OH: 5:30-6:30 M
8:30-9:30 W

Summary: stress distribution due to combined shear force and bending couple at cut

At a cut through a section of a beam experiencing both a shear force V and bending moment M, and abiding by the Euler-Bernoulli assumptions, we can make the following observations (see following figure):

a) Both normal stresses σ_x and shear stresses τ exist at the cut.

b) The normal stresses vary linearly in the *y*-direction as in the pure bending case. All previous observations about the normal stresses due to pure bending also apply in the case.

c) The shear stresses are approximately constant in the *z*-direction (into the depth of the beam) for "narrow beams", $t > 2h$.

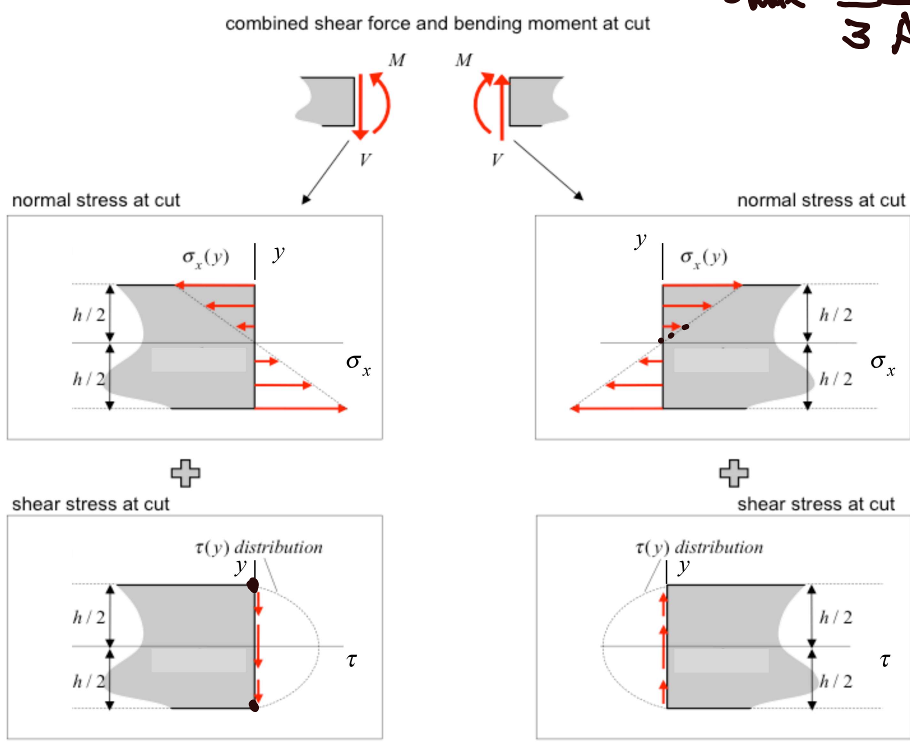
d) The shear stress is zero at the outer surfaces of the beam.

e) For rectangular cross-section beams, the shear stress distribution at a cut is parabolic in the y-direction:

$$\tau = \frac{6}{Ah^2}\left(\frac{h^2}{4} - y^2\right)V$$

where A is the area of the cross section. The maximum shear stress, $\tau_{max} = 3V/2A$, occurs at the neutral axis ($y = 0$).

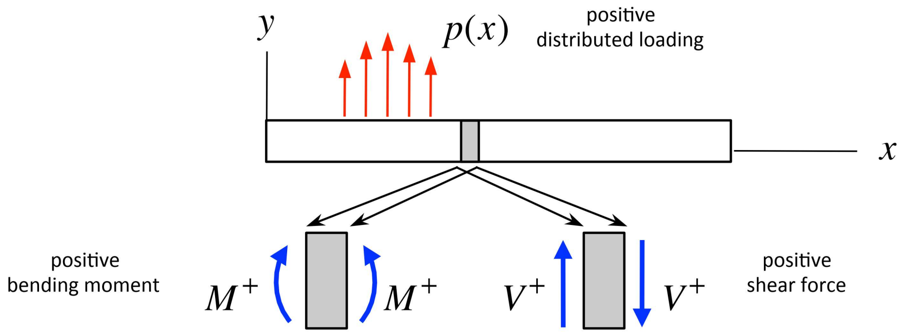
$\tau_{max} = \frac{4V}{3A}$ Circular.



$\tau = \frac{VA^*\bar{y}^*}{It}$

Graphical method for constructing shear force and bending moment diagrams

Sign conventions:



Basic relationships (as derived via equilibrium relations):

$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V_2 = V_1 + \int_{x_1}^{x_2} p(x)dx$$

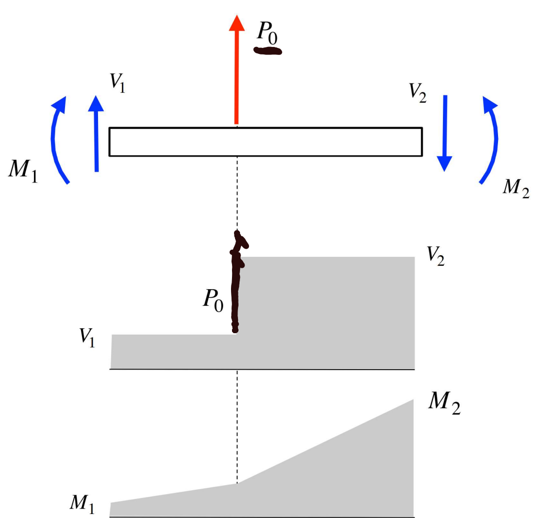
$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M_2 = M_1 + \int_{x_1}^{x_2} V(x)dx$$

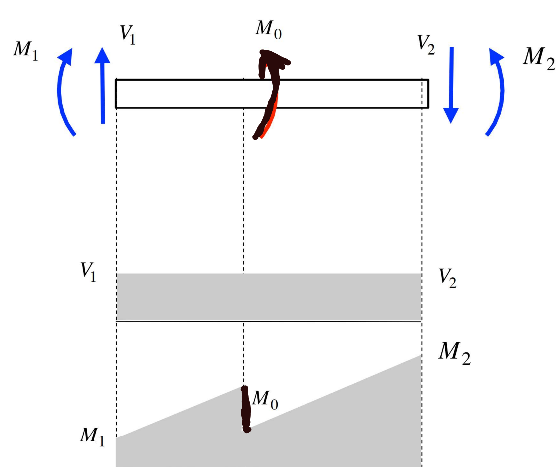
Concentrated shear force V_0 applied at location x:

$V(x^+) = V(x^-) + V_0$ (jump UP in shear force)

Concentrated moment M_0 applied at location x:

$M(x^+) = M(x^-) - M_0$ (jump DOWN in moment)





Deflections of statically-determinate beams – DEFINITE INTEGRAL APPROACH

Recall that for statically-determinate beams, we can determine the external reactions on the beam using the rigid body equilibrium equations. Assume that for a given determinate problem we have already determined these external reactions through equilibrium analysis. Using these, our goal is to determine the deflection of the beam over the full length of the beam.

To this end, we will now reconsider equations (1), (2), (5) and (7) above. We will integrate these equations over a given segment $x_1 < x < x_2$ of the beam. Note that the following results assume that the cross sectional and material properties are constant throughout a given segment.

Equation (1):

$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V(x) = V(x_1) + \int_{x_1}^{x} p(s)\,ds \tag{10}$$

Equation (2):

$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M(x) = M(x_1) + \int_{x_1}^{x} V(s)\,ds \tag{11}$$

Equation (7):

$$EI\frac{d\theta}{dx} = M(x) \qquad \Rightarrow \qquad \theta(x) = \theta(x_1) + \frac{1}{EI}\int_{x_1}^{x} M(s)\,ds \tag{12}$$

Equation (5):

$$\frac{dv}{dx} = \theta(x) \qquad \Rightarrow \qquad v(x) = v(x_1) + \int_{x_1}^{x} \theta(s)\,ds \tag{13}$$

p(x)

V(x)

M(x)

EI

θ(x)

v(x)

These results can be used two alternate ways for determining the deflection of a beam:

i) *Fourth-order approach* – Here we start with the loading $p(x)$ and perform the four integrations of (10)-(15) to obtain v(x).

ii) *Second-order approach* – Here we determine bending moment distribution $M(x)$ through FBDs and equilibrium analysis. With this $M(x)$, equations (12)-(13) are used to produce the deflection $v(x)$.

Note that with this *definite integral* approach, the boundary conditions such as $\theta(x_1)$ and $v(x_1)$ naturally appear in the solutions.

Summary: beam deflection – second-order integration method

The procedure to determine the deflection of a bending beam using the second-order integration method:

i) Before starting, write down the boundary conditions (BCs) for the problem.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions. If *DETERMINATE*, solve these equations for the external reactions.

iii) Divide beam into sections: $x_i < x < x_{i+1}$, where this section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).

iv) For each section, draw free an FBD of either the left of right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M(x)$ through that section of the beam.

v) Use the following integrals to determine slope and deflection of the beam over $x_i < x < x_{i+1}$:

$$\theta(x) = \theta(x_i) + \frac{1}{EI}\int_{x_i}^{x} M(x)\,dx$$

$$v(x) = v(x_i) + \int_{x_i}^{x} \theta(x)\,dx$$

vi) The final values of slope and displacement for one section $\theta(x_{i+1})$ and $v(x_{i+1})$ become the initial values of slope and displacement for the next section (continuity conditions).

vii) Enforce any remaining boundary conditions to determine any remaining integrations constants. For the case of *INDETERMINATE* beams, additional equations needed for determining external reactions are also produced through the enforcement of boundary conditions. These equations are solved with the equilibrium equations in ii) above.

$\Delta = \frac{\partial U}{\partial P}$

Deflection analysis – Castigliano's method

The procedure for deflection analysis using Castigliano's method:

i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). *Add in ALL of the needed dummy loads from the start*; this can save you a lot of time down the road.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.

- If *DETERMINATE*, solve these equations for the external reactions.
- If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of n redundant reactions (R_i ; $i = 1,2,...,N_R$). *Write the remaining reactions in terms of these N_R redundant reactions*.

iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).

iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI}\int_{x_i}^{x_{i+1}} M_i^2\,dx + \frac{f_s}{2GA}\int_{x_i}^{x_{i+1}} V_i^2\,dx + \frac{1}{2EA}\int_{x_i}^{x_{i+1}} F_{Ni}^2\,dx$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \ldots$. It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1,2,...,N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.

$$\boxed{\Delta_i = \frac{\partial U}{\partial P_i} \qquad 0 = \frac{\partial U}{\partial R_i} \qquad i = 1, \ldots R_i}$$

ME 323 Examination #2 | **Name** ______________________________
(Print) (Last) (First)

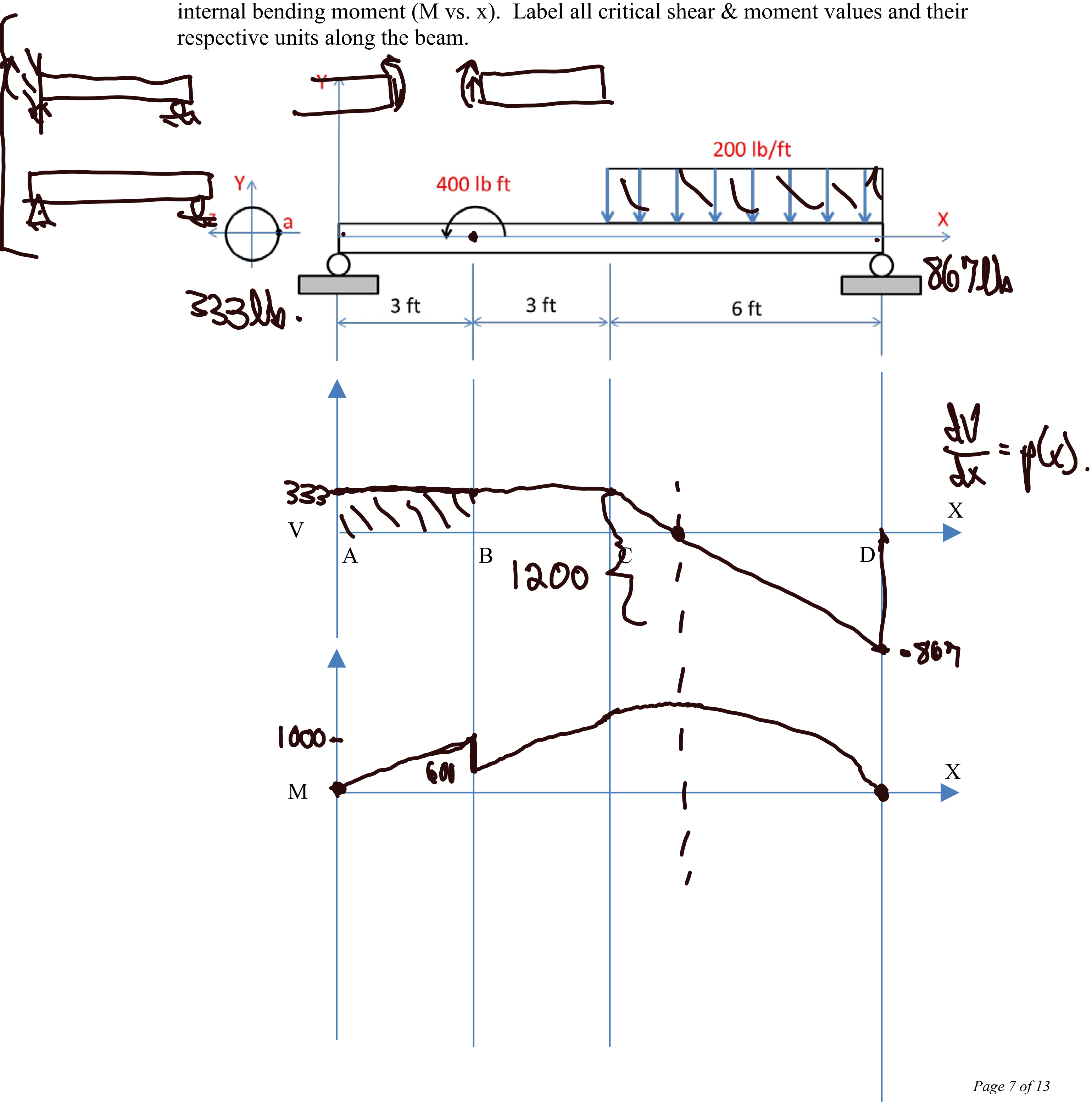
March 29, 2016 | **Instructor** ______________________________

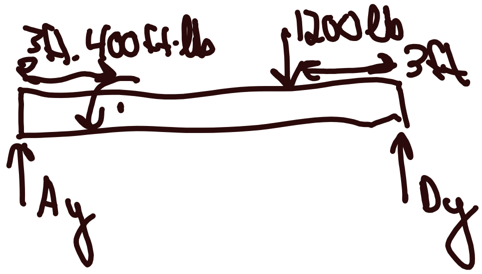
<u>PROBLEM #2 (22 points)</u>

(ΣF_y). (ΣM)

The beam is subjected to the loading condition as shown below.

1. On the following blank page, draw the Free-Body Diagram of the beam and determine the reactions acting on the beam.
2. On the axes shown below, construct to scale plots of the internal shear force (V vs. x), and the internal bending moment (M vs. x). Label all critical shear & moment values and their respective units along the beam.





$$\Sigma F_y = A_y + D_y - 1200 = 0$$

$$(\Sigma M)_A = 400 - 1200(9) + D_y(12) = 0$$

$$A_y = 333 \text{ lb}$$

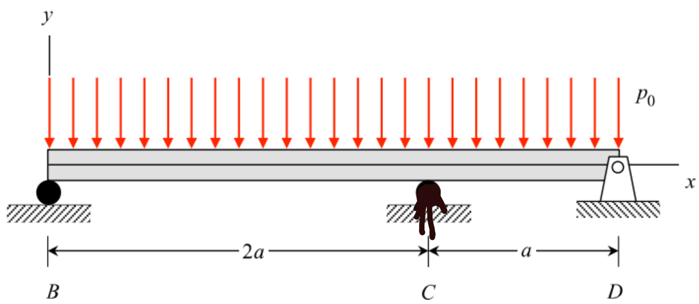
$$D_y = 867 \text{ lb.}$$

ME 323 Examination #2 **SOLUTION**

April 11, 2018

PROBLEM NO. 4 - PART A – 7 points max.

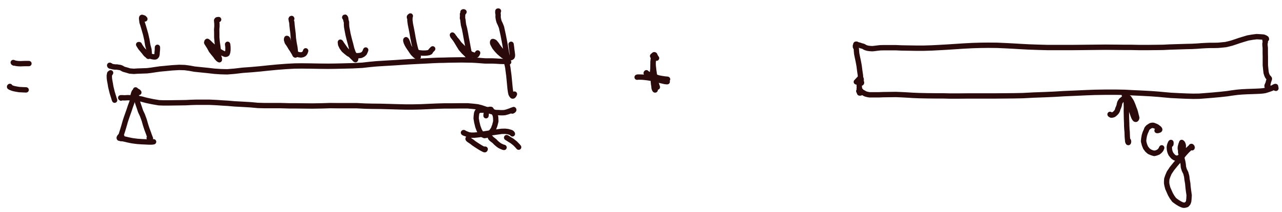
A beam is made up a material with a Young's modulus of E and has a constant cross section with a second area moment of I. A downward, constant line load p_0 (force/length) acts along the full length of the beam. The beam has roller supports at B and C, along with a pin joint support at end D. Using the *superposition* approach, determine the reaction force acting on the beam at the roller support C.



$v(0)=0$ ✓

$v(2a)=0$

$v(3a)=0$ ✓



$$0 = v(2a) = v_{p_0}(2a) + v_{C_y}(2a)$$

$$= -\frac{1}{24}(2a)\left[(3a)^3 - 2(3a)(2a)^2 + (2a)^3\right]\frac{p_0}{EI}$$

$$+ \frac{1}{6}(a)(2a)\left[(3a)^2 - a^2 - (2a)^2\right]\frac{C_y}{(3a)EI}$$

ME 323 Examination #2 **Name** ______________________________

November 14, 2017 **Instructor** ______________________________

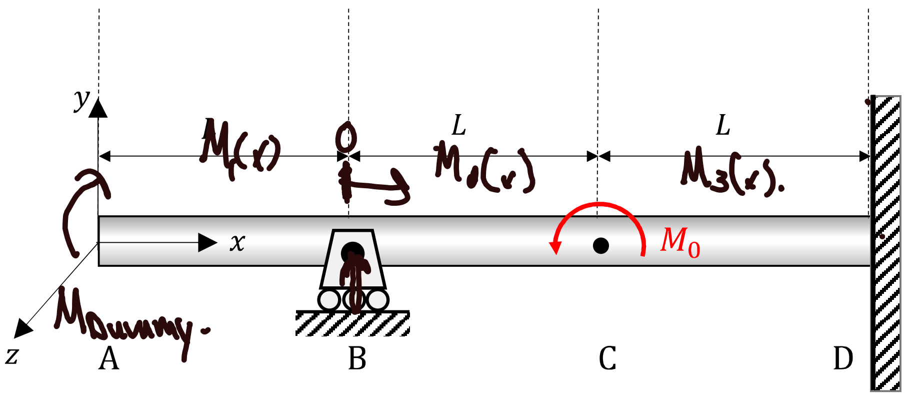
PROBLEM NO. 1 – 30 points max.

The cantilever beam AD of the bending stiffness EI is subjected to a concentrated moment M_0 at C. The beam is also supported by a roller at B. Using Castigliano's theorem:

a) Determine the reaction force at the roller B.

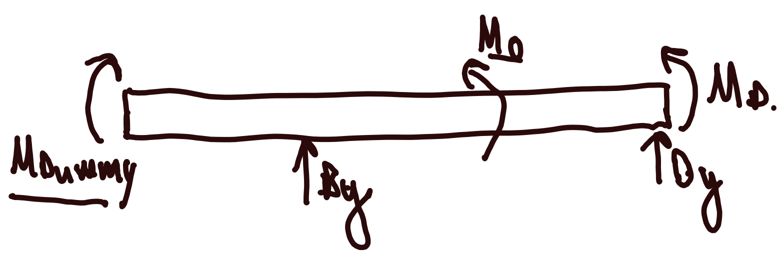
b) Determine the rotation angle of the beam about z axis at the end A.

Ignore the shear energy due to bending. Express your answers in terms of M_0, E, and I.



(i) Dummy?.

(ii) FBD



ΣF_y

ΣM

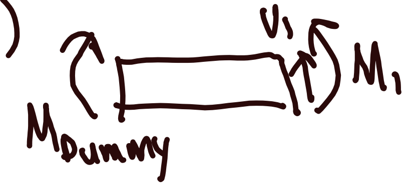
3 unknowns

− 2 equations

1 redundant. (B_y)

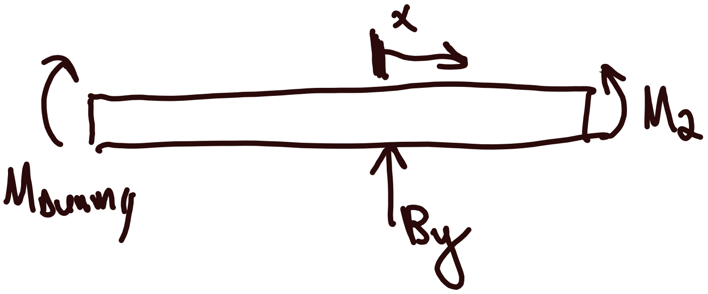
(iii) Sections?. 3.

(iv)



$$\Sigma M = -M_{Dummy} + M_1 = 0$$

$$M_1(x) = M_{Dummy}.$$



$$(\Sigma M)_2 = -M_{Dummy} - B_y x + M_2 = 0$$

$$M_2(x) = M_{Dummy} + B_y x$$

(v) $U_{total} = U_1 + U_2 + U_2.$

$$U_{total} = \frac{1}{2EI}\int_{-L}^{0} M_1^2\,dx + \frac{1}{2EI}\int_{0}^{L} M_2^2\,dx + \frac{1}{2EI}\int_{L}^{2L} M_3^2\,dx.$$

$$\left[\frac{\partial U}{\partial B_y}\right]_{M_{Dummy}=0} = 0$$

Solve for angle:

$$\theta_A = \left[\frac{\partial U}{\partial M_{Dummy}}\right]_{M_{Dummy}=0} \qquad \frac{\partial M_2}{\partial M_{Dummy}} = 1$$

Summary

The strain energy functions for the three types of members investigated here (axially-loaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$U = \frac{1}{2}\int_0^L \frac{F^2 dx}{EA} = \frac{1}{2}\frac{F^2 L}{EA}.$	$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U = \frac{1}{2}\int_0^L \frac{T^2}{GI_p} dx = \frac{1}{2}\frac{T^2 L}{GI_p}.$	$U = \frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$U_\sigma = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx$	$U_\sigma \frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$U_\tau = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.

PROBLEM #2 (25 points)

The beam BCD is fixed to the wall at B and supported by a roller at C. An external moment M_0 is applied at C. The beam has Young's modulus E and second moment of area I.

a) Draw a free-body diagram of the entire beam, and write down the equilibrium equations.

b) Use the second-order (or fourth-order) integration method to find the slope $v'(x)$ and deflection $v(x)$ of each segment of the beam. These can be left in terms of the unknown support reactions.

c) Write down the relevant boundary conditions and continuity conditions for the beam.

d) Use the boundary/continuity conditions to determine the reactions at B and C in terms of M_0 and L.

e) Determine the deflection at the free end D. Sketch the deflection curve over the length of the beam. The sketch does not need to be exact; show enough detail to indicate the boundary conditions.

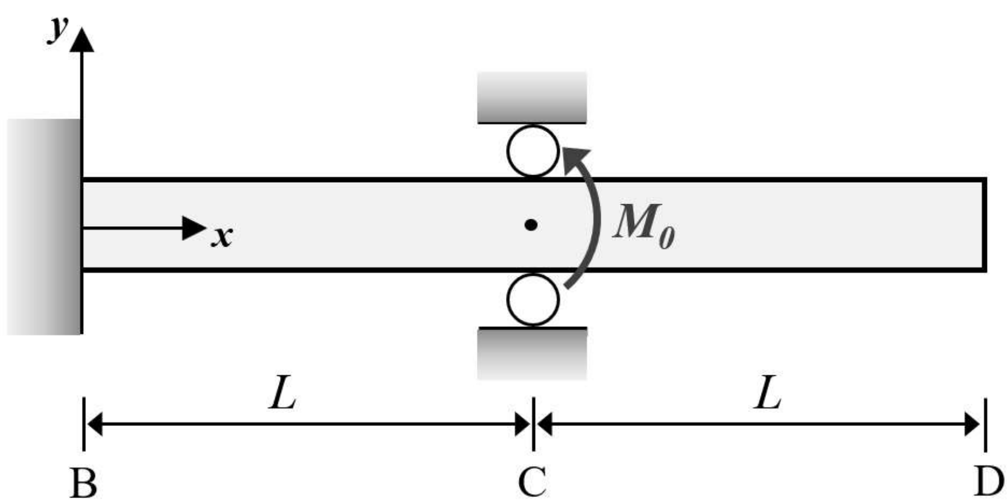


Figure 2

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<u>PROBLEM #4 (25 Points)</u>:

PART A – 4 points

Figure 4A shows a beam that is subjected to point load at multiple locations. The beam has a T-shaped cross section as shown in Figure 4B. Circle the correct answer for the following questions:

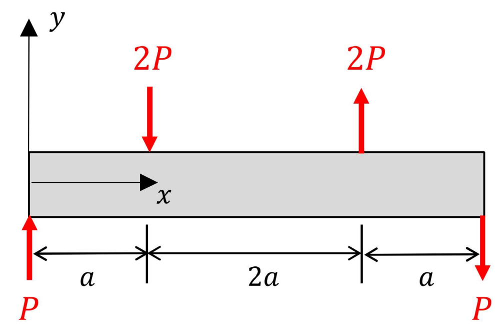


Figure 4A

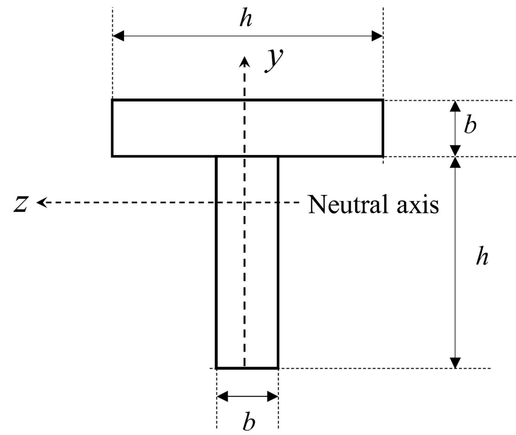


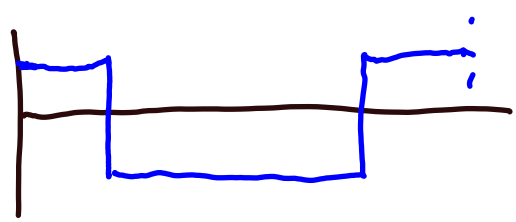
Figure 4B

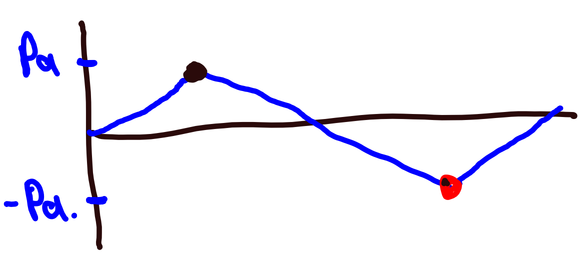
a) On which cross section the maximum tensile stress is attained?
   (1) $x = 0$
   $\boxed{(2)\ x = a}$
   (3) $x = 2a$
   (4) $x = 3a$
   (5) $x = 4a$

b) On which cross section the maximum compressive stress is attained?
   (1) $x = 0$
   (2) $x = a$
   (3) $x = 2a$
   $\boxed{(4)\ x = 3a}$
   (5) $x = 4a$

$\sigma = -\frac{My}{I}$

V

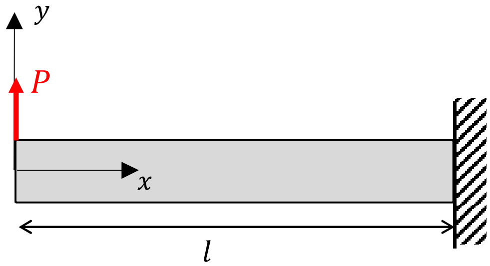




PROBLEM #4 (cont.):

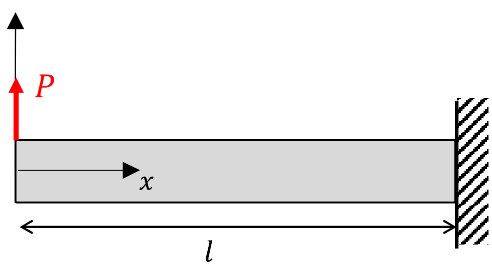
PART B – 9 points

Beam (i) and (ii) shown below are identical, except that beam (i) is made of steel, and beam (ii) is made of aluminum. Note that $E_{\text{steel}} > E_{\text{aluminum}}$.



Beam (i) – steel

$I = \frac{bh^3}{12}.$



Beam (ii) – aluminum

$\sigma = -\frac{My}{I}.$

a) **TRUE** or **FALSE**: The two beams have the same second moment of area.

b) **TRUE** or **FALSE**: The two beams have the same magnitude of the maximum normal stress.

c) **TRUE** or **FALSE**: The two beams have the same magnitude of the maximum shear stress.

d) **TRUE** or **FALSE**: The two beams have the same magnitude of the maximum deflection. EI

e) Let v_{max} be the maximum deflection in beam (i). If the length of beam (i) increases from its original value l to a new value $2l$, and the same load is applied at the free end. The new value of the maximum deflection becomes v^*_{max}. Circle the correct answer:

(1) $v^*_{max} = v_{max}$.

(2) $v^*_{max} = 2v_{max}$.

(3) $v^*_{max} = 4v_{max}$.

(4) $v^*_{max} = 8v_{max}$.

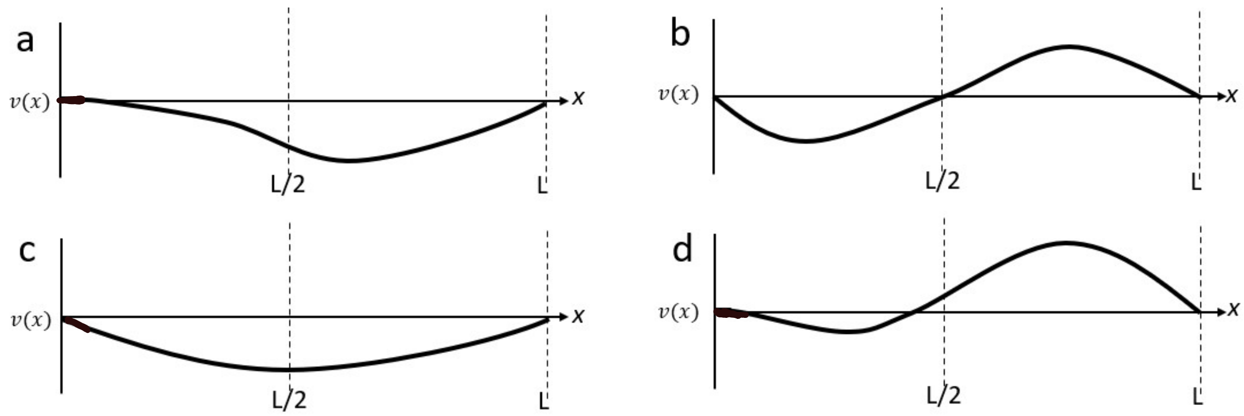
(5) $v^*_{max} = 16v_{max}$.

$\Delta = \frac{PL^3}{3EI}$ $\quad 2^3 = 8.$

 Name (Print)________________________________

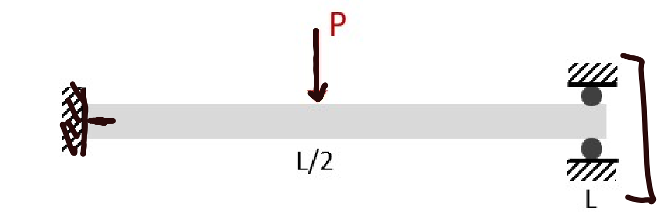
PROBLEM 4 – PART B (6 points)

Figures a-d indicate the deflection curve along four different beams.



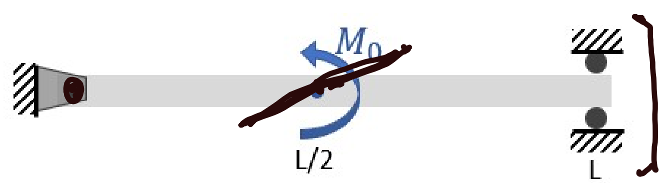
(i) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a   b   c   d



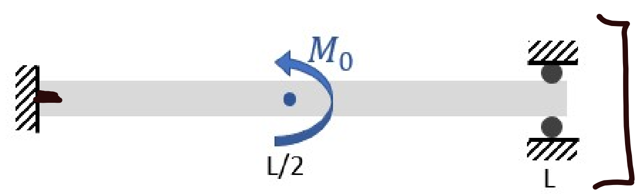
(ii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a   b   c   d



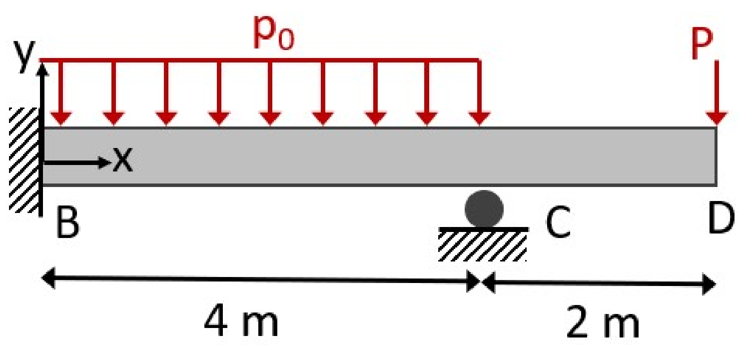
(iii) Circle the deflection curve that corresponds to the given beam and loading conditions (**2 points**):

a   b   c   d



PROBLEM 4 – PART D (6 points)

A beam is loaded with a distributed load from 0 to 4 m and a point load at 6 m.



Circle the value(s) that will be zero at x = 0m (**2 points**):

V(0) M(0) θ(0) v(0)

Circle the value(s) that will be zero at x = 4m (**2 points**):

V(4) M(4) θ(4) v(4)

Circle the value(s) that will be zero at x = 6m (**2 points**):

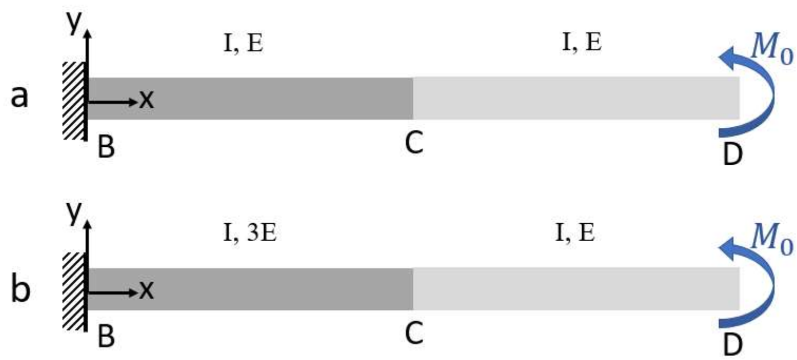
V(6) M(6) θ(6) v(6)

Exam 2

March 29, 2023 Name (Print)____________________________________

PROBLEM 4 – PART E (5 points)

A simple cantilever is composed of two sections with an applied moment at the end.



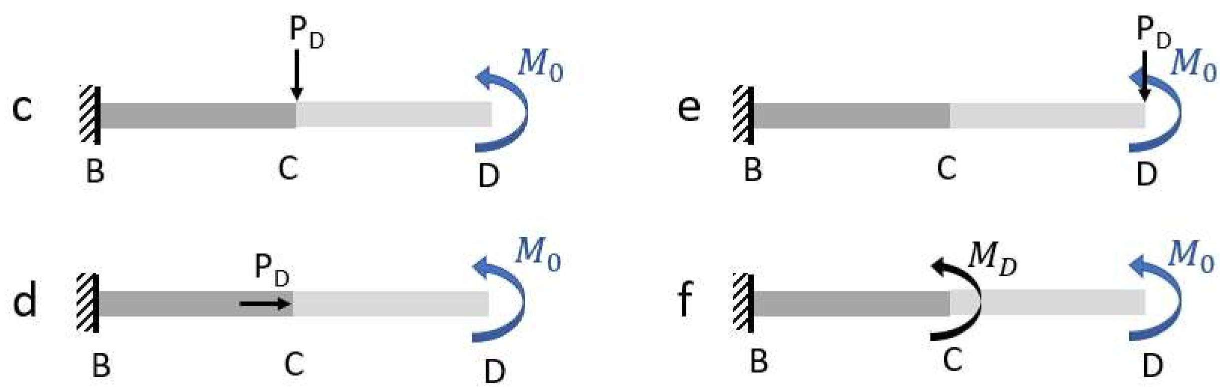
(i) **(3 points)** In beam (a), the two sections both have the same Young's moduli of E. In beam (b), one of the sections has a Young's modulus of 3E, while one has a Young's modulus of E. How does the total strain energy of these two beams compare?:

$U_{total,a} > U_{total,b}$

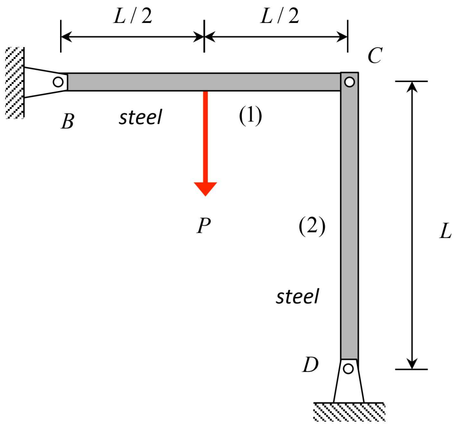
$U_{total,a} = U_{total,b}$

$U_{total,a} < U_{total,b}$

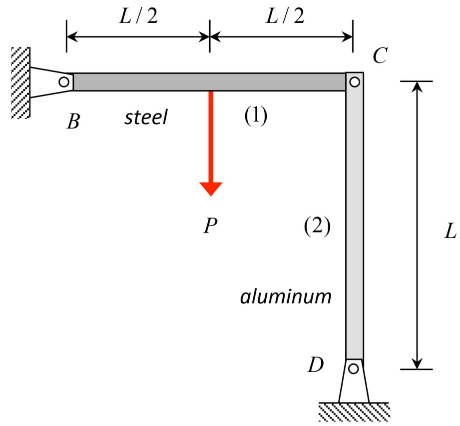
(ii) **(2 points)** Circle the loading condition below (c to f) that would be used if we want to calculate the deflection at point C in the y-direction.



Conceptual question 11.1



Structure (a)



Structure (b)

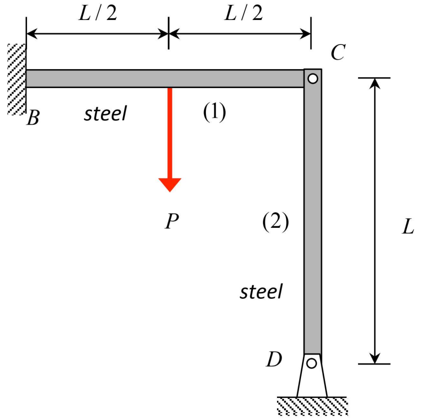
Structures (a) and (b) are identical, except that member (2) in Structure (a) is made of steel and member (2) in Structure (b) is made of aluminum. Let $(\sigma_2)_a$ and $(\sigma_2)_b$ represent the axial stresses in member (2) of Structures (a) and (b), respectively, due to the load P acting on member (1). Circle the item below that describes the relative sizes of these stresses:

i) $\left|(\sigma_2)_a\right| = \left|(\sigma_2)_b\right|$

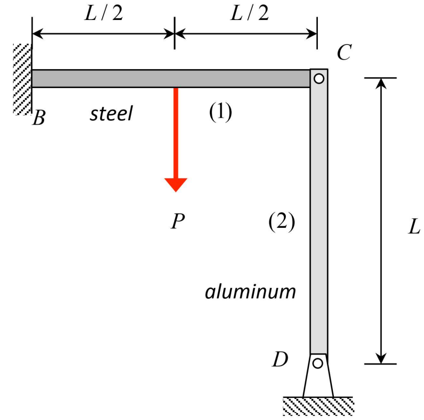
ii) $\left|(\sigma_2)_a\right| \neq \left|(\sigma_2)_b\right|$

iii) more information about the structures is needed in order to answer this question

Conceptual question 11.2



Structure (a)



Structure (b)

Structures (a) and (b) are identical, except that member (2) in Structure (a) is made of steel and member (2) in Structure (b) is made of aluminum. Let $(\sigma_2)_a$ and $(\sigma_2)_b$ represent the axial stresses in member (2) of Structures (a) and (b), respectively, due to the load P acting on member (a). Circle the item below that describes the relative sizes of these stresses:

i) $|(\sigma_2)_a| = |(\sigma_2)_b|$

ii) $|(\sigma_2)_a| \neq |(\sigma_2)_b|$

iii) more information about the structures is needed in order to answer this question

ME 323 Examination # 2

Name ________________________________
(Print) (Last) (First)

Circle Instructor Koslowski Gonzalez

PROBLEM # 1 (25 points)

Using Castigliano's theorem determine the vertical displacement at **C** for the member shown in the figure. A moment M is applied at C. The cross section of the member is square and constant. Use A, I, and E constant along the member ABC. Ignore shear stress in the energy calculation.