MECHANICS OF MATERIALS

Fall 2023

ME 323- 005

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Lecture 26: Example Problems

PURDUE UNIVERSITY | School of Mechanical Engineering

Example 1

Consider the propped-cantilevered beam shown, with the roller support at location C along the beam. The beam is made up of a material having a Young's modulus of $\boxed{E}$, with the beam having a (b x b) square cross-section, where a = 10b. A constant line lone p_0 acts along the length of the beam, and a concentrated load P acts at end H, where $P = 2p_0a$.

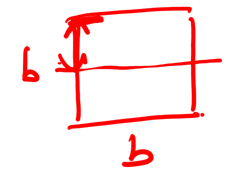
a) Using either integration techniques, Castigliano's theorem or superposition, determine the support reactions on the beam at B, C and H. If Castigliano's theorem is used, consider the contributions from shear to be negligible.

b) Draw the shear force V(x) and bending moment M(x) diagrams in the plot axes provided.

V, M

c) Determine the maximum flexural stress in the beam.

$$\sigma_{max} = \frac{|M_{max}|(b/2)}{I}$$

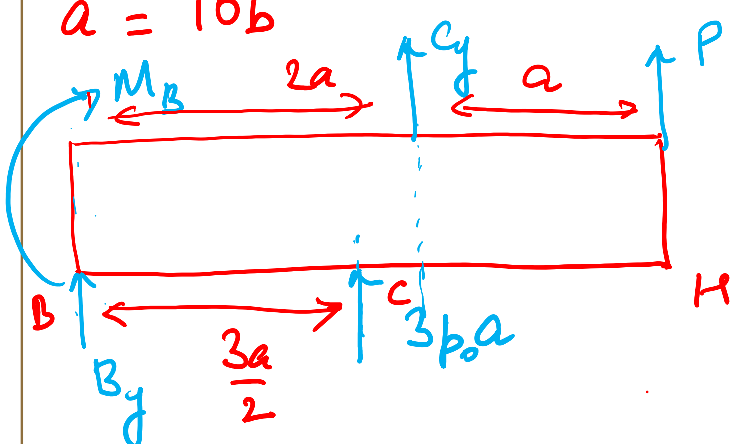


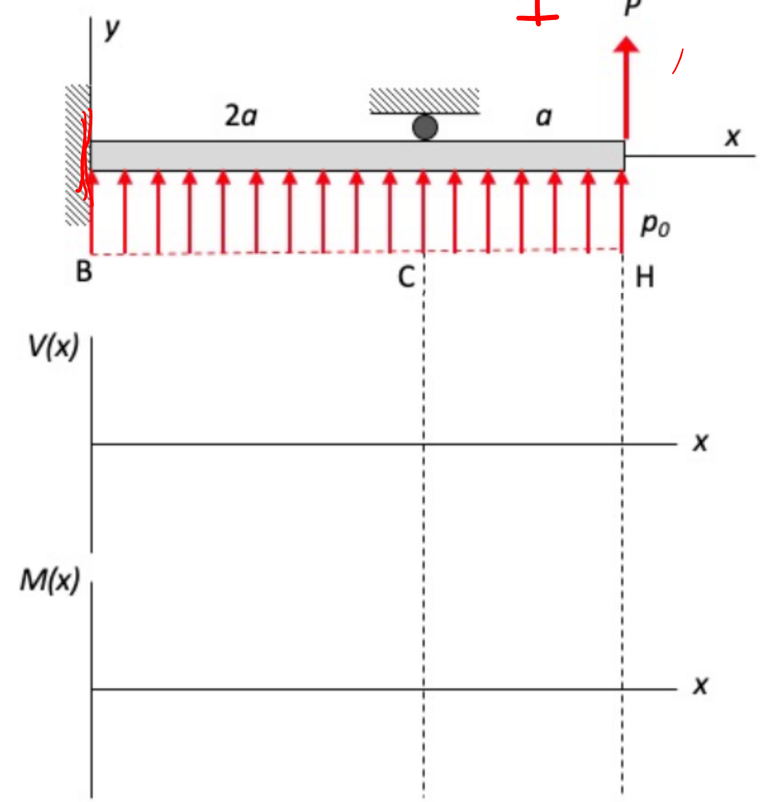
Leave your answers in terms of, at most: a, E and p_0.

$3p_0a$

$P = 2p_0a$

$a = 10b$





equilibrium equations -

$$(\Sigma M)_B = -M_B + C_y(2a) + P(3a) + 3ap_0\left(\frac{3a}{2}\right) = 0$$

$$M_B = 2C_y.a + 3Pa + \frac{9}{2}ap_0 \quad \text{—— } 1$$

$$(\Sigma F)_y = B_y + 3ap_0 + C_y + P = 0 \quad \text{—— } 2$$

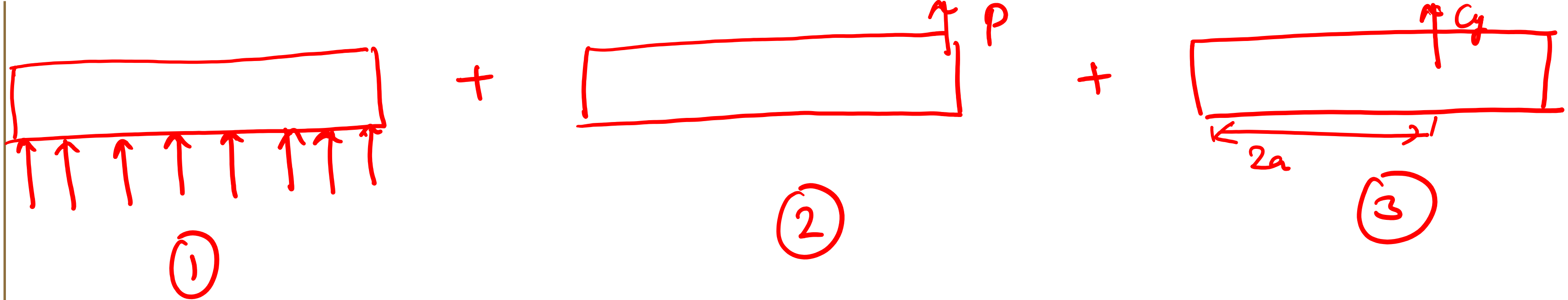
$M_B, B_y, C_y \rightarrow$ 3 unknowns, 2 equations] Indeterminate

Second order integration $\rightarrow v(x), \theta(x)$

Superposition $\rightarrow$ simpler loading conditions

Castigliano's method $\rightarrow$ U, redundant load (B_y)

$$\frac{\partial U}{\partial B_y} = 0$$



$$v(x) = v_1(x) + v_2(x) + v_3(x)$$

$$v_1(x) = \frac{1}{24} x^2 \left[6(3a)^2 - 4(3a)x + x^2 \right] \frac{p_0}{EI} \quad \text{---} \quad 0 < x < 3a$$

$$v_2(x) = \frac{1}{6} x^2 \left[3(3a) - x \right] \frac{P}{EI} \quad \text{------} \quad 0 < x < 3a$$

$$v_3(x) = \frac{1}{6} x^2 \left[3(2a) - x \right] \frac{C_y}{EI} \quad \text{-----} \quad 0 < x < 2a$$

$$v(x) = v_1(x) + v_2(x) + v_3(x)$$

Compatibility; $v(2a) = 0$

$$v_1(2a) + v_2(2a) + v_3(2a) = 0$$

$$v_1(2a) = \frac{1}{24}(2a)^2\left[54a^2 - 24a^2 + 4a^2\right]\frac{p_0}{EI} = \frac{17a^4 p_0}{3EI}$$

$$v_2(2a) = \frac{14a^3 P}{3EI}$$

$$v_3(2a) = \frac{8a^3 C_y}{3EI}$$

$$\frac{17a^4 p_0}{3EI} + \frac{14a^3 P}{3EI} + \frac{8a^3 C_y}{3EI} = 0$$

$$17ap_0 + 14P + 8C_y = 0 \qquad — \ 3$$

$$17p_0 a + 28p_0 a + 8C_y = 0$$

$$\boxed{C_y = -\frac{45p_0 a}{8}}$$

$$\boxed{B_y = \frac{5p_0 a}{8}} \qquad \boxed{M_B = -\frac{3p_0 a^2}{4}}$$

$$V(0) = B_y = \frac{5p_0a}{8}$$

$$V(2a^-) = V(0) + p_0(2a)$$

$$= \frac{5p_0a}{8} + 2p_0a = \frac{21}{8}p_0a$$

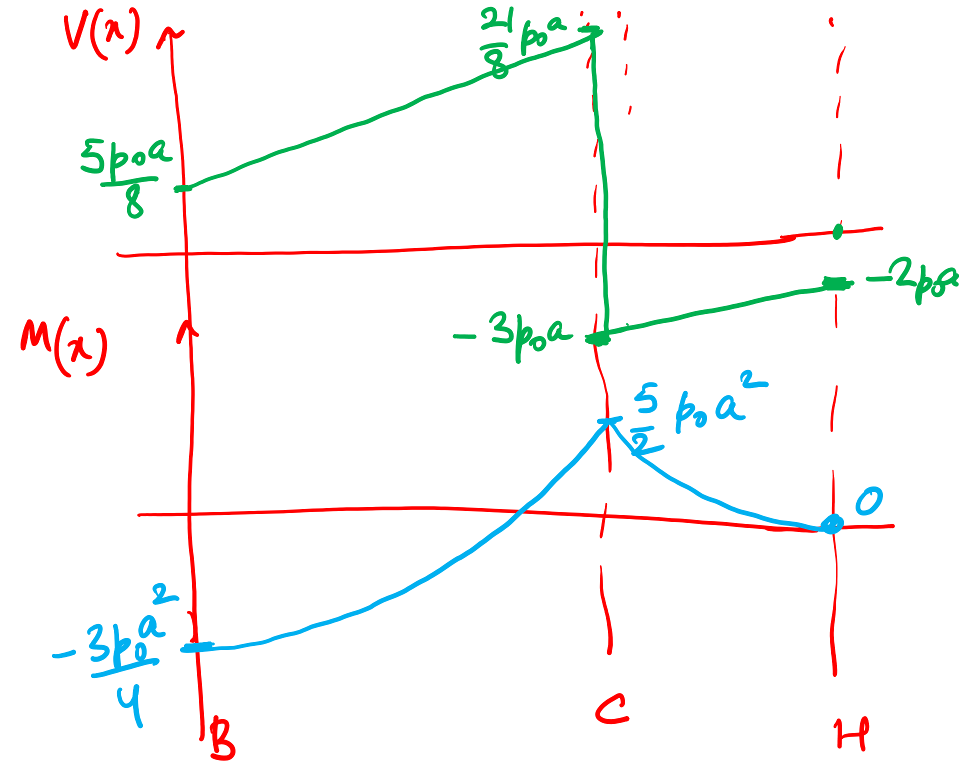
$$V(2a^+) = V(2a^-) + C_y =$$

$$= \frac{21}{8}p_0a - \frac{45}{8}p_0a$$

$$= -3p_0a$$

$$V(3a) = V(2a^+) + p_0(a)$$

$$= -3p_0a + p_0a = -2p_0a$$



$$M(0) = M_B = -\frac{3p_0a^2}{4}$$

$$M(2a) = M(0) + B_y(2a) + \frac{1}{2}\left[V(2a^-) - V(0)\right](2a)$$

$$= -\frac{3p_0a^2}{4} + \frac{10p_0a^2}{8} + \frac{1}{\cancel{2}}\left[\frac{16}{8}p_0a\right](\cancel{2}a)$$

$$= -\frac{3p_0a^2}{4} + \frac{5p_0a^2}{4} + 2p_0a^2$$

$$M(2a) = \frac{10}{4}p_0a^2 = \frac{5}{2}p_0a^2$$

$$M(3a) = M(2a) + (-2p_0a)(a) + \frac{1}{2}(-p_0a)(a)$$

$$= \frac{5}{2}p_0a^2 - 2p_0a^2 - \frac{p_0a^2}{2} = 0$$

$M_H = 0$ [correct]

$$|M_{max}| = \frac{5}{2}p_0a^2$$

$$\sigma_{max} = \frac{|M_{max}|(b/2)}{(b^4/12)} = \frac{\left(\frac{5}{2}p_0a^2\right)\left(\frac{b}{2}\right)}{(b^4/12)}$$

$$= 15\left(\frac{a}{b}\right)^2 \cdot \frac{p_0}{b} = \boxed{1500\frac{p_0}{b} = \sigma_{max}}$$

Example 2

The two-span, simply-supported beam shown is made up of a material having a Young's modulus of E, with the beam having a circular cross-section with a radius r, where $a = 10r$. A constant line lone p_0 acts along the length of the beam, and a concentrated couple M_0 acts at the roller support C, where $\boxed{M_0 = 2p_0a^2}$.

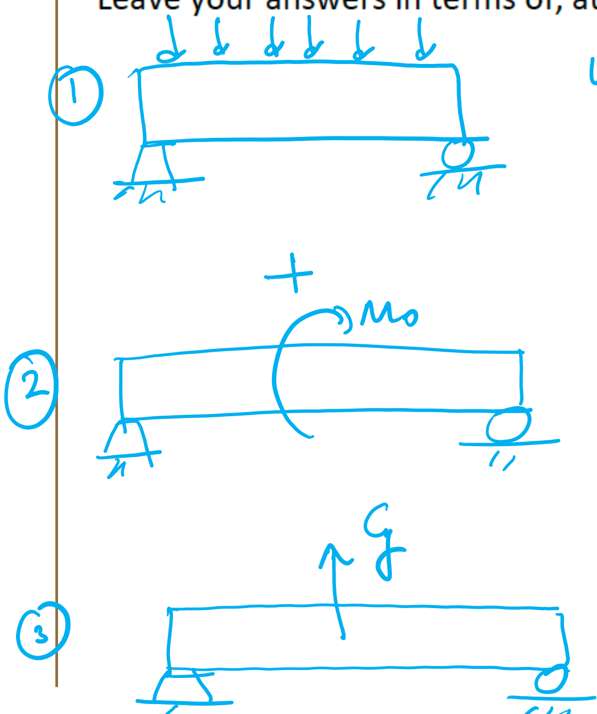
By, Cy, Hy

a) Using either integration techniques, Castigliano's theorem or superposition, determine the support reactions on the beam at B, C and H. If Castigliano's theorem is used, consider the contributions from shear to be negligible.

b) Draw the shear force $V(x)$ and bending moment $M(x)$ diagrams in the plot axes provided.

c) Determine the maximum flexural stress in the beam. $\sigma = -\frac{My}{I}$ compatibility $v(a) = 0$

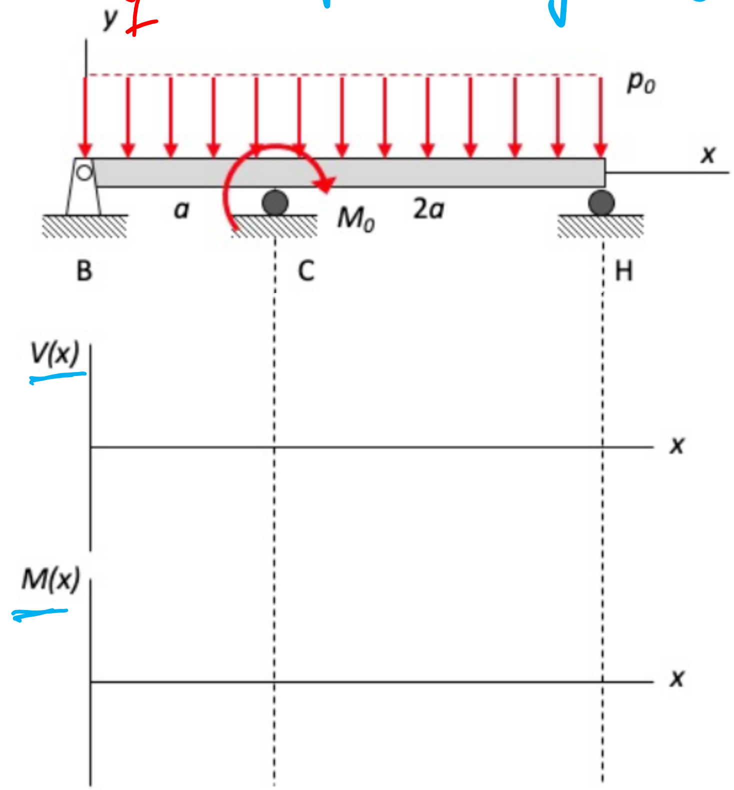
Leave your answers in terms of, at most: a, E and p_0.



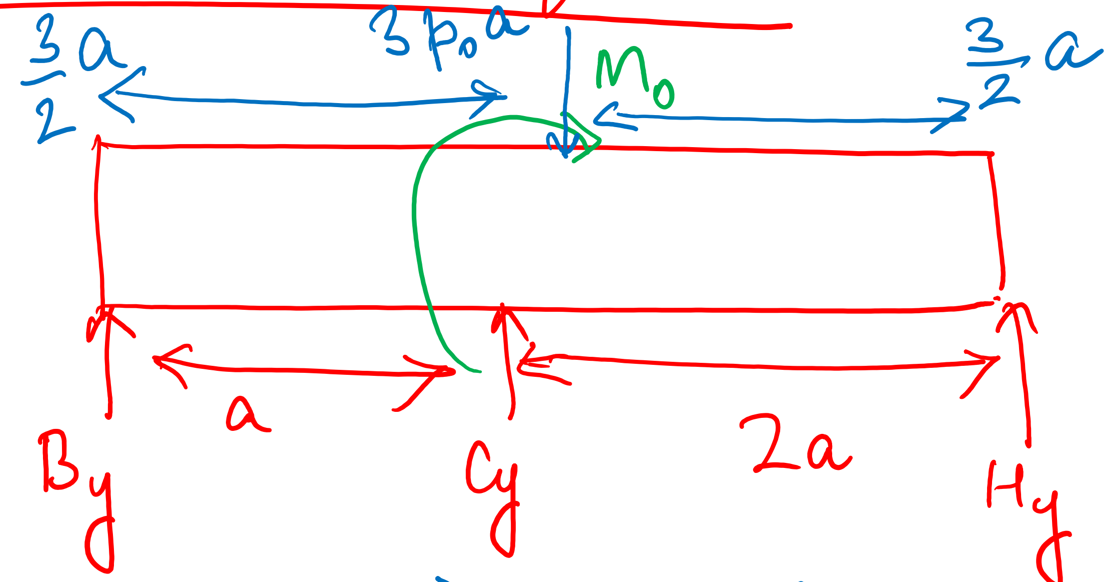
$v_1(x)$

$v_2(x)$

$v_3(x)$



FBD and equilibrium equations -



$$(\Sigma M)_B = -M_0 + C_y(a) + H_y(3a) - (3p_0 a)\left(\frac{3}{2}a\right)$$

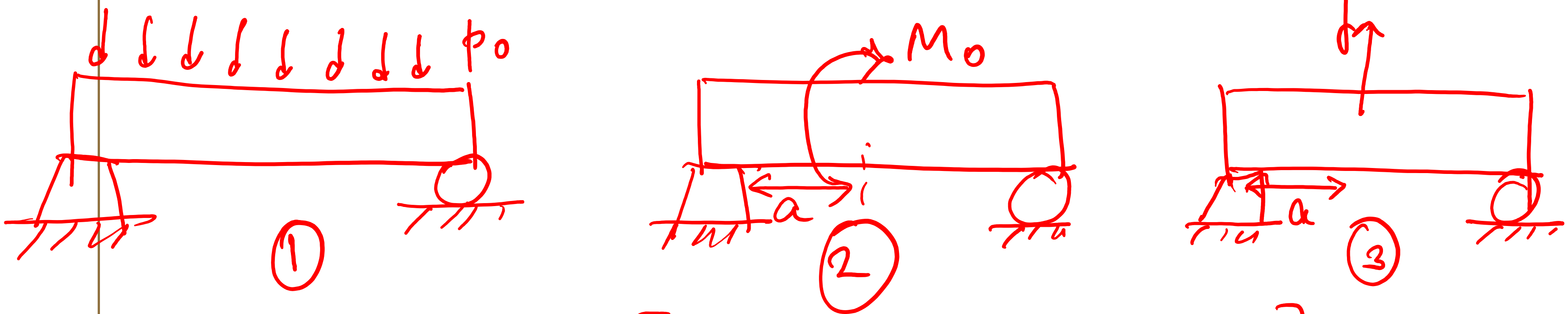
$$M_0 = C_y a + 3H_y a - \frac{9}{2}p_0 a^2 \quad \text{——} \; (1)$$

$$(\Sigma F)_y = B_y + C_y + H_y - 3p_0 a = 0 \quad \text{——} \; (2)$$

B_y, C_y, H_y (unknowns) ; 2 equations

Superposition -

$$v(x) = v_1(x) + v_2(x) + v_3(x)$$



$$v_1(x) = -\frac{1}{24}x\left[(3a)^3 - 2(3a)x^2 + x^3\right]\frac{p_0}{EI}$$

$$0 < x < 3a$$

$$v_2(x) = -\frac{1}{6}x\left[-6a(3a) + 3a^2 + 2(3a)^2 + x^2\right]\left[\frac{M_0}{(3a)EI}\right]$$

$$= -\frac{1}{6}x\left[3a^2 + x^2\right]\frac{M_0}{(3a)(EI)}$$

$$v_3(x) = \frac{1}{6}(2a)(x)\left[(3a)^2 - (2a)^2 - x^2\right]\frac{C_y}{(3a)EI}$$

$$v_3(x) = \frac{ax}{3}\left[5a^2 - x^2\right]\frac{C_y}{(3a)(EI)}$$

$$v(x) = v_1(x) + v_2(x) + v_3(x)$$

Compatibility; $v(a) = 0$

$x = a$

$M_0 = 2p_0 a^2$

$$v_1(a) + v_2(a) + v_3(a) = 0$$

$$-\frac{a}{24}\left[27a^3 - 6a^3 + a^3\right]\frac{p_0}{\cancel{EI}} - \frac{a}{6}\left[3a^2 + a^2\right]\frac{M_0}{(3a)\cancel{EI}}$$

$$+\frac{a^2}{3}\left[5a^2 - a^2\right]\frac{C_y}{(3a)\cancel{EI}} = 0$$

$$C_y = \frac{49}{16} p_0 a$$

$$H_y = \frac{55}{48} p_0 a$$

$$B_y = -\frac{29}{24} p_0 a$$

(b) $V(x)$, $M(x)$

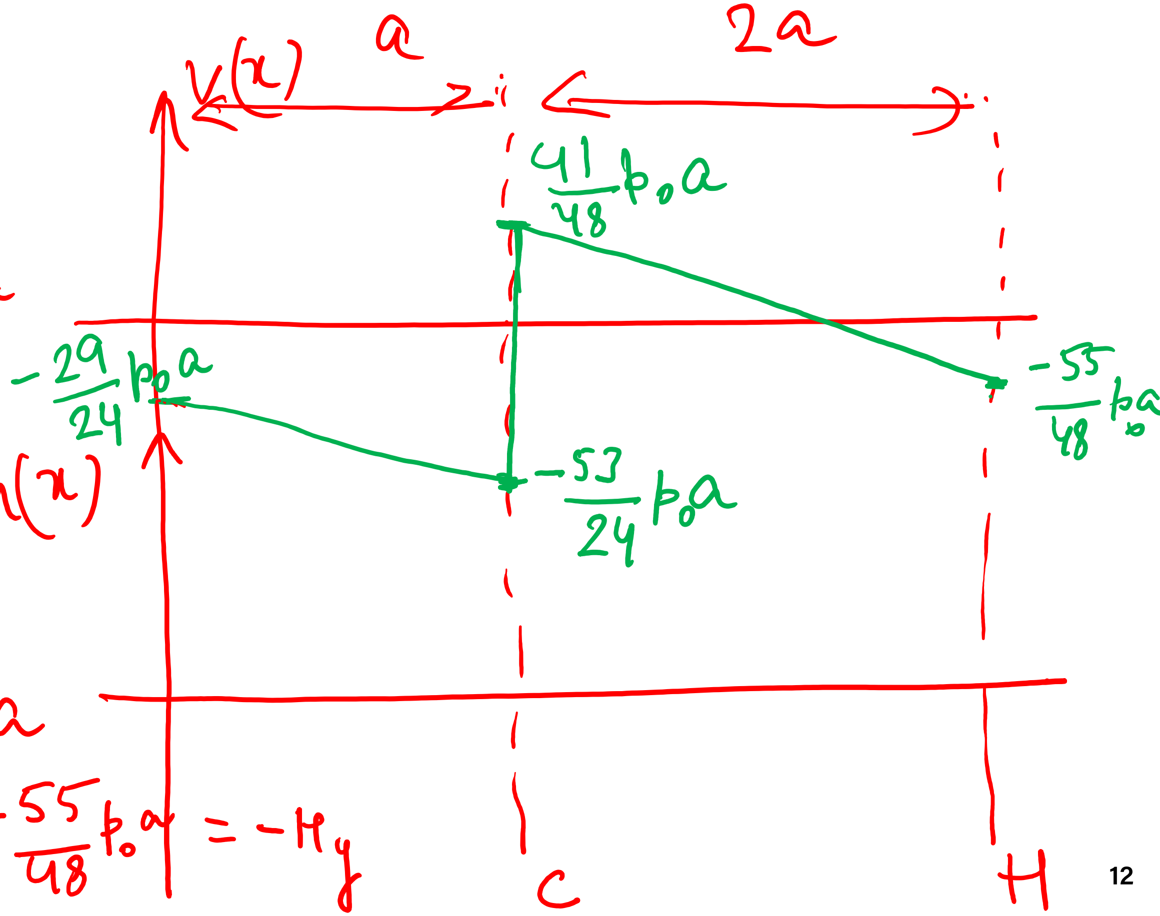
$$V(0) = B_y = -\frac{29}{24} p_0 a$$

$$V(a^-) = V(0) - p_0 a$$

$$= -\frac{53}{24} p_0 a$$

$$V(a^+) = V(a^-) + C_y = \frac{41}{48} p_0 a$$

$$V(3a) = V(a^+) - p_0(2a) = -\frac{55}{48} p_0 a = -H_y$$



$$M(0) = 0$$

$$M(a^-) = M(0) + (-B_y)(a) + \left(-\frac{58}{48}p_0 a\right)(a) + \frac{1}{2}(a)\left[-\frac{106}{48} - \left(-\frac{58}{48}\right)\right]p_0 a$$

$$= -\frac{82}{48}p_0 a^2$$

$$M(a^+) = M(a^-) + M_0 = \frac{14}{48}p_0 a^2$$

similar triangles.

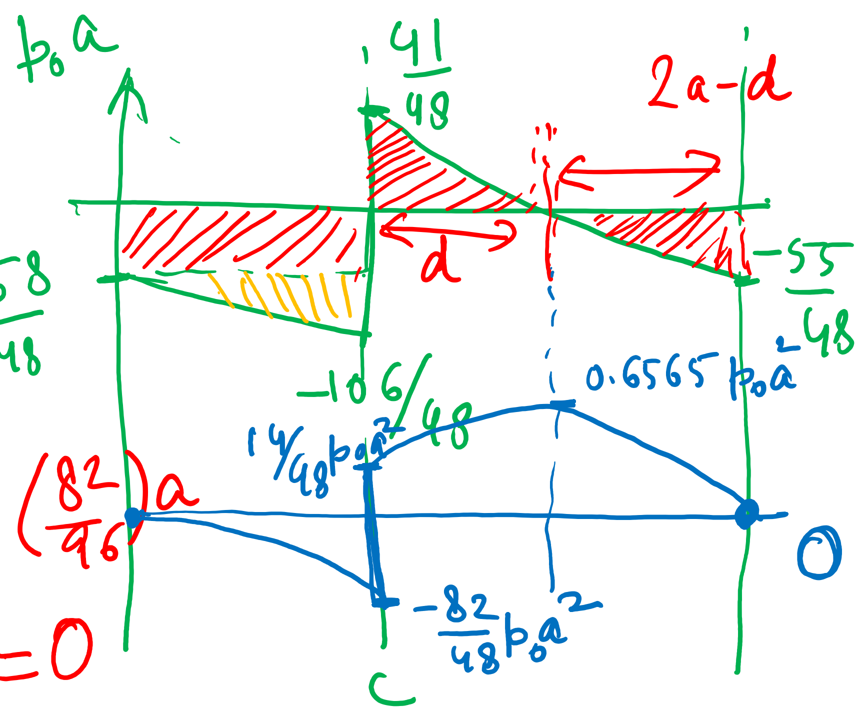
$$\frac{d}{(41/\cancel{48})} = \frac{(2a-d)}{(55/\cancel{48})}$$

$$d = \frac{82}{96}a$$

$$M(d+a) = M(a^+) + \frac{1}{2}\left(\frac{41}{48}p_0 a\right)\left(\frac{82}{96}\right)a$$

$$M(d+a) = 0.6565\, p_0 a^2$$

$$M(3a) = 0$$



(c) maximum flexural stress σ_{max}

$$\sigma = \frac{-My}{I}$$

$$\sigma_{max} = \frac{|M_{max}|\, r}{I} \qquad I = \frac{\pi}{4} r^4$$

$$\sigma_{max} = \frac{\left(\frac{82}{\cancel{48}\,12} p_0 a^2\right) \cancel{r}}{\frac{\pi}{\cancel{4}} r^{\cancel{4}\,3}}$$

$$= \frac{41 p_0}{6\pi} \left(\frac{a}{r}\right)^2 \cdot \frac{1}{r} = \frac{4100\, p_0}{6\pi r}$$

$$\boxed{\sigma_{max} = \frac{4100\, p_0}{6\pi r}}$$

Example 3

The propped cantilevered beam shown is made up of two sections: section BC with a material having a Young's modulus of E, and section CH with a material having a Young's modulus of $2E$. The beam has a circular cross-section throughout with a radius r, where $a = 10r$. Concentrated couples of M_0 and $2M_0$ act at locations B and C, respectively, on the beam.

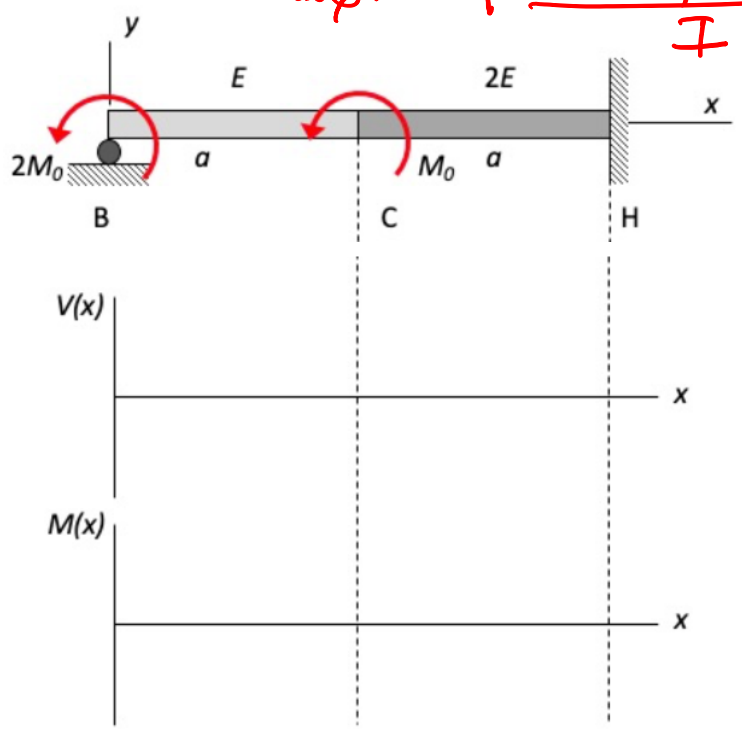
a) Using either integration techniques, Castigliano's theorem or superposition, determine the support reactions on the beam at B and H. If Castigliano's theorem is used, consider the contributions from shear to be negligible.
b) Draw the shear force $V(x)$ and bending moment $M(x)$ diagrams in the plot axes provided.
c) Determine the maximum flexural stress in the beam.

$\sigma_{max.} = \frac{|M_{max}| r}{I}$

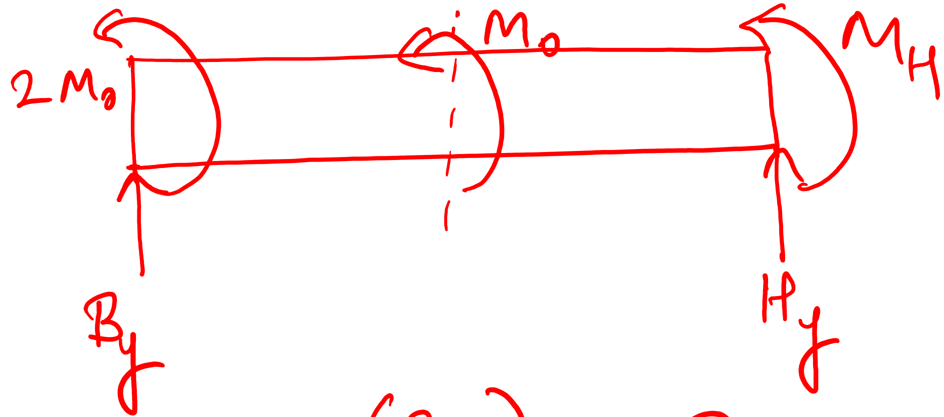
Leave your answers in terms of, at most: a, E and M_0.

Castigliano's theorem

B and H



FBD and equilibrium



$$(\Sigma M)_B = 2M_0 + M_0 + M_H + H_y(2a) = 0$$

$$M_H = -3M_0 - 2H_y a \quad \text{--- (1)}$$

$$(\Sigma F)_y = B_y + H_y = 0 \Rightarrow B_y = -H_y \quad \text{——— (2)}$$

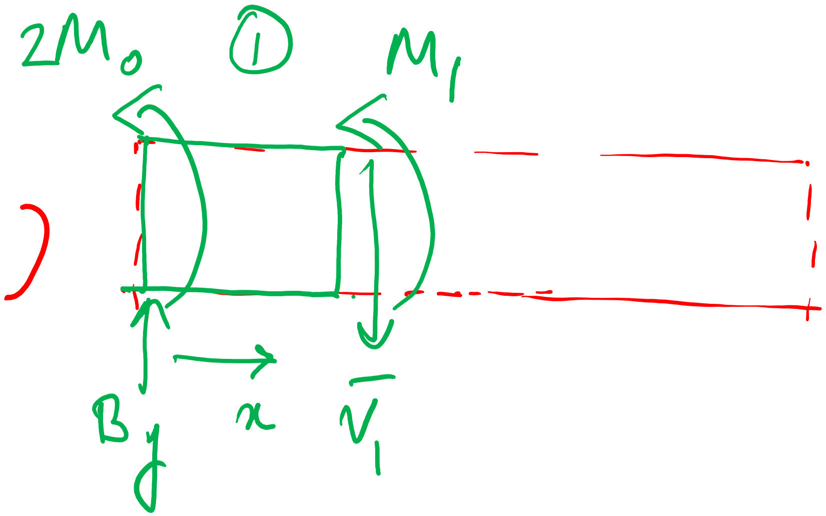
M_H, B_y, H_y (3 unknowns), 2 equations, indeterminate.

Castigliano's theorem $\Rightarrow$ redundant load (B_y)

displacement at B $\quad \dfrac{\partial U}{\partial B_y} = 0$ [Compatibility condition]

$U = U_1 + U_2$

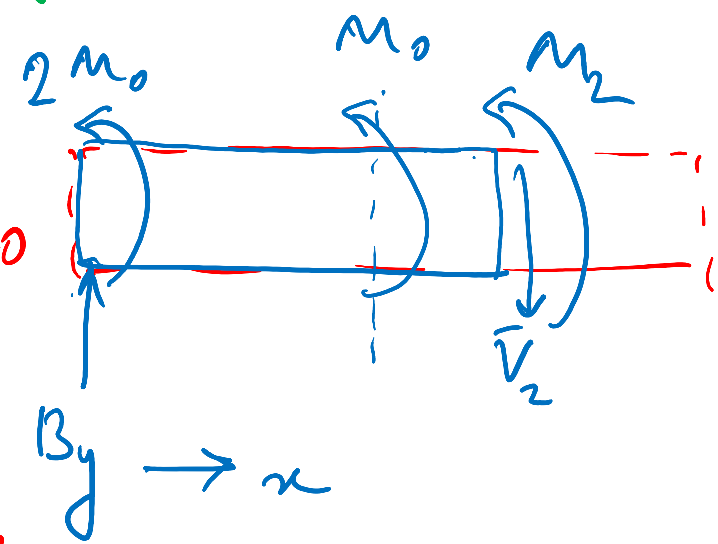
Section 1 $\longrightarrow$ between B & C



$(\Sigma M)_x = 2M_0 + M_1 - B_y(x) = 0$

$M_1(x) = B_y x - 2M_0$

Section 2 $\longrightarrow$ between C & H



$(\Sigma M)_x = 2M_0 + M_0 + M_2 - B_y x = 0$

$M_2(x) = B_y x - 3M_0$

$U = U_1 + U_2$

$$= \frac{1}{2EI}\int_0^a M_1^2 \cdot dx + \frac{1}{2EI}\int_a^{2a} M_2^2 \cdot dx$$

$$\frac{\partial U}{\partial B_y} = 0$$

$$\frac{\partial U}{\partial B_y} = \frac{1}{\cancel{2}EI}\int_0^a (\cancel{2}M_1)\cdot\left(\frac{\partial M_1}{\partial B_y}\right).dx + \frac{1}{\cancel{2}EI}\int_a^{2a}(\cancel{2}M_2)\left(\frac{\partial M_2}{\partial B_y}\right).dx$$

$$0 = \int_0^a M_1\left(\frac{\partial M_1}{\partial B_y}\right)dx + \int_a^{2a} M_2\left(\frac{\partial M_2}{\partial B_y}\right).dx$$

$$\frac{\partial M_1}{\partial B_y} = x \quad ; \quad \frac{\partial M_2}{\partial B_y} = x$$

$$\int_0^a \left(B_y x - 2M_0\right)x.dx + \int_a^{2a}\left(B_y x - 3M_0\right).x\,dx = 0$$

$$B_y = \frac{13}{6}\frac{M_0}{a} \quad ; \quad H_y = -\frac{13}{6}\frac{M_0}{a}$$

from Eq (1) —

$$M_H = -3M_0 + \frac{13}{3}M_0 = \frac{4}{3}M_0$$

$$B_y = \frac{13}{6}\frac{M_0}{a} \; ; \; H_y = -\frac{13}{6}\frac{M_0}{a} \; ; \; M_H = \frac{4}{3}M_0$$

(5)

$$V(0) = B_y = \frac{13}{6}\frac{M_0}{a}$$

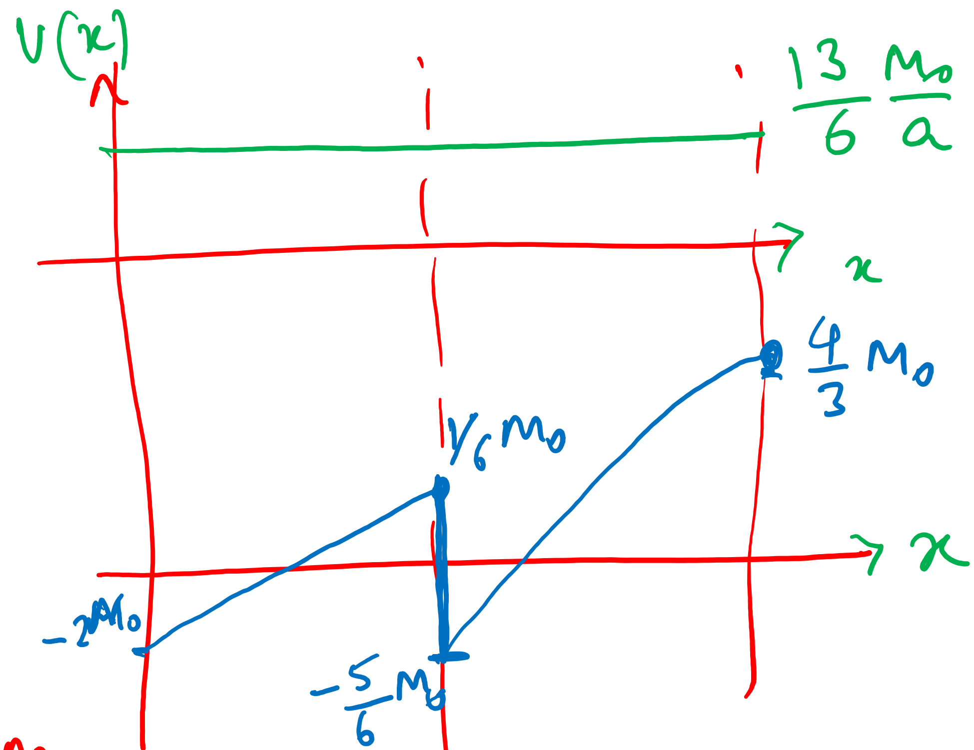
$$V(2a) = V(0) = \frac{13}{6}\frac{M_0}{a}$$

$$M(0) = -2M_0$$

$$M(a^-) = M(0) + B_y(a) = \frac{1}{6}M_0$$

$$M(a^+) = M(a^-) - M_0 = -\frac{5}{6}M_0$$

$$M(2a) = M(a^+) + (-H_y)(a) = \frac{4}{3}M_0$$



(c) $$\sigma_{max} = \frac{|M_{max}|\, r}{I}$$

$$M_{max} = M_H = \frac{4}{3} M_0$$

$$\sigma_{max} = \frac{\left(\frac{4}{3} M_0\right)(\cancel{r})}{\left(\frac{\pi}{4} r^{\cancel{4}\,3}\right)}$$

$a = 10r$

$r = \frac{a}{10}$

$$\sigma_{max} = \frac{16 M_0}{3\pi r^3}$$

$$= \frac{16 M_0}{3\pi \left(a/10\right)^3}$$

$$\boxed{\sigma_{max} = \frac{16{,}000\, M_0}{\pi a^3}}$$ ✓

THANK YOU