



APPENDIX

DEFLECTIONS AND SLOPES OF BEAMS; FIXED-END ACTIONS

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E

E.1. Deflections and Slopes of Uniform Cantilever Beams*

		Notation
		$v(x)$ = deflection in the y direction $v'(x)$ = slope of the deflection curve $\delta_B = v(L)$ = deflection at end B $\theta_B = v'(L)$ = slope at end B
1		$v = \frac{M_0 x^2}{2EI} \quad v' = \frac{M_0 x}{EI}$ $\delta_B = \frac{M_0 L^2}{2EI} \quad \theta_B = \frac{M_0 L}{EI}$
2		$v = \frac{M_0 x^2}{2EI} \quad v' = \frac{M_0 x}{EI} \quad 0 \leq x \leq a$ $v = \frac{M_0 a}{2EI}(2x - a) \quad v' = \frac{M_0 a}{EI} \quad a \leq x \leq L$ $\delta_B = \frac{M_0 a}{2EI}(2L - a) \quad \theta_B = \frac{M_0 a}{EI}$
3		$v = \frac{Px^2}{6EI}(3L - x) \quad v' = \frac{Px}{2EI}(2L - x)$ $\delta_B = \frac{PL^3}{3EI} \quad \theta_B = \frac{PL^2}{2EI}$
4		$v = \frac{Px^2}{6EI}(3a - x) \quad v' = \frac{Px}{2EI}(2a - x) \quad 0 \leq x \leq a$ $v = \frac{Pa^2}{6EI}(3x - a) \quad v' = \frac{Pa^2}{2EI} \quad a \leq x \leq L$ $\delta_B = \frac{Pa^2}{6EI}(3L - a) \quad \theta_B = \frac{Pa^2}{2EI}$

5

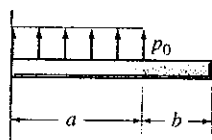


$$v = \frac{p_0 x^2}{24EI} (6L^2 - 4Lx + x^2)$$

$$v' = \frac{p_0 x}{6EI} (3L^2 - 3Lx + x^2)$$

$$\delta_B = \frac{p_0 L^4}{8EI} \quad \theta_B = \frac{p_0 L^3}{6EI}$$

6



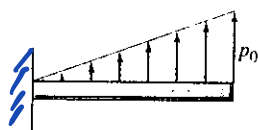
$$v = \frac{p_0 x^2}{24EI} (6a^2 - 4ax + x^2) \quad 0 \leq x \leq a$$

$$v' = \frac{p_0 x}{6EI} (3a^2 - 3ax + x^2) \quad 0 \leq x \leq a$$

$$v = \frac{p_0 a^3}{24EI} (4x - a) \quad v' = \frac{p_0 a^3}{6EI} \quad a \leq x \leq L$$

$$\delta_B = \frac{p_0 a^3}{24EI} (4L - a) \quad \theta_B = \frac{p_0 a^3}{6EI}$$

7

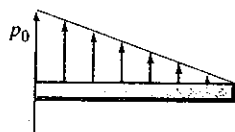


$$v = \frac{p_0 x^2}{120LEI} (20L^3 - 10L^2x + x^3)$$

$$v' = \frac{p_0 x}{24LEI} (8L^3 - 6L^2x + x^3)$$

$$\delta_B = \frac{11p_0 L^4}{120EI} \quad \theta_B = \frac{p_0 L^3}{8EI}$$

8

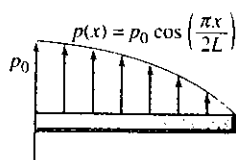


$$v = \frac{p_0 x^2}{120LEI} (10L^3 - 10L^2x + 5Lx^2 - x^3)$$

$$v' = \frac{p_0 x}{24LEI} (4L^3 - 6L^2x + 4Lx^2 - x^3)$$

$$\delta_B = \frac{p_0 L^4}{30EI} \quad \theta_B = \frac{p_0 L^3}{24EI}$$

9



$$v = \frac{p_0 L}{3\pi^4 EI} \left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3 \right)$$

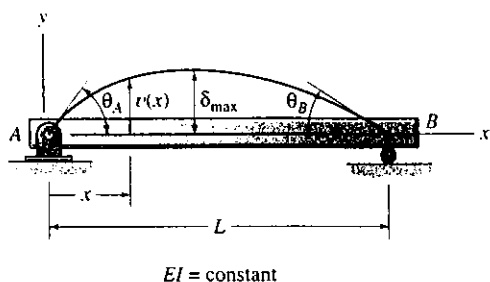
$$v' = \frac{p_0 L}{\pi^3 EI} \left(2\pi^2 Lx - \pi^2 x^2 - 8L^2 \sin \frac{\pi x}{2L} \right)$$

$$\delta_B = \frac{2p_0 L^4}{3\pi^4 EI} (\pi^3 - 24) \quad \theta_B = \frac{p_0 L^3}{\pi^3 EI} (\pi^2 - 8)$$

*Beam-deflection theory is covered in Chapter 7. The sign convention used here is the same as in Chapter 7.

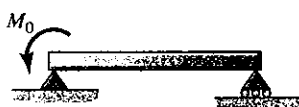
E.2. Deflections and Slopes of Uniform Simply-Supported Beams*

Notation



$v(x)$ = deflection in the y direction
 $v'(x)$ = slope of the deflection curve
 $\theta_A \equiv v'(0)$ = slope (angle) at end A
 $\theta_B \equiv -v'(L)$ = angle of rotation at end B
 x_m = distance from end A to the point of maximum deflection
 $\delta_C \equiv |v(L/2)|$ = deflection at the center of the beam
 $\delta_{\max} \equiv \max |v(x)|$ = maximum deflection

1



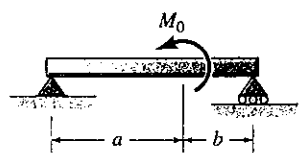
$$v = \frac{M_0 x}{6LEI} (2L^2 - 3Lx + x^2)$$

$$v' = \frac{M_0}{6LEI} (2L^2 - 6Lx + 3x^2)$$

$$\theta_A = \frac{M_0 L}{3EI} \quad \theta_B = \frac{M_0 L}{6EI}$$

$$x_m = L \left(1 - \frac{\sqrt{3}}{3} \right) \text{ and } \delta_{\max} = \frac{M_0 L^2}{9\sqrt{3}EI}$$

2

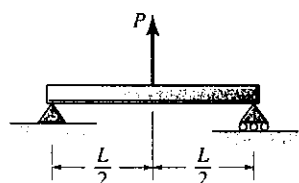


$$v = \frac{-M_0 x}{6LEI} (6aL - 3a^2 - 2L^2 - x^2) \quad 0 \leq x \leq a$$

$$v' = \frac{-M_0}{6LEI} (6aL - 3a^2 - 2L^2 - 3x^2) \quad 0 \leq x \leq a$$

$$\theta_A = \frac{-M_0}{6LEI} (6aL - 3a^2 - 2L^2) \quad \theta_B = \frac{-M_0}{6LEI} (3a^2 - L^2)$$

3

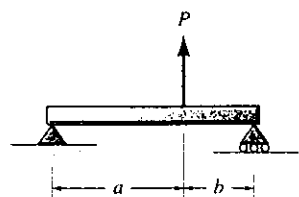


$$v = \frac{Px}{48EI} (3L^2 - 4x^2) \quad 0 \leq x \leq \frac{L}{2}$$

$$v' = \frac{P}{16EI} (L^2 - 4x^2) \quad 0 \leq x \leq \frac{L}{2}$$

$$\delta_C = \delta_{\max} = \frac{PL^3}{48EI} \quad \theta_A = \theta_B = \frac{PL^2}{16EI}$$

4



$$v = \frac{Pbx}{6LEI} (L^2 - b^2 - x^2) \quad 0 \leq x \leq a$$

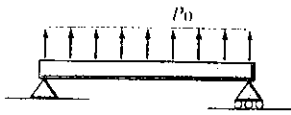
$$v' = \frac{Pb}{6LEI} (L^2 - b^2 - 3x^2) \quad 0 \leq x \leq a$$

$$\theta_A = \frac{Pab(L + b)}{6LEI}$$

$$\theta_B = \frac{Pab(L + a)}{6LEI}$$

$$\text{If } a \geq b, x_m = \sqrt{\frac{L^2 - b^2}{3}} \text{ and } \delta_{\max} = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EI}$$

5

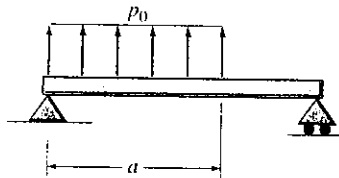


$$v = \frac{p_0 x}{24EI} (L^3 - 2Lx^2 + x^3)$$

$$v' = \frac{p_0}{24EI} (L^3 - 6Lx^2 + 4x^3)$$

$$\delta_C = \delta_{\max} = \frac{5p_0 L^4}{384EI} \quad \theta_A = \theta_B = \frac{p_0 L^3}{24EI}$$

6



$$v = \frac{p_0 x}{24LEI} (a^4 - 4a^3L + 4a^2L^2 + 2a^2x^2 - 4aLx^2 + Lx^3) \quad 0 \leq x \leq a$$

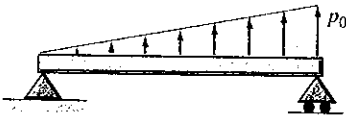
$$v' = \frac{p_0}{24LEI} (a^4 - 4a^3L + 4a^2L^2 + 6a^2x^2 - 12aLx^2 + 4Lx^3) \quad 0 \leq x \leq a$$

$$v = \frac{p_0 a^2}{24LEI} (-a^2L + 4L^2x + a^2x - 6Lx^2 + 2x^3) \quad a \leq x \leq L$$

$$v' = \frac{p_0 a^2}{24LEI} (4L^2 + a^2 - 12Lx + 6x^2) \quad a \leq x \leq L$$

$$\theta_A = \frac{p_0 a^2}{24LEI} (2L - a)^2 \quad \theta_B = \frac{p_0 a^2}{24LEI} (2L^2 - a^2)$$

7



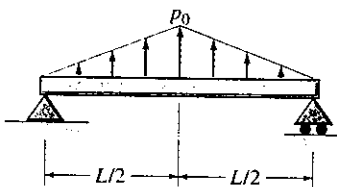
$$v = \frac{p_0 x}{360LEI} (7L^4 - 10L^2x^2 + 3x^4)$$

$$v' = \frac{p_0}{360LEI} (7L^4 - 30L^2x^2 + 15x^4)$$

$$\theta_A = \frac{7p_0 L^3}{360EI} \quad \theta_B = \frac{p_0 L^3}{45EI}$$

$$x_m = 0.5193 L \quad \delta_{\max} = 0.00652 \frac{p_0 L^4}{EI}$$

8

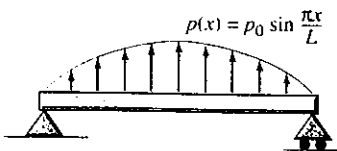


$$v = \frac{p_0 x}{960LEI} (5L^2 - 4x^2)^2 \quad 0 \leq x \leq \frac{L}{2}$$

$$v' = \frac{p_0}{192LEI} (5L^2 - 4x^2)(L^2 - 4x^2) \quad 0 \leq x \leq \frac{L}{2}$$

$$\delta_C = \delta_{\max} = \frac{p_0 L^4}{120LEI} \quad \theta_A = \theta_B = \frac{5p_0 L^3}{192EI}$$

9



$$v = \frac{p_0 L^4}{\pi^4 EI} \sin\left(\frac{\pi x}{L}\right)$$

$$v' = \frac{p_0 L^3}{\pi^3 EI} \cos\left(\frac{\pi x}{L}\right)$$

$$\delta_C = \delta_{\max} = \frac{p_0 L^4}{\pi^4 EI} \quad \theta_A = \theta_B = \frac{p_0 L^3}{\pi^3 EI}$$

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