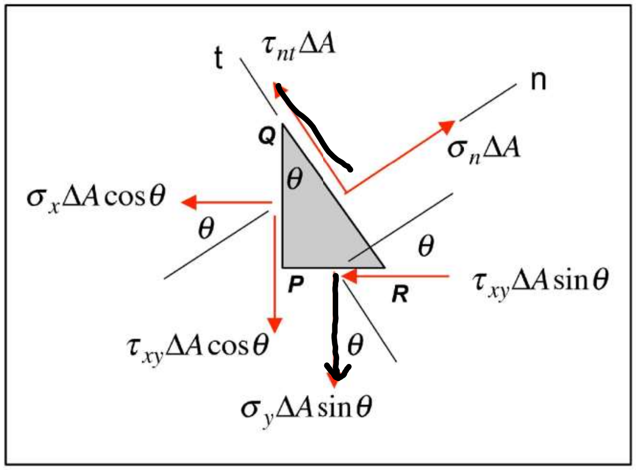
Let ΔA be the area of the cut face QR. Therefore, the area of faces PQ and PS are $\Delta A\cos\theta$ and $\Delta A\sin\theta$, respectively.



cut through stress element at an angle of θ w.r.t. the y-axis



At this point, we will perform an equilibrium analysis of the cut section PQR (left side of the cut) to determine the stress components σ_n and τ_{nt}.



Summing forces in the n-direction on the cut section gives:

$$\sum F_n = \sigma_n\Delta A - \left(\sigma_x\Delta A\cos\theta\right)\cos\theta - \left(\tau_{xy}\Delta A\cos\theta\right)\sin\theta$$
$$-\left(\sigma_y\Delta A\sin\theta\right)\sin\theta - \left(\tau_{xy}\Delta A\sin\theta\right)\cos\theta$$
$$0 = \left(\sigma_n - \sigma_x\cos^2\theta - \sigma_y\sin^2\theta - 2\tau_{xy}\cos\theta\sin\theta\right)\Delta A$$

$$\sigma_n = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta \quad (1)$$

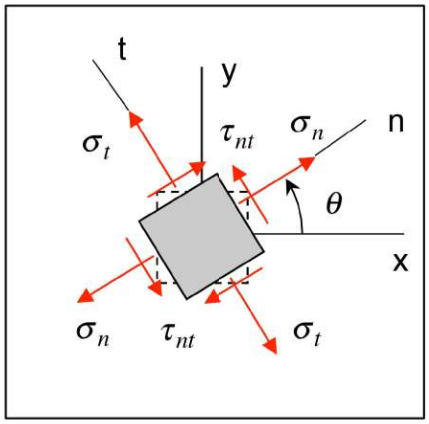
Similarly, summing forces in the t-direction on the cut section gives:

$$\sum F_t = \tau_{nt}\Delta A + \left(\sigma_x\Delta A\cos\theta\right)\sin\theta - \left(\tau_{xy}\Delta A\cos\theta\right)\cos\theta$$
$$-\left(\sigma_y\Delta A\sin\theta\right)\cos\theta + \left(\tau_{xy}\Delta A\sin\theta\right)\sin\theta$$
$$0 = \left[\tau_{nt} + \left(\sigma_x - \sigma_y\right)\cos\theta\sin\theta - \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)\right]\Delta A \quad \Rightarrow$$

$$\tau_{nt} = -\left(\sigma_x - \sigma_y\right)\cos\theta\sin\theta - \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \quad (2)$$

Equations (1) and (2) provide us with equations for determining two of the three components of stress, σ_n and τ_{nt}, on the rotated stress element. The remaining state of stress, σ_t, can be found from equation (1) by substituting $\theta + 90°$ in for θ (since the "*t*" face of the stress element is a $90°$ CCW rotation from the "*n*" face); that is,

$$\sigma_t = \sigma_x \cos^2(\theta + 90°) + \sigma_y \sin^2(\theta + 90°) + 2\tau_{xy} \cos(\theta + 90°)\sin(\theta + 90°)$$
$$= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \cos\theta \sin\theta \qquad (3)$$



With the use of some trigonometric identities[1], equations (1) and (2) can be written in a slightly modified form, a form that we will find useful later on in interpreting the results of stress transformations:

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta \qquad (1a)$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy} \cos 2\theta \qquad (2a)$$

[1] Here, we use the trig identities: $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos^2\theta = (1+\cos 2\theta)/2$ and $\sin^2\theta = (1-\cos 2\theta)/2$.

c) Principal normal stresses and maximum shear stress

Equations (1a) and (2a) show how the normal and shear stresses on the n-axis face of a 2D stress element varies with the rotation angle θ of the stress element:

$$\sigma_n = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta \tag{1a}$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta \tag{2a}$$

where:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

The question for us in this section is to determine the maximum and minimum values of these normal and shear stress components as the stress element is rotated. The maximum and minimum values of the normal stress are known as the "principal" stresses.

σ_{P1} = most positive σ_{P2} = most negative.

Principal stresses

To determine the rotations that correspond to the principal stresses we need to set $\dfrac{d\sigma_n}{d\theta} = 0$ and solve for the rotation angle. To this end, we write from equation (1a):

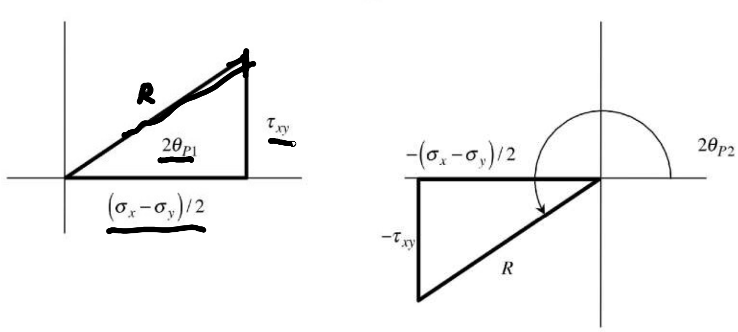
$$\frac{d\sigma_n}{d\theta} = -2\left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0 \Rightarrow$$

$$\tan 2\theta_p = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \quad \Rightarrow \quad \theta_P = \frac{1}{2} \tan^{-1}\left[\frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \right] \tag{5}$$

- We see that there are two values of $2\theta_P$ ($2\theta_{P1}$ and $2\theta_{P2}$) separated by 180° that satisfy equation (5): $2\theta_{P2} = 2\theta_{P1} \pm 180°$, or:

  $$\theta_{P2} = \theta_{P1} \pm 90°$$

- Substitution of these two angles back into equation (1a) gives the two values for the principal stresses, σ_{P1} and σ_{P2}. For given numerical values for the stress state, this process of calculating principal stresses is straightforward. However, we desire to develop general expressions for these principal stresses. To this end, consider a right triangle having τ_{xy} and $\left(\sigma_x - \sigma_y\right)/2$ as opposite and adjacent sides, respectively, for the angle $2\theta_{P1}$ shown below left:



From this figure we see that:

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R} \tag{6}$$

$$\cos 2\theta_{p1} = \frac{\left(\sigma_x - \sigma_y\right)/2}{R} \tag{7}$$

where $R = \sqrt{\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2 / 4}$. Substituting (6) and (7) into equation (1a) gives:

$$\sigma_{P1} = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\left[\frac{\left(\sigma_x - \sigma_y\right)/2}{R}\right] + \tau_{xy}\left[\frac{\tau_{xy}}{R}\right] = \sigma_{ave} + R \tag{8}$$

For the triangle corresponding to the angle θ_{P2} we have:

$$\sin 2\theta_{p2} = -\frac{\tau_{xy}}{R} \tag{6a}$$

$$\cos 2\theta_{p2} = -\frac{\left(\sigma_x - \sigma_y\right)/2}{R} \tag{7a}$$

Substituting (6a) and (7a) into equation (1a) gives:

$$\sigma_{P2} = \sigma_{ave} - \left(\frac{\sigma_x - \sigma_y}{2}\right)\left[\frac{\left(\sigma_x - \sigma_y\right)/2}{R}\right] - \tau_{xy}\left[\frac{\tau_{xy}}{R}\right] = \sigma_{ave} - R \tag{9}$$

- If we substitute either (6) and (7), or (6a) and (7a), into equation (2a) we see that:
$$\tau_{nt}\left(\theta_{P1}\right) = \tau_{nt}\left(\theta_{P2}\right) = 0 \tag{10}$$

In summary:

a) The two "principal" stress components σ_{P1} and σ_{P2} are given by:

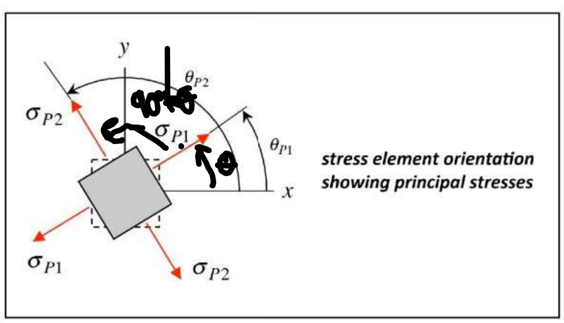
$$\sigma_{P} = \sigma_{ave} \pm R$$

$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

b) These stress states occur on faces whose rotations are separated by 90° :

$$\theta_{P2} = \theta_{P1} \pm 90°$$

c) Equation (10) shows that the shear stress on the faces corresponding to principal stresses is ZERO.



Maximum in-plane shear stress

To determine the rotations that correspond to the maximum shear stress in the plane we need to set $\frac{d\tau_{nt}}{d\theta} = 0$ and solve for the rotation angle. To this end, we write from equation (1b):

$$\frac{d\tau_{nt}}{d\theta} = -2\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \quad \Rightarrow$$

$$\tan 2\theta_s = \frac{\sin 2\theta_s}{\cos 2\theta_s} = -\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \quad \Rightarrow \quad \theta_s = \frac{1}{2}\tan^{-1}\left[-\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}}\right] \qquad (11)$$

Using a procedure similar to that above for principal stresses (and detailed in the textbook), we can show that there are two orientations 90° apart producing maximum shear stresses of:

$$\tau_{s1,s2} = \pm R$$

where, as before, $R = \sqrt{\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2 / 4}$.

In summary:

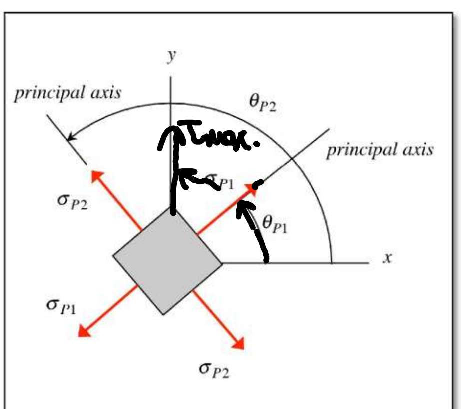
a) The two maximum shear stress values τ_{s1} and τ_{s2} are given by:

$$\tau_{s1,s2} = \pm R$$

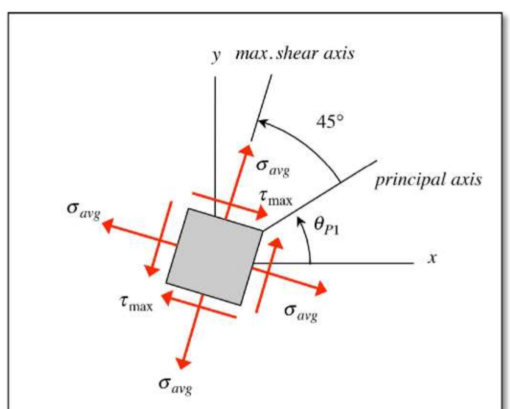
b) These stress states occur on faces whose rotations are separated by 90° : $\theta_{s2} = \theta_{s1} \pm 90°$

These orientations are 45° rotations from the principal stress axes.

c) The normal stress on the faces corresponding to maximum shear stress is NOT zero, rather they are given by: $\sigma_n\left(\theta_{s1}\right) = \sigma_n\left(\theta_{s2}\right) = \sigma_{avg}$, where, as before, $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$.



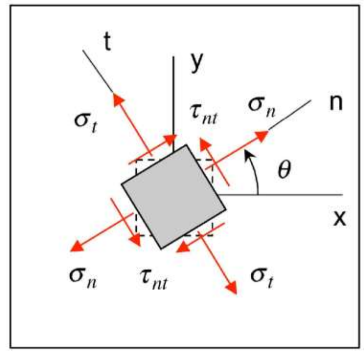
stress element orientation showing principal axes



stress element orientation showing axes of max. shear stress

d) Mohr's circle: visualizing the stress transformation

As seen in equations (1a) and (2a), for a given state of plane stress at a point (σ_x, σ_y, τ_{xy}), we have the following stress transformation equations:



$$\sigma_n - \sigma_{ave} = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$.

Suppose we take the square of both sides of the above two equations and add together the results:

$$\left(\sigma_n - \sigma_{ave}\right)^2 + \tau_{nt}^2 = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$

$$+\left[-\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$

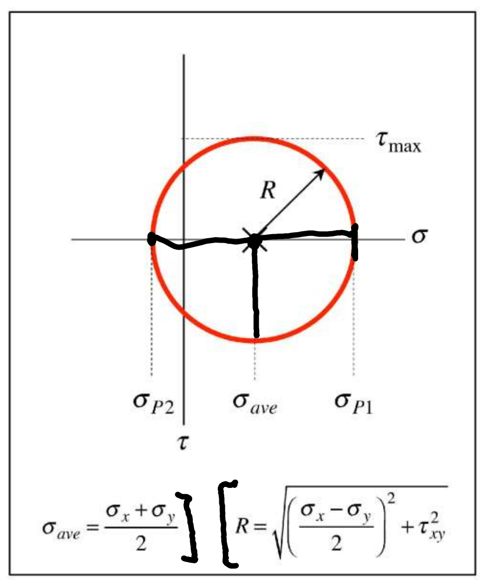
$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\left(\cos^2 2\theta + \sin^2 2\theta\right) + \tau_{xy}^2\left(\sin^2 2\theta + \cos^2 2\theta\right)$$

$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \triangleq R^2 \quad x^2 + y^2 = R^2$$

The above shows us that if the results of the stress transformation equations (1a) and (2a) are plotted in the (σ, τ) space, the result is a circle:

- whose center is located at $(\sigma_{ave}, 0)$, where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$, and
- whose radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

as shown in the figure to the right. This representation is known as "Mohr's circle" for a given state of plane stress, $(\sigma_x, \sigma_y, \tau_{xy})$.



What can we learn from Mohr's circle?

a) The principal stresses, σ_{P1} and σ_{P2}, are given by:

$$\sigma_{P1} = \sigma_{ave} + R$$
$$\sigma_{P2} = \sigma_{ave} - R$$

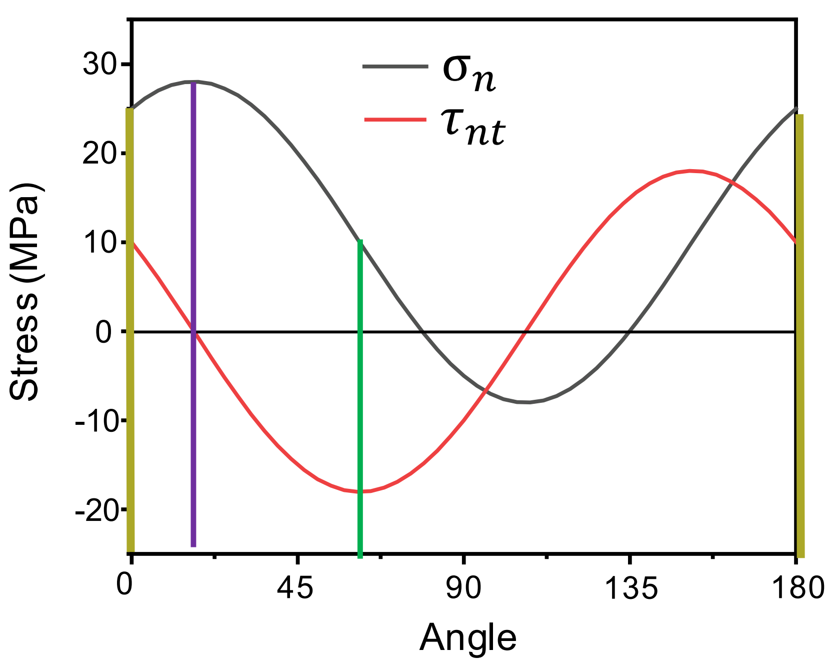
b) The principal stresses occur at stress element orientations at which the shear stress is zero, $\tau = 0$.

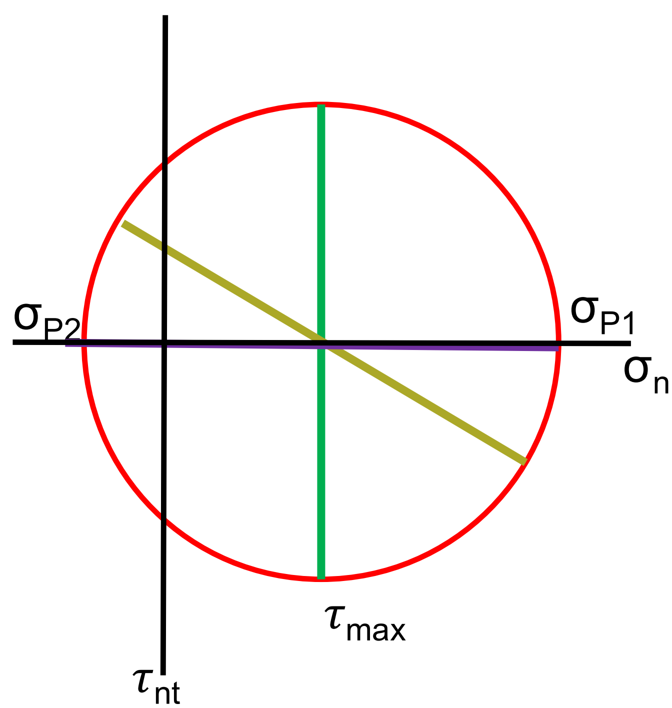
c) The maximum shear stress is given by:

$$\tau_{\max} = \underline{R}$$

d) The maximum shear stress occurs at stress element orientations at which the normal stress is $\underline{\sigma = \sigma_{ave}}$.

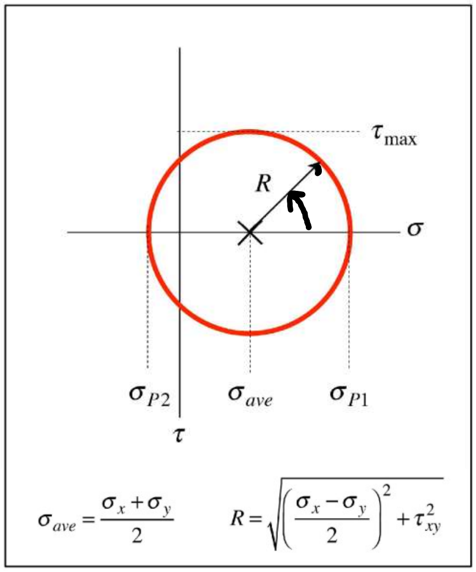
These are all things that we discovered from analysis earlier when considering principal stresses and maximum shear stress. The Mohr's circle simply allows us to visualize these results and will help us to remember these important relations.

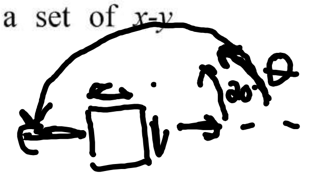




Using Mohr's circle to locate planes of principal stresses and in-plane maximum shear stress

Up to this point we have seen that Mohr's circle in the $\sigma - \tau$ plane provides us with information on the description of the state of plane stress: the stress states lie on a circle of radius $R = \sqrt{\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2 / 4}$ and centered on $(\sigma, \tau) = (\sigma_{ave}, 0)$, where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$, and where σ_x, σ_y and τ_{xy} are the two normal components of stress and shear component of stress, respectively, corresponding to a set of *x-y* coordinate axes.





What we have *not* done at this point is discussed how to relate a transformed stress state through a rotation angle of θ to its location on the Mohr's circle in the $\sigma - \tau$ plane.

Before attempting this, let's review a couple points related to what we already know about stress states and Mohr's circle.

- <u>Direction of positive shear stress in Mohr's circle</u>. We have defined a positive shear stress on the x-face of the stress cube as being in the positive y-direction. Once we rotate this stress cube, this notation is equivalently stated as being positive in the n-face pointing in the t-direction. We will continue that here. However, here we will point the <u>positive</u> τ direction DOWNWARD in the $\sigma - \tau$ plane when constructing our Mohr's circle diagram. The reasoning behind this somewhat odd choice of positive direction is to maintain an equivalence in the direction of rotation of the element in the physical space with the direction of rotation (e.g., to insure that a CCW rotation in the physical space corresponds to a CCW rotation in Mohr's circle plane).

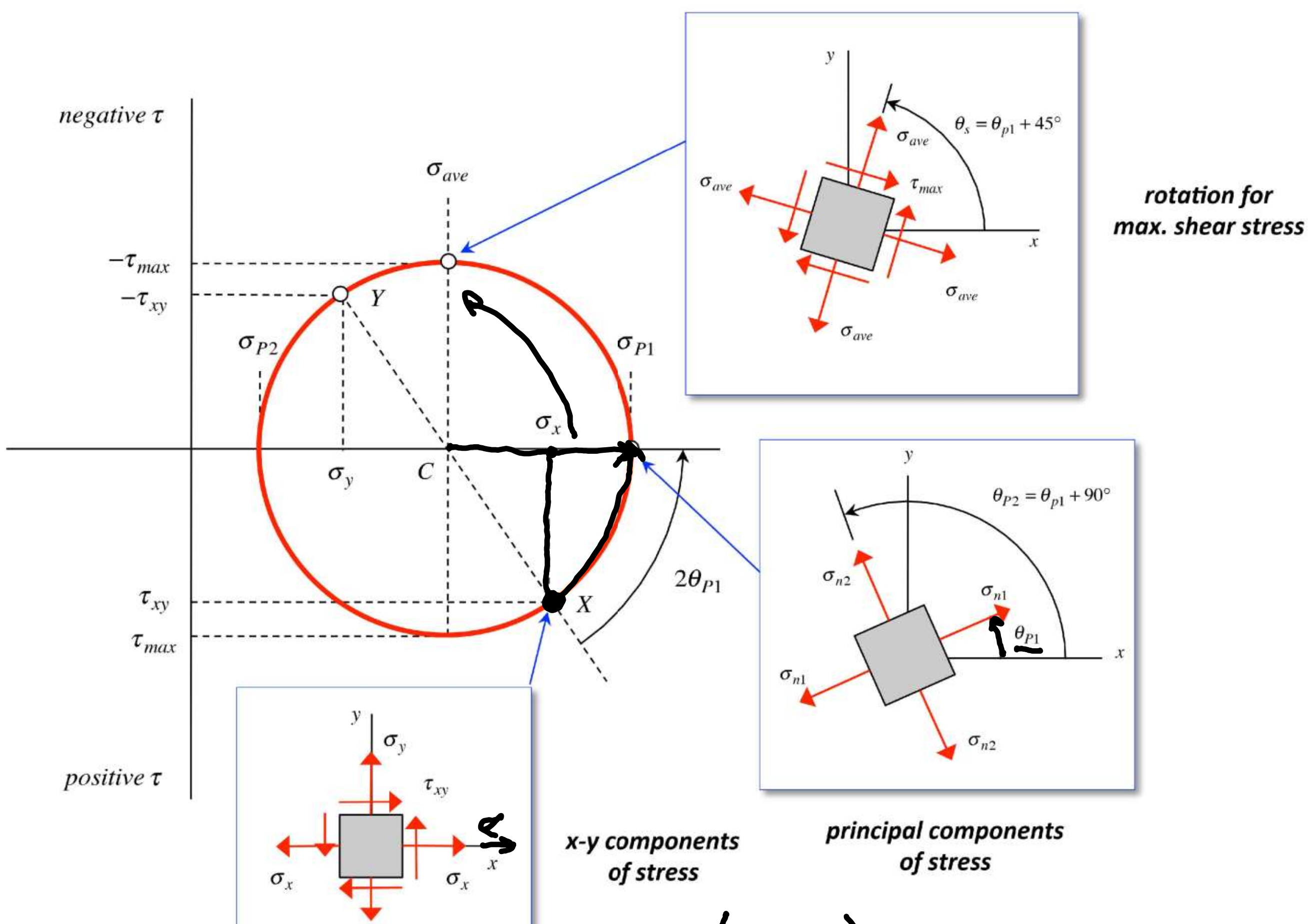
- Angle of rotation vs. angle in the $\sigma - \tau$ plane. Note that the stress transformation equations are all written in terms of the angle 2θ, where θ is the physical angle of rotation in the *x-y* plane:

$$\sigma = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

  As a result, a physical angle of rotation of θ corresponds to an angle of rotation of 2θ in the $\sigma - \tau$ plane.

Both of these are demonstrated in the following figure. This figure contains much detailed information concerning the construction of Mohr's circle and the relationship of rotations in the physical $x - y$ plane to rotations in the $\sigma - \tau$ plane. In addition, we can readily see the locations of the principal stresses and maximum in-plane shear stress. Study this figure, and then move onto the next page where we have listed a series of steps that are convenient for constructing Mohr's circle from a state of stress $(\sigma_x, \sigma_y, \tau_{xy})$.

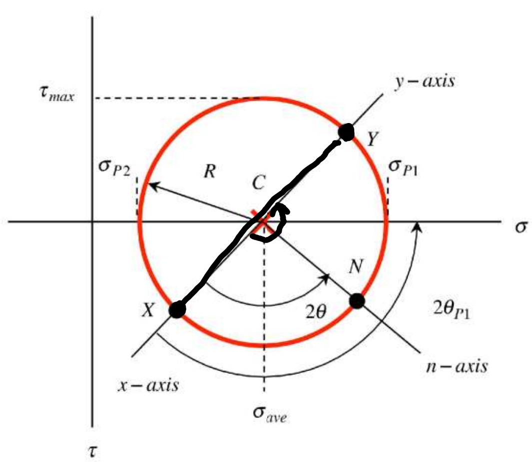


$X = (\sigma_x, \tau_{xy})$

Construction of Mohr's circle for a general state of plane stress

For a given state of stress (σ_x, σ_y, τ_{xy}) for a point:

1) Establish a set of σ-τ axes (be sure to use the same scale on each axis)::
   - +σ points to *right*
   - +τ points *down*
2) Calculate the two parameters that define the location and size of Mohr's circle:
   - $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2} = \textit{average normal stress}$
   - $R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
3) Draw a circle in the $\sigma - \tau$ plane with its center C at $(\sigma, \tau) = (\sigma_{ave}, 0)$ and having a radius of R.
4) Show the point X given by the coordinates $(\sigma, \tau) = (\sigma_x, \tau_{xy})$ on the Mohr's circle. Line OX is the x-axis. (Note that the y-axis is at a $180°$ from the x-axis in the $\sigma - \tau$ plane.)
5) The components of stress on the face of a stress element rotated through an angle of θ corresponds to a point N on Mohr's circle found through a rotation of 2θ on the circle.
6) The angle from the x-axis to the $\sigma -$axis in the Mohr's circle plane is $2\theta_{P1}$, where θ_{P1} the rotation angle for the stress element that produces the largest principal stress σ_{P1}. It is readily seen from the figure that the principal stresses are given by: $\sigma_{P1,P2} = \sigma_{ave} \pm R$.

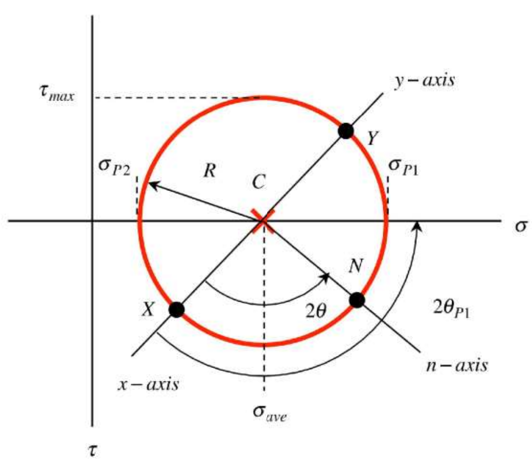


$X = (\sigma_x, \tau_{xy})$
$Y = (\sigma_y, -\tau_{xy})$

Alternate (graphical) construction of Mohr's circle for a general state of plane stress

For a given state of stress (σ_x, σ_y, τ_{xy}) for a point:

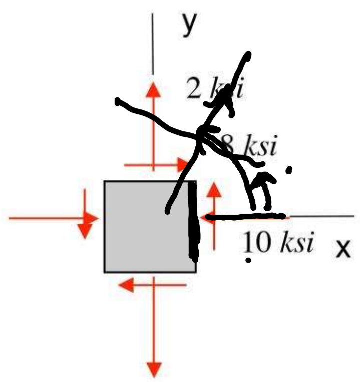
1) Establish a set of σ-τ axes (be sure to use the same scale on each axis):
   - +σ points to *right*
   - +τ points *down*
2) Locate points X and Y at locations $\left(\sigma_x, \tau_{xy}\right)$ and $\left(\sigma_y, -\tau_{xy}\right)$ on your set of axes.
3) Connect points X and Y with a straight line, and locate the center of the Mohr's circle at location C where this line crosses the $\sigma - axis$. This intersection occurs at $\left(\sigma_{ave}, 0\right)$. Note that the x- and y-axes correspond to lines CX and CY, respectively.
4) Draw a circle with its center at $\left(\sigma_{ave}, 0\right)$ and passing through points X and Y.
5) Calculate the radius of the circle using $R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.
6) The components of stress on the face of a stress element rotated through an angle of θ corresponds to a point N on Mohr's circle found through a rotation of 2θ on the circle.
7) The angle from the x-axis to the $\sigma -$axis in the Mohr's circle plane is $2\theta_{P1}$, where θ_{P1} the rotation angle for the stress element that produces the largest principal stress σ_{P1}. It is readily seen from the figure that the principal stresses are given by: $\sigma_{P1,P2} = \sigma_{ave} \pm R$.

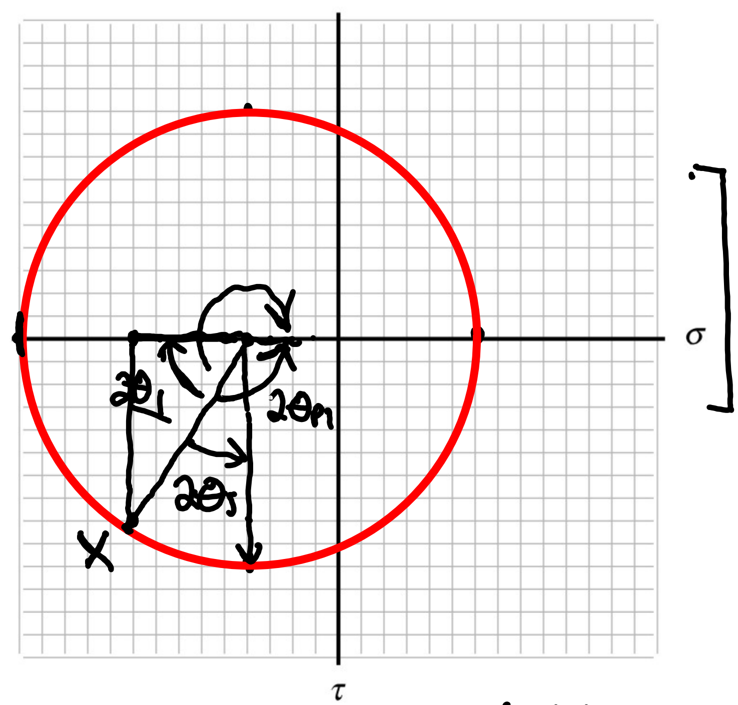


Example 13.2

For the given state of stress shown, determine the principal stresses, the maximum in-plane shear stress and the stress element rotation angles corresponding to these stresses.

$\sigma_p = \sigma_{avg} \pm R.$





1 unit = 1 ksi

$$\sigma_{avg} = \frac{-10+2}{2} = -4 \text{ ksi}$$

$$R = \sqrt{\left(\frac{-10-2}{2}\right)^2 + 8^2} = 10 \text{ ksi}$$

$$\sigma_{p1} = -4 + 10 = 6 \text{ ksi}$$

$$\sigma_{p2} = -4 - 10 = -14 \text{ ksi}$$

$$\tau_{max} = 10 \text{ ksi}$$

$$X = (\sigma_x, \tau_{xy}) = (-10, 8)$$

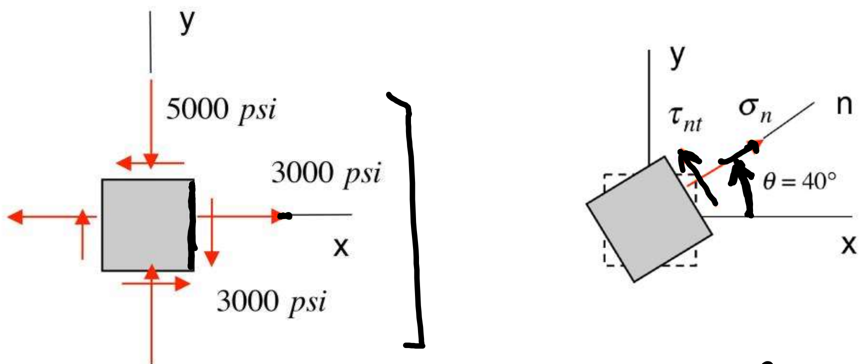
$$2\theta_{p1} = 180 - 2\theta_1 = 180 - \sin^{-1}\left(\frac{8}{10}\right) = 126.87^\circ$$

$$\theta_{p1} = 63.43^\circ \text{ CCW}$$

$$\theta_s = 63.43^\circ - 45^\circ = 18.43^\circ$$

Example 13.1

Consider the state of plane stress shown below. Determine the stresses on the plane shown below whose orientation is a 40° CCW rotation from the x-axis.



1 units = 1000 psi

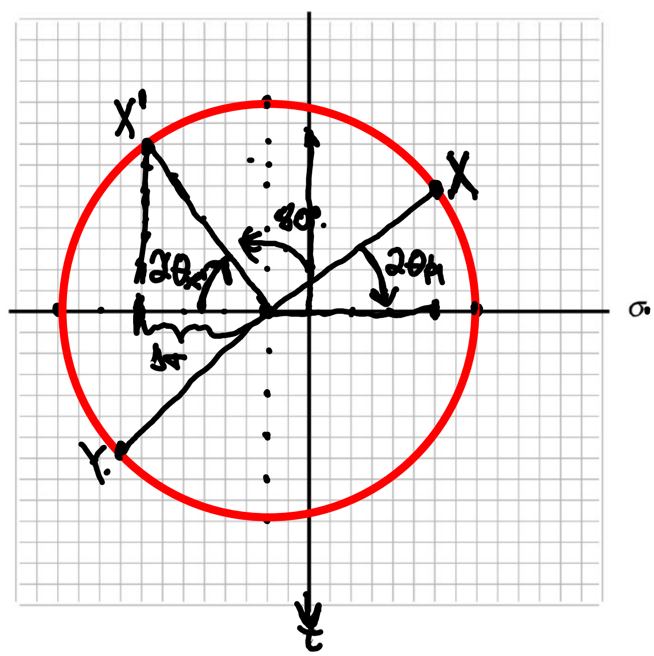
$$\sigma_{avg} = \frac{3000-5000}{2} = -1000 \text{ psi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$R = \sqrt{\left(\frac{3000+5000}{2}\right)^2 + 3000^2}$$

$$R = 5000 \text{ psi}$$

$$X = (\sigma_x, \tau_{xy}) = (3000, -3000)$$



$$2\theta_{p1} = \tan^{-1}\left(\frac{3000}{4000}\right) = 36.87°$$

$$2\theta_{x'} = 180 - 80 - 37 = 63°$$

$$\sin(2\theta_{x'}) = \frac{\tau_{x'}}{R} \qquad \tau_{x'} = -4460 \text{ psi}$$

$$\cos(2\theta_{x'}) = \frac{\Delta\sigma}{R} \qquad \Delta\sigma = 2260 \text{ psi} \qquad \sigma_n = -1000 - 2260 = -3260 \text{ psi}$$