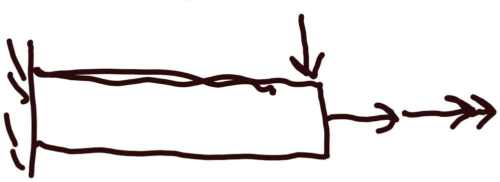
Summary

The strain energy functions for the three types of members investigated here (axially-loaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$U = \frac{1}{2}\int_0^L \frac{F^2 dx}{EA}$	$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U = \frac{1}{2}\int_0^L \frac{T^2}{GI_p} dx$	$U = \frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$U_\sigma = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx$	$U_\sigma \frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$U_\tau = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.



$U_{tot} = U_{axial} + U_{torsion} + U_M + U_V$

Castigliano's

$\Delta = \frac{\partial U}{\partial P} \qquad \phi = \frac{\partial U}{\partial T} \qquad \theta = \frac{\partial U}{\partial M}$

Example 16.3

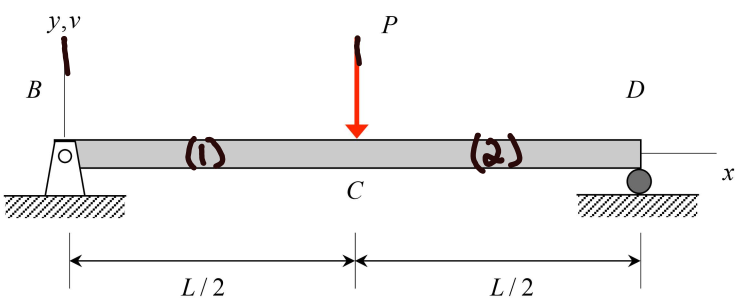
A load P is applied at the midspan of the beam of length L. The beam has a rectangular cross section, with the cross-sectional dimensions shown. The beam is made up of a material with a Young's modulus E and Poisson's ratio of ν.

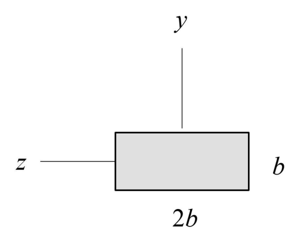
a) Determine the strain energy stored in the beam in terms of the load P and the work done by the applied load P under static equilibrium conditions.

b) ~~Write down the work-energy equation for the system under static equilibrium~~ conditions. Use ~~the *work-energy method* to determine the static deflection of ppoint C~~ of the beam.

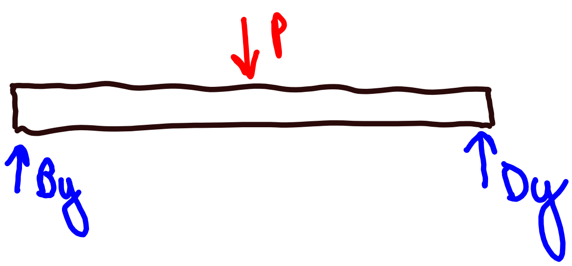
c) Use *Castigliano's theorem* to determine the static deflection of C.

d) Identify the contributions to your solution in c) above that come from flexure stresses and those contributions that come from shear stresses. Compare the sizes of these contributions for $b/L = 0.05$ and $\nu = 0.4$. Are the contributions from shear effects signficant?





beam cross section



$$\left.\begin{matrix}\Sigma F_y \\ \Sigma M\end{matrix}\right\} \quad B_y = D_y = \frac{P}{2}$$

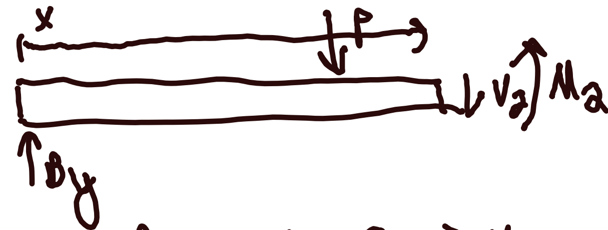
$$U = U_{(1)} + U_{(2)} \qquad U_{(1)} = \cancel{U_F}^{0} + \cancel{U_T}^{0} + U_M + U_V$$

Section BC

$$\Sigma F_y = 0 = B_y - V_1 = 0 \Rightarrow B_y = \frac{P}{2} = V_1$$

$$(\Sigma M)_1 = -B_y x + M_1 = 0 \Rightarrow M_1 = \frac{P}{2}x$$

Section CD



$$\Sigma F_y = \frac{P}{2} - P - V_2 = 0 \Rightarrow V_2 = -\frac{P}{2}$$

$$\Sigma M = -B_y x + P\left(x - \frac{L}{2}\right) + M_2 = 0$$

$$M_2 = (B_y - P)x + P\left(\frac{L}{2}\right) = \frac{P}{2}(L-x)$$

$$U = U_{1,M} + U_{1,V} + U_{2,M} + U_{2,V}$$

$$U = \frac{1}{2EI}\left[\int_0^{L/2} M_{1,M}^2\,dx + \int_{L/2}^{L} M_{2,M}^2\,dx\right] + \frac{f_s}{2GA}\left[\int_0^{L/2} V_1^2\,dx + \int_{L/2}^{L} V_2^2\,dx\right]$$

$$U = \frac{1}{2EI}\left[\int_0^{L/2}\left(\frac{P}{2}x\right)^2 dx + \int_{L/2}^{L}\left(\frac{P}{2}(L-x)\right)^2 dx\right] + \frac{f_s}{2GA}\left[\underbrace{\int_0^{L/2}\left(\frac{P}{2}\right)^2 dx} + \int_{L/2}^{L}\left(-\frac{P}{2}\right)^2 dx\right]$$

$$U = \frac{1}{EI}\frac{P^2L^3}{96} + \frac{1}{GA}\frac{P^2 L f_s}{8}$$

$$\left[\frac{P^2}{4}x\right]_0^{L/2} = \frac{P^2 L}{8}$$

$$\Delta_c = \frac{\partial U}{\partial P} = \underbrace{\frac{PL^3}{48EI}}_{\Delta_M} + \underbrace{\frac{PLf_s}{4GA}}_{\Delta_V}$$

d) $$\frac{(\Delta_c)_V}{(\Delta_c)_M} = \frac{\dfrac{PLf_s}{4GA}}{\dfrac{PL^3}{48EI}} = 12 f_s \left(\frac{E}{G}\right) \frac{I}{AL^2}$$

$$f_s = 6/5$$

$$E/G = 2(1+\overset{0.4.}{\nu}) = 2.8$$

$$\frac{I}{AL^2} = \frac{\frac{1}{12}(2b)(b^3)}{(2b)(b)L^2} = \frac{1}{12}\left(\frac{b^2}{L^2}\right) = \frac{1}{12}(0.05)^2$$

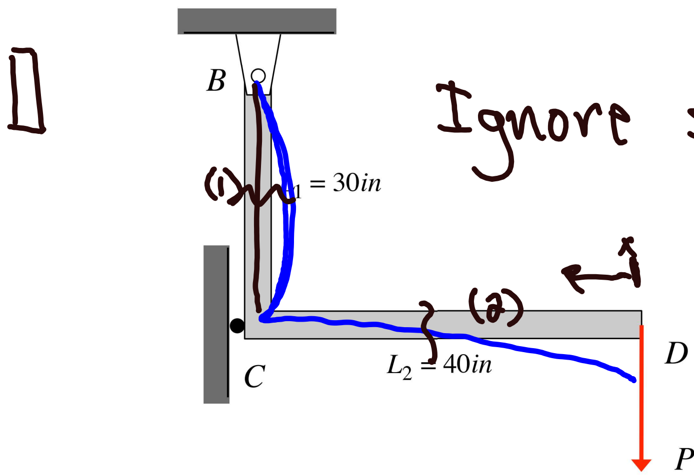
$$\frac{(\Delta_c)_V}{(\Delta_c)_M} = 12\left(\tfrac{6}{5}\right)(2.8)\left(\tfrac{1}{12}\right)(0.05)^2 = 0.0084$$

Deflection due to bending is ~120 x the deflection due to shear.

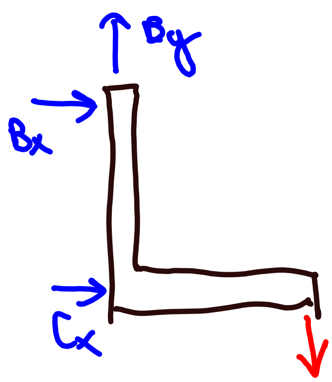
$\Rightarrow$ Many questions will tell you to neglect shear

Example 16.5

Determine the vertical deflection of point D ofnthe structural member shown. The cross section of the member is rectangular and constant throughout. Use $A = 2in^2$, $I = 1.3in^4$, $E = 30\times10^6\,psi$ and $\nu = 0.3$.



Ignore shear effects.



$$(\Sigma M)_B = C_x L_1 - P L_2 = 0$$

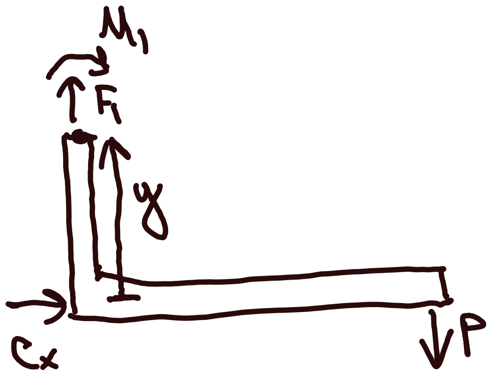
$$C_x = P\left(\frac{L_2}{L_1}\right)$$

$$\Sigma F_y = B_y - P = 0 \Rightarrow B_y = P$$

$$\Sigma F_x = B_x + C_x = 0 \Rightarrow B_x = -P\left(\frac{L_2}{L_1}\right)$$

3 unknowns
3 eqns

Section BC



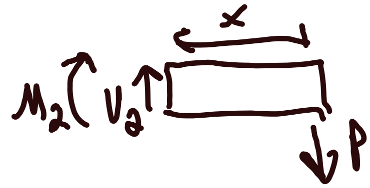
$$(\Sigma M)_1 = -M_1 + yC_x - PL_2$$

$$M_1(y) = \frac{L_2}{L_1}Py - L_2P$$

$$\Sigma F_y = F_1 - P = 0$$

$$F_1 = P$$

Section CD



$$(\Sigma M)_2 = -M_2 - Px = 0$$

$$M_2(x) = -Px.$$

$$V_2 = P$$

Neglect V

$$U = U_1 + U_2$$

$$U = \frac{1}{2EA}\int_0^{L_1} F_1^2\,dy + \frac{1}{2EI}\int_0^{L_1} M_1^2\,dy + \frac{1}{2EI}\int_0^{L_2} M_2^2\,dx.$$

$$\Delta = \frac{\partial U}{\partial P}$$

$$\boxed{\begin{array}{l}\text{Important point} \\ \dfrac{\partial}{\partial P}\displaystyle\int M_1^2\,dx = 2\int M_1 \frac{\partial M_1}{\partial P}\,dx.\end{array}}$$

$$\frac{\partial U}{\partial P} = \frac{1}{EA}\int_0^{L_1} F_1 \frac{\partial F_1}{\partial P}\,dy + \frac{1}{EI}\int_0^{L_1} M_1 \frac{\partial M_1}{\partial P}\,dy + \frac{1}{EI}\int_0^{L_2} M_2 \frac{\partial M_2}{\partial P}\,dx.$$

$$\frac{\partial M_1}{\partial P} = \frac{L_2}{L_1}y - L_2 \qquad \frac{\partial M_2}{\partial P} = -x \qquad \frac{\partial F_1}{\partial P} = 1$$

$$\Delta_D = \frac{PL_1}{EA} + \frac{P}{EI}\int_0^{L_1}\left(\frac{L_2}{L_1}y - L_2\right)^2 dy + \frac{1}{EI}\int_0^{L_2} Px^2\,dx.$$

$$\Delta_D = \frac{PL_1}{EA} + \frac{P}{EI}\left[\left(\frac{L_2}{L_1}\right)^2\left(\frac{L_1^3}{3}\right) - 2\left(\frac{L_2}{L_1}\right)L_2\left(\frac{L_1^2}{2}\right) + L_2^2 L_1\right] + \frac{P}{EI}\frac{L_2^3}{3}$$

$$\Delta_D = \frac{PL_1}{EA} + \frac{PL_2^2 L_1}{3EI} + \frac{PL_2^3}{3EI}$$

Deflection analysis – Castigliano's method

The procedure for deflection analysis using Castigliano's method:

i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). *Add in ALL of the needed dummy loads from the start*; this can save you a lot of time down the road.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.

- If *DETERMINATE*, solve these equations for the external reactions.
- If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of *n* redundant reactions (R_i ; $i = 1,2,...,N_R$). *Write the remaining reactions in terms of these N_R redundant reactions*.

iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).

iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI}\int_{x_i}^{x_{i+1}} M_i^2\,dx + \frac{f_s}{2GA}\int_{x_i}^{x_{i+1}} V_i^2\,dx + \frac{1}{2EA}\int_{x_i}^{x_{i+1}} F_{Ni}^2\,dx$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \ldots$. It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1,2,...,N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.