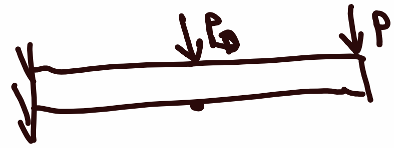


$\Delta = \frac{\partial U}{\partial P}$

Deflection analysis – Castigliano's method

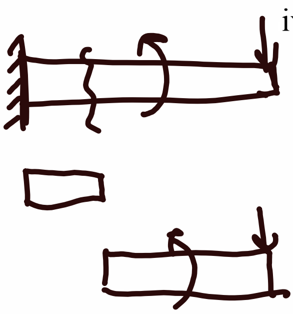
The procedure for deflection analysis using Castigliano's method:

i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). *Add in ALL of the needed dummy loads from the start*; this can save you a lot of time down the road.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.

- If *DETERMINATE*, solve these equations for the external reactions.
- If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of n redundant reactions (R_i ; $i = 1, 2, ..., N_R$). *Write the remaining reactions in terms of these N_R redundant reactions.*

iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).



iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through — $T(x)$ that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI}\int_{x_i}^{x_{i+1}} M_i^2\,dx + \frac{f_s}{2GA}\int_{x_i}^{x_{i+1}} V_i^2\,dx + \frac{1}{2EA}\int_{x_i}^{x_{i+1}} F_{Ni}^2\,dx + \frac{1}{2GI_p}\int_{x_i}^{x_{i+1}} T_i^2\,dx.$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \ldots$. It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$\boxed{0 = \frac{\partial U}{\partial R_i}}\ ; \quad i = 1, 2, ..., N_R$$

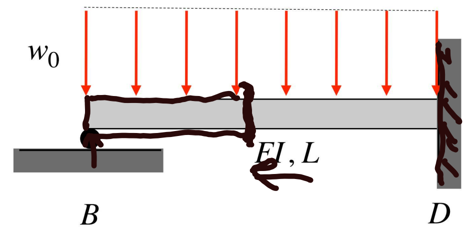
U=f(redundant reactions, known)

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.

Example 16.7

Determine the reaction at end B of the beam shown.



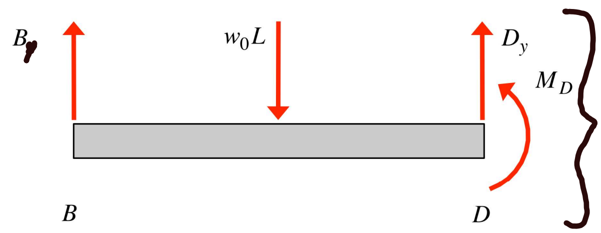
BC

$V_B = 0$

$V_D = 0$

$\theta_D = 0$

Equilibrium – FBD of entire beam



From here, we see that the problem is statically indeterminate: 3 unknowns (B_y , D_y and M_D) and only two equations. Here, we will choose B_y to be our redundant reaction:

$$\sum F_y = B_y - w_0 L + D_y = 0 \;\Rightarrow\; D_y = w_0 L - B_y$$

$$\sum M_B = -(w_0 L)\left(\frac{L}{2}\right) + D_y L + M_D = 0 \;\Rightarrow\; M_D = -\left(w_0 L - B_y\right)L + \frac{1}{4} w_0 L^2$$

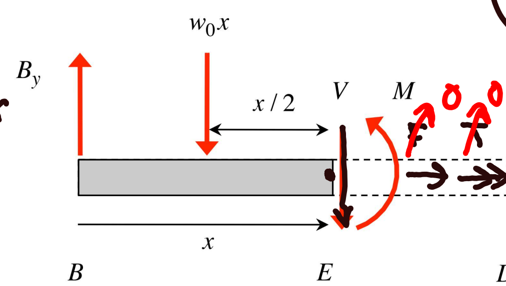
3 unknowns

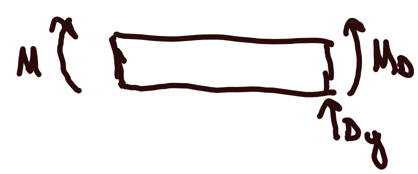
2 eqns

1.

Determining internal bending moment

Ignore shear





$M(x) = f(D_y, M_D)$

$$\sum M_E = M - B_y x + (w_0 x)\left(\frac{x}{2}\right) = 0 \;\Rightarrow\; M(x) = -\frac{w_0 x^2}{2} + B_y x$$

Strain energy in beam (ignoring contributions from shear stress/strain)

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}dx = \frac{1}{2EI}\int_0^L \left(-\frac{w_0 x^2}{2} + B_y x\right)^2 dx$$

$$\frac{\partial}{\partial P}\int M^2 dx = 2\int M\frac{\partial M}{\partial P}dx$$

<u>*Castigliano's theorem*</u>

With B_y being our choice for the redundant reaction:

Enforcing B.C.

$$\boxed{0=\frac{\partial U}{\partial B_y}}=\frac{1}{2EI}\int_0^L 2\left(-\frac{w_0x^2}{2}+B_yx\right)(x)\,dx=\frac{1}{EI}\int_0^L\left(-\frac{w_0x^3}{2}+B_yx^2\right)dx$$

$$0=\frac{1}{EI}\left[-\frac{w_0x^4}{8}+\frac{B_yx^3}{3}\right]_{x=0}^{x=L}=\underbrace{\frac{1}{EI}\left[-\frac{w_0L^4}{8}+\frac{B_yL^3}{3}\right]}\;\Rightarrow\; B_y=\frac{3}{8}w_0L$$

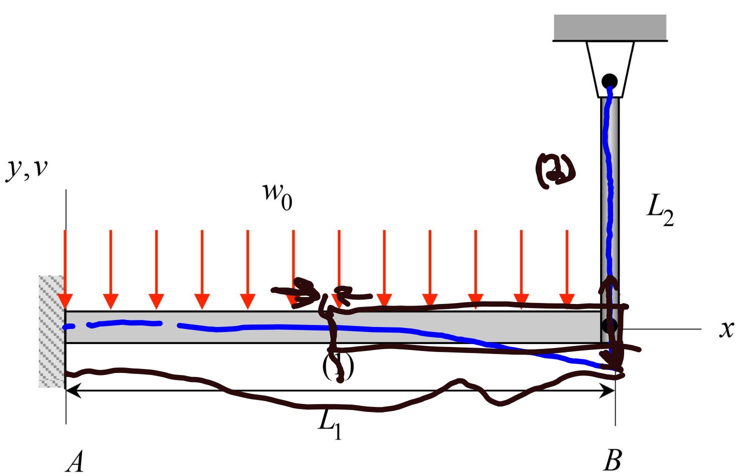
Example 16.8

For the following examples, ***set up*** the problem for determining the requested deflections using Castigliano's method:

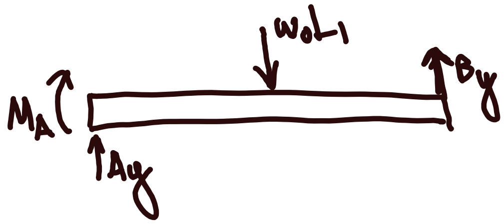
- draw appropriate FBDs;
- determine internal results for each section;
- set up the integrals for calculating the required deflections;
- explain how Castigliano's method is used to solve. Discuss the application of dummy forces (when needed) and how to handle redundant forces for indeterminate structures.

Problem A

Find the load carried by member (2) of the structure below. Let E and A be the Young's modulus and cross-sectional area, respectively, of member (2), whereas E and I are the Young's modulus and second area moment of the cross section of (1), respectively.



$\Delta = -e$



$(\Sigma M)_A = -M_A - \frac{1}{2} w_0 L_1^2 + B_y L_1 = 0$

$\Sigma F_y = A_y - w_0 L_1 + B_y = 0$

3 unknowns
2 eqns
1

$$\Sigma F = V_1 + B_y - w_0 x = 0$$

$$\underline{V_1} = -B_y + w_0 x$$

$$(\Sigma M)_1 = -M_1 + B_y x - w_0 x \left(\frac{x}{2}\right) = 0$$

$$\underline{M_1(x)} = B_y x - w_0 \frac{x^2}{2}$$

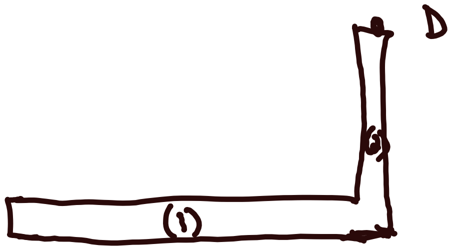
$$U = \frac{1}{2EI}\int_0^{L_1} M_1^2\,dx + \frac{f_s}{2GA}\int_0^{L_1} V_1^2\,dx.$$

$$-\underline{e} = \Delta = \frac{\partial U}{\partial B_y} = \frac{1}{EI}\int_0^{L_1} M_1 \frac{\partial M_1}{\partial B_y}\,dx + \frac{f_s}{GA}\int_0^{L_1} V_1 \frac{\partial V_1}{\partial B_y}\,dx.$$

$$\frac{\partial M_1}{\partial B_y} = x \qquad \frac{\partial V_1}{\partial B_y} = -1$$

Super important
Only in terms of
redundant load.

$$\underbrace{-\frac{B_y L_2}{EA}} = \frac{\partial U}{\partial B_y}.$$

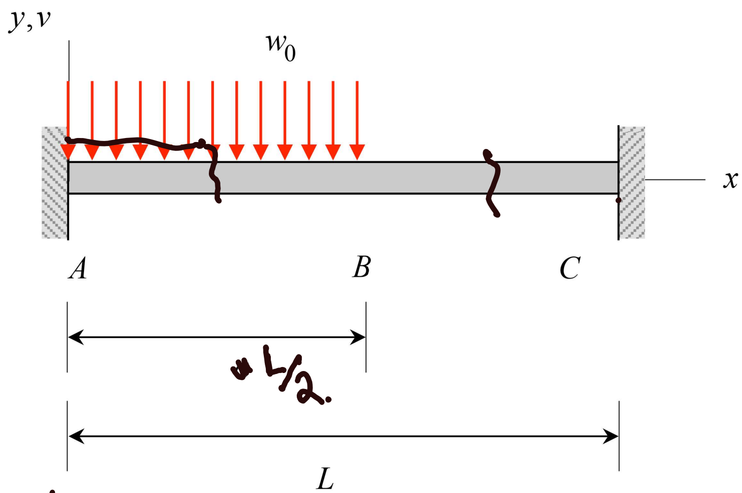


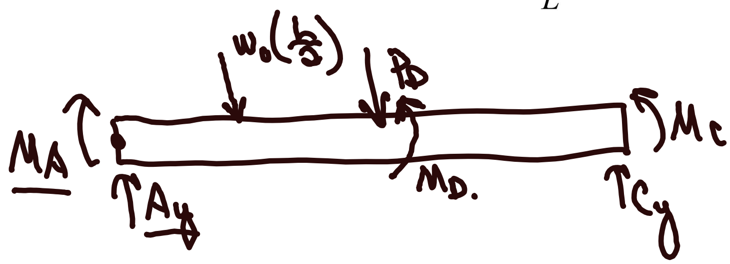
$$U = \underline{U_1} + \underline{U_2}. \qquad U_2 = \frac{F^2 L}{EA}.$$

$$0 = \frac{\partial U}{\partial D_y} =$$

Problem B

Find the vertical deflection of the beam at point B and the angle of rotation of the beam at B. Let E and I be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.



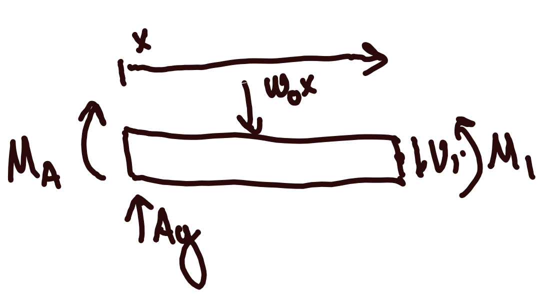


(1) $(\Sigma M)_A = -M_A + M_C - w_0\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - P_D\left(\frac{L}{2}\right) + M_D + C_y L = 0$

(2) $\Sigma F_y = A_y + C_y - w_0\left(\frac{L}{2}\right) - P_D = 0$

4 unknowns
2 eqns
2.

Section AB



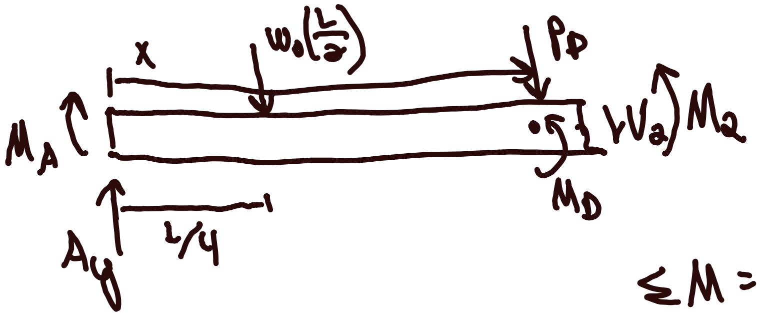
$$\Sigma F_y = A_y - w_0 x - V_1 = 0$$

$$V_1 = -A_y + w_0 x$$

$$\Sigma M = -M_A + M_1 - A_y x + w_0 x\left(\frac{x}{2}\right) = 0$$

$$M_1(x) = M_A + A_y x - w_0 \frac{x^2}{2}.$$

Section BC



$$\Sigma F_y = A_y - w_0\left(\frac{L}{2}\right) - P_D - V_2 = 0$$

$$V_2 = A_y - w_0\left(\frac{L}{2}\right) - P_D$$

$$\Sigma M = -M_A + M_2 + M_D - A_y x + w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right) + P_D\left(x - \frac{L}{2}\right)$$

$$M_2(x) = M_A - M_D + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right) - P_D\left(x - \frac{L}{2}\right)$$

$$\frac{\partial U}{\partial P} \qquad \frac{\partial}{\partial P}\int M^2\,dx \qquad \int M\frac{\partial M}{\partial P}\,dx.$$

$U = U_1 + U_2$

Ignore shear.

$$U = \frac{1}{2EI}\int_0^{L/2} M_1^2\,dx + \frac{1}{2EI}\int_{L/2}^{L} M_2^2\,dx.$$

$$\rightarrow \left[\frac{\partial U}{\partial A_y}\right]_{P_D, M_D = 0} = 0 \qquad \frac{\partial M_1}{\partial A_y} = x \qquad \frac{\partial M_2}{\partial A_y} = x.$$

$$0 = \frac{1}{EI}\int_0^{L/2}\left(M_A + A_y x - w_0\frac{x^2}{2}\right)x\,dx + \frac{1}{EI}\int_{L/2}^{L}\left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]x\,dx. \quad (3)$$

$$\rightarrow \left[\frac{\partial U}{\partial M_A}\right]_{P_D, M_D = 0} = 0 \qquad \frac{\partial M_1}{\partial M_A} = 1 \qquad \frac{\partial M_2}{\partial M_A} = 1$$

$$0 = \frac{1}{EI}\int_0^{L/2}\left(M_A + A_y x - w_0\frac{x^2}{2}\right)dx + \frac{1}{EI}\int_{L/2}^{L}\left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]dx \quad (4)$$

(1-4) solve for reactions.

$$\left[\frac{\partial U}{\partial P_D}\right]_{P_D, M_D = 0} \qquad \frac{\partial M_1}{\partial P_D} = 0 \qquad \frac{\partial M_2}{\partial P_D} = -\left(x - \frac{L}{2}\right)$$

$$\left[\frac{\partial U}{\partial P_D}\right]_{P_D, M_D = 0} = 0 + \frac{1}{EI}\int_{L/2}^{L} \left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]\left[-\left(x - \frac{L}{2}\right)\right] dx.$$

$$\left[\frac{\partial U}{\partial M_D}\right]_{P_D, M_D = 0} = \theta_B$$