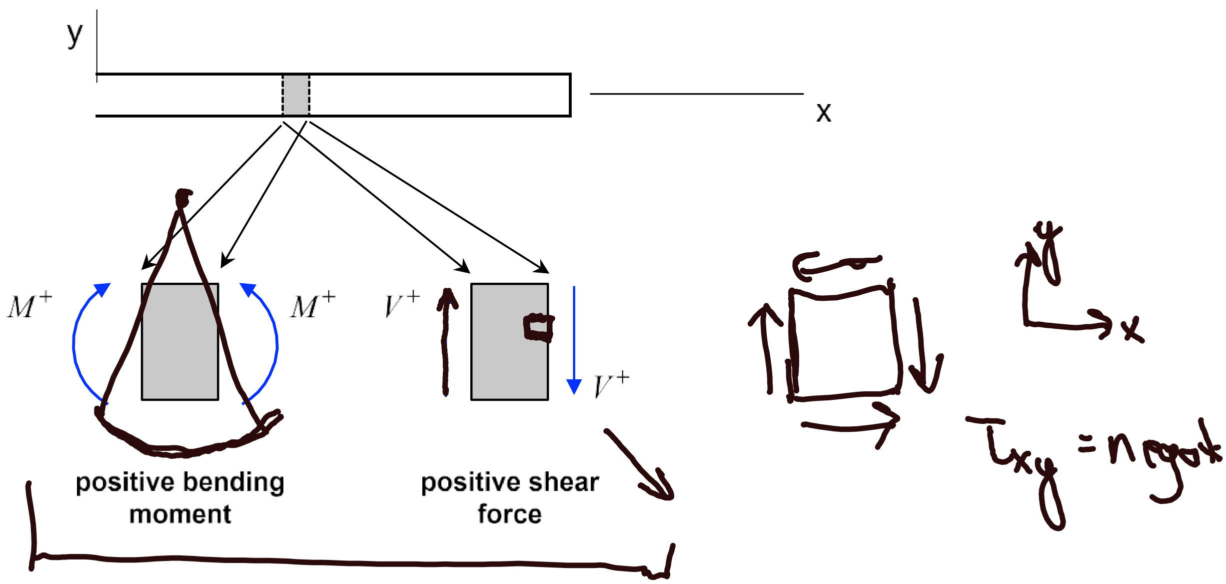
Lecture Notes

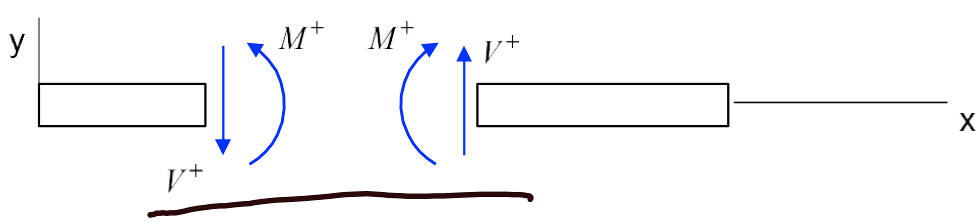
a) *Sign conventions for bending moments and shear forces*

Sign conventions to be used in this course for internal bending moments and shear forces (see following figure):

- A positive bending moment M on the left face (negative x-face) of a section is CW. A positive bending moment M on the right face (positive x-face) of a section is CCW. Such a positive bending moment creates a concave curvature in the deflection of the beam.
- A positive shear force V on the left face (negative x-face) of a section is in positive y-direction. A positive shear force V on the right face (positive x-face) of a section is in negative y-direction.



When making a cut through a cross section of the beam, the positive sign conventions for the bending moment and shear force are as shown below:

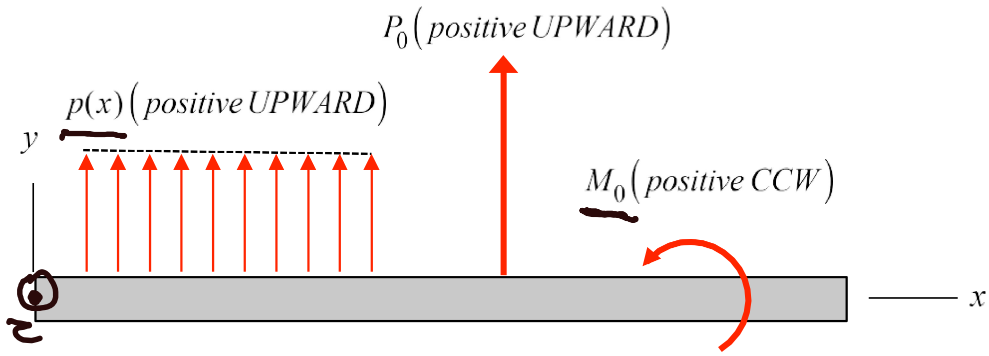


positive bending moment and shear force at cut in beam

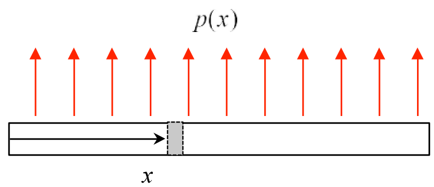
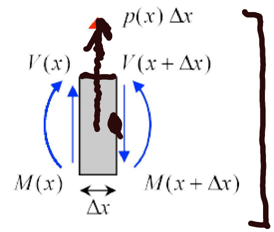
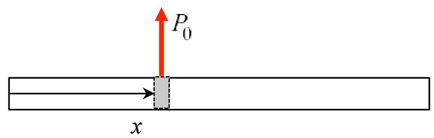
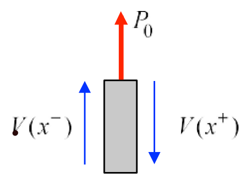
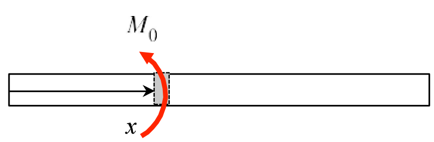
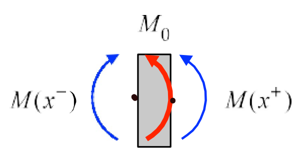
Sign conventions for *external* loadings on beams:

- Positive *EXTERNAL distributed loads* p(x) and *EXTERNAL concentrated loads* P_0 act in the "+" y-direction:

- Positive *EXTERNAL couples* are in the "+" z-direction (CCW by the right hand rule):



b) Equilibrium relations for bending moments and shear forces

applied loading	FBD	*key relationship(s)*
		$\dfrac{dV}{dx} = p(x)$ $\dfrac{dM}{dx} = V(x)$
	 ΣF_y	$V(x^+) = V(x^-) + P_0$
		$M(x^+) = M(x^-) - M_0$

The *derivations* of the above key relationships are to be added below:

$$\Sigma F_y = V(x) + p(x)\Delta x - V(x+\Delta x) = 0 \Rightarrow p(x) = \frac{V(x+\Delta x) - V(x)}{\Delta x}.$$

$$p(x) = \frac{dV}{dx}$$

$$(\Sigma M) = -M(x) + M(x+\Delta x) - V(x)\Delta x - p(x)\Delta x\left(\frac{\Delta x}{2}\right) = 0$$

$$V(x) = \frac{M(x+\Delta x) - M(x)}{\Delta x} + p(x)\Delta x. \Rightarrow V(x) = \frac{dM}{dx}.$$

Geometric meaning of the equilibrium relationships for beams

- $\frac{dV}{dx} = p(x)$

  The slope of the shear force diagram at any location x equals the value of the distributed external loading p at that location.

- $V(x_2) = V(x_1) + \int_{x_1}^{x_2} p(\xi)\,d\xi$ (integral form of the above)

  The shear force at point x_2 is equal to the shear force at x_1 plus the area under the external loading curve between these two points.

- $\frac{dM}{dx} = V(x)$

  The slope of the bending moment diagram at any location x equals the value of the shear force at that location.

- $M(x_2) = M(x_1) + \int_{x_1}^{x_2} V(\xi)\,d\xi$ (integral form of the above)

  The bending moment at point x_2 is equal to the bending moment at x_1 plus the area under the external loading curve between these two points.

- $V(x^+) = V(x^-) + P_0$

  The shear force diagram has an *upward* step jump at location x where an external point force is applied. The value of the shear force jump *increase* equals the value of the external point force.

- $M(x^+) = M(x^-) - M_0$

  The bending moment diagram has a *downward* step jump at location x where an external point moment is applied. The value of the bending moment jump *decrease* equals value of the external point moment.

$$\begin{array}{ccc} & p(x) & \\ \text{integ} \downarrow & & \uparrow \text{deriv.} \\ & V(x) & \\ \text{integ.} \downarrow & & \uparrow \text{deriv.} \\ & M(x) & \end{array}$$

c) *Bending-moment and shear-force diagrams*

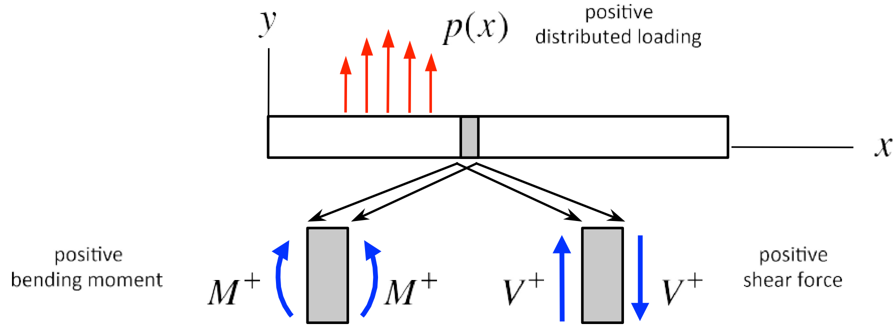
Three methods for determining the internal shear force and bending moment resultants:

- Using free body diagrams with cut sections, as demonstrated in the earlier examples of this section of notes $\Rightarrow$ good for understanding
- Using equilibrium relationships among applied loads, shear force and bending moments derived earlier and summarized above (integration and discontinuities). $\frac{dV}{dx} = p(x)$ $\Rightarrow$ good for complex loadings
- Using a graphical method based on the integration and discontinuity equations from the equilibrium method. The description of this method follows.

  good for simple loading + applied moments.

Graphical method for constructing shear force and bending moment diagrams

Sign conventions:



Basic relationships (as derived via equilibrium relations):

$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V_2 = V_1 + \int_{x_1}^{x_2} p(x)\,dx$$

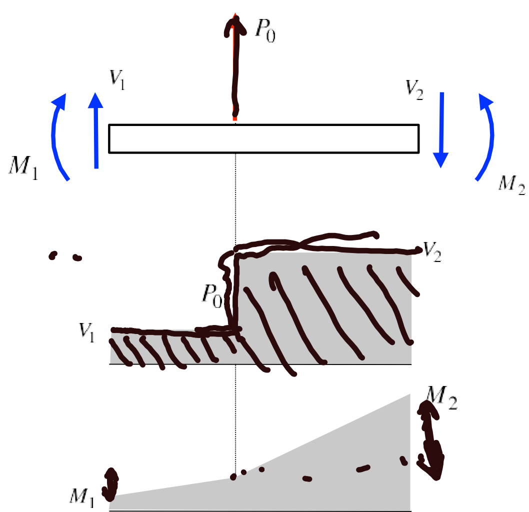
$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M_2 = M_1 + \int_{x_1}^{x_2} V(x)\,dx$$

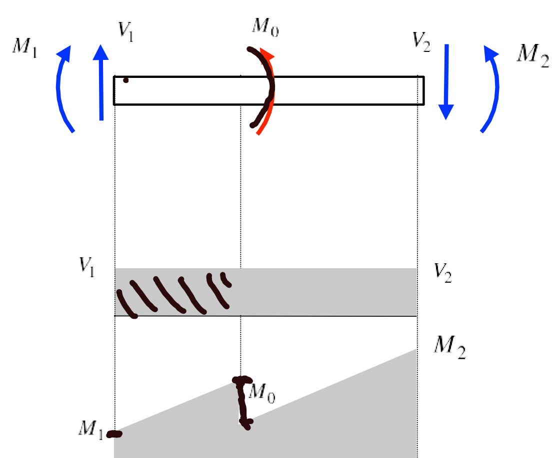
Concentrated shear force V_0 applied at location x:

$$V(x^+) = V(x^-) + V_0 \text{ (jump UP in shear force)}$$

Concentrated moment M_0 applied at location x:

$$M(x^+) = M(x^-) - M_0 \text{ (jump DOWN in moment)}$$





$$\frac{dM}{dx} = V$$

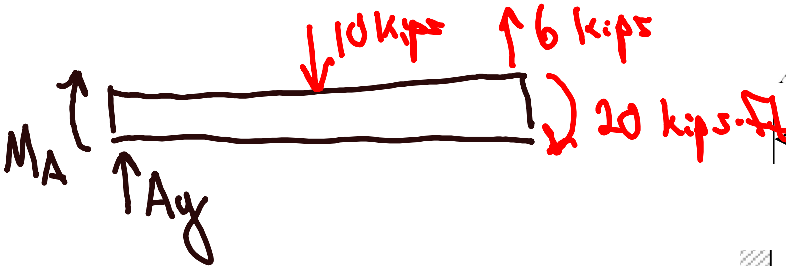
$$M(x_2) = M(x_1) + \int V\,dx.$$

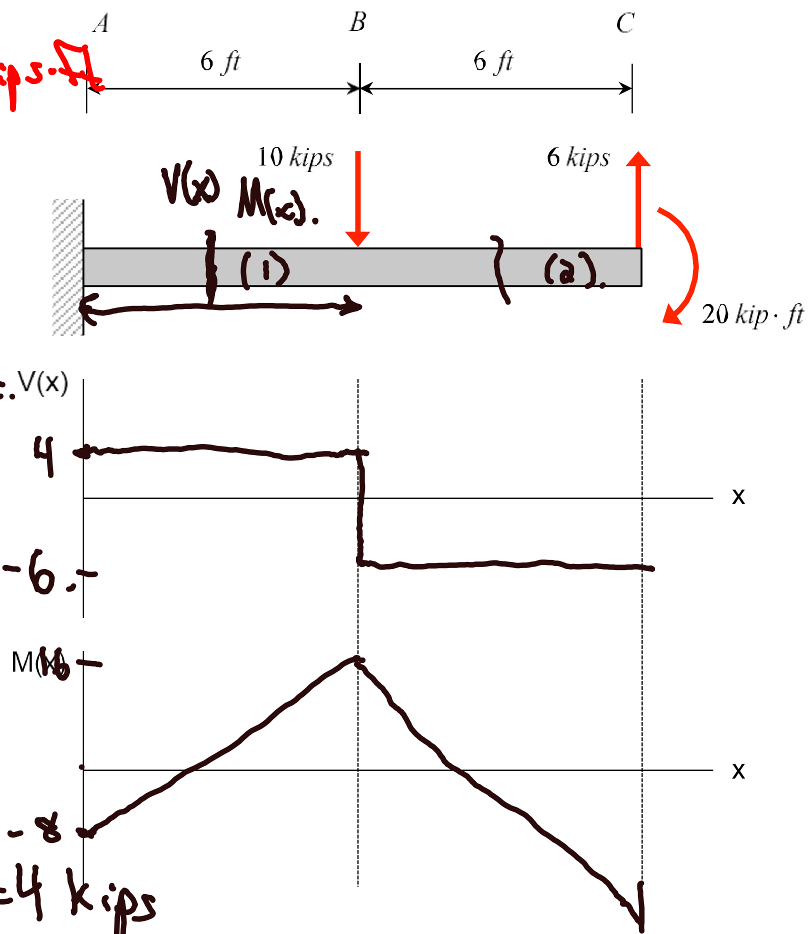
1.) Find resultants

3.) Graphical.

Example 9.3

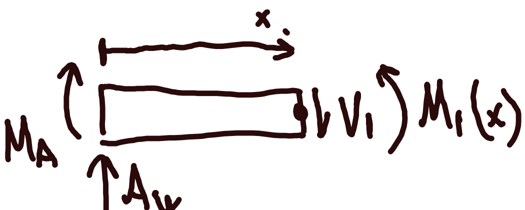
Two transverse forces and a couple are applied as external loads to the cantilevered beam AC. Draw the shear force and bending moment diagrams in the plot axes below.





$$\left.\begin{array}{l}\Sigma F_y \\ (\Sigma M)_A\end{array}\right\} \Rightarrow \begin{array}{l}A_y = 4 \text{ kips} \\ M_A = -8 \text{ kip·ft.}\end{array}$$

Section AB

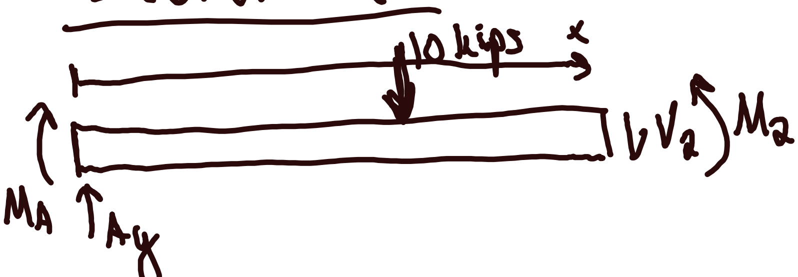


$$\Sigma F_y = A_y - V_1 = 0 \Rightarrow V_1 = A_y = 4 \text{ kips}$$

$$(\Sigma M)_1 = -M_A + M_1 - A_y x = 0$$

$$M_1(x) = M_A + A_y x = -8 + 4x. \text{ [kips·ft]}. \quad M_1(6) = -8 + 24 = 16$$

Section BC



$$\Sigma F_y = A_y - V_2 - 10 = 0 \Rightarrow V_2 = A_y - 10$$

$$V_2 = -6 \text{ kips}$$

$$(\Sigma M)_2 = -M_A + M_2(x) - A_y x + 10(x-6) = 0$$

$$M_2(x) = -8 + 4x - 10x + 60 = 52 - 6x.$$

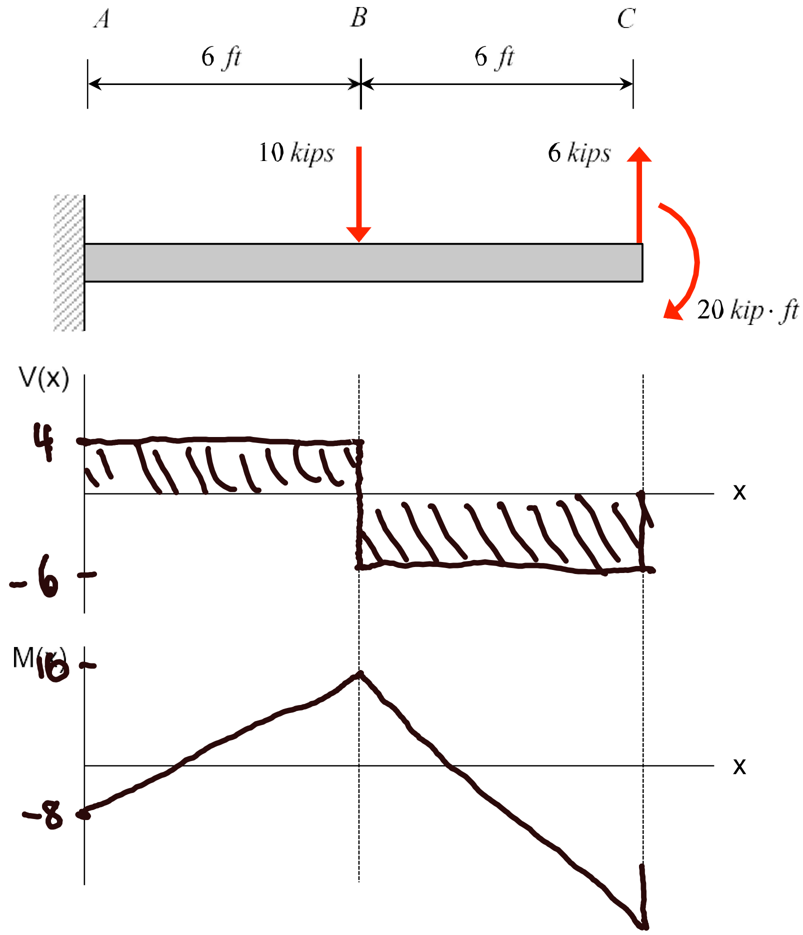
$$M_2(6) = 16 \qquad M_2(12) = 52 - 72 = -20 \text{ kip·ft.}$$

Example 9.3

Two transverse forces and a couple are applied as external loads to the cantilevered beam AC. Draw the shear force and bending moment diagrams in the plot axes below.

$A_y = 4$ kips

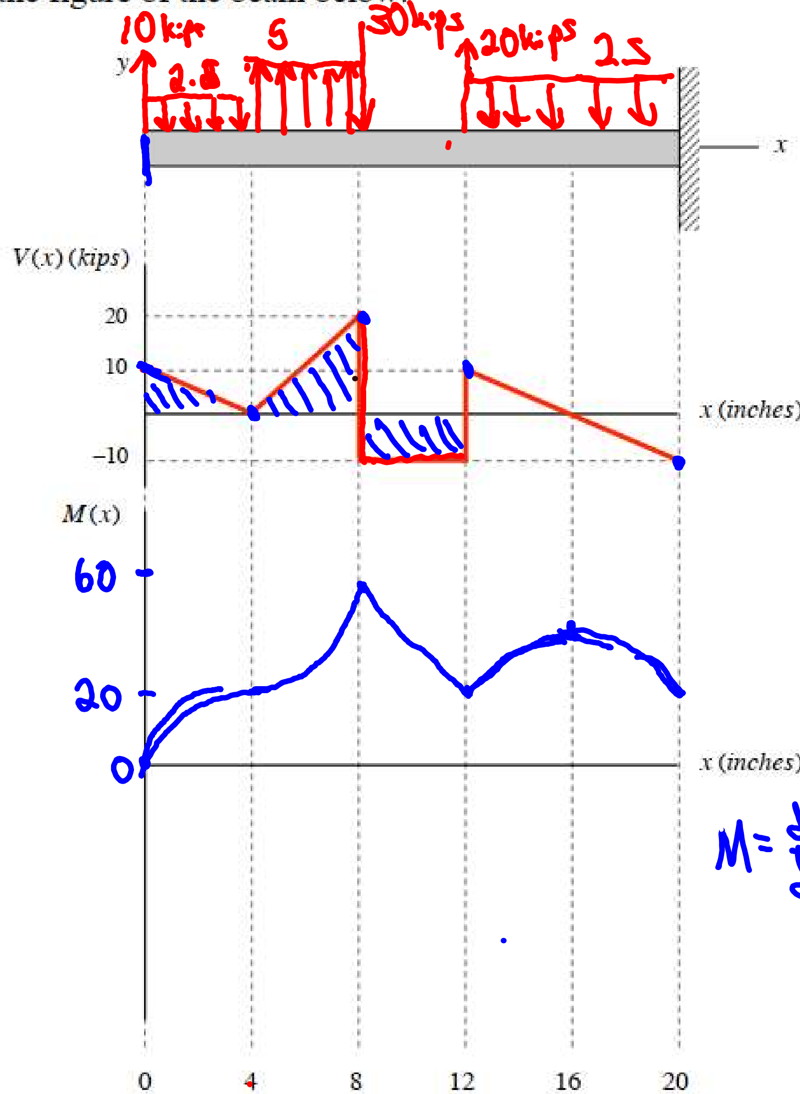
$M_A = -8$ kips·ft.



Example 9.11

Consider the cantilevered beam shown below that is loaded only by concentrated and distributed forces (no external couples applied). The loading is not shown in the figure of the beam. The internal shear force distribution in the beam is shown below. For this beam:

a) Determine the internal bending moment M(x) in the beam and show M(x) in the plot below.
b) Determine the external loading (both concentrated and distributed forces) acting on the beam and show these on the figure of the beam below.



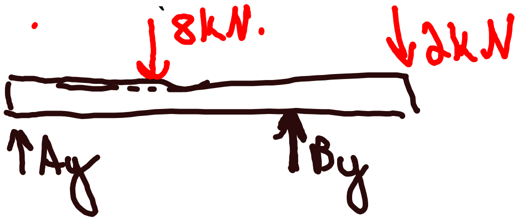
$M = \frac{dV}{dx}$.

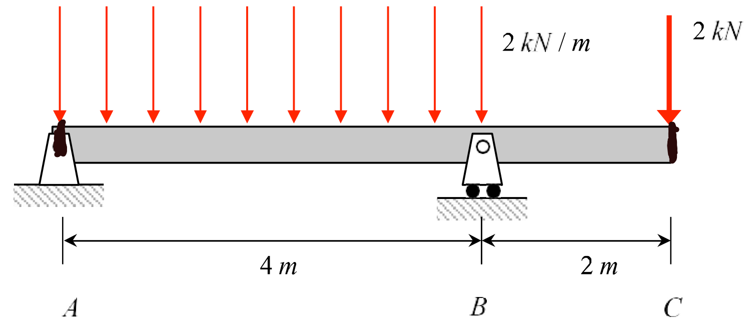
2.) Equations

3.) Graphical.

Example 9.4

Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.





$$\left.\begin{array}{l}\Sigma F_y \\ (\Sigma M)_A\end{array}\right\} \begin{array}{l} A_y = 3\ kN \\ B_y = 7\ kN. \end{array}$$

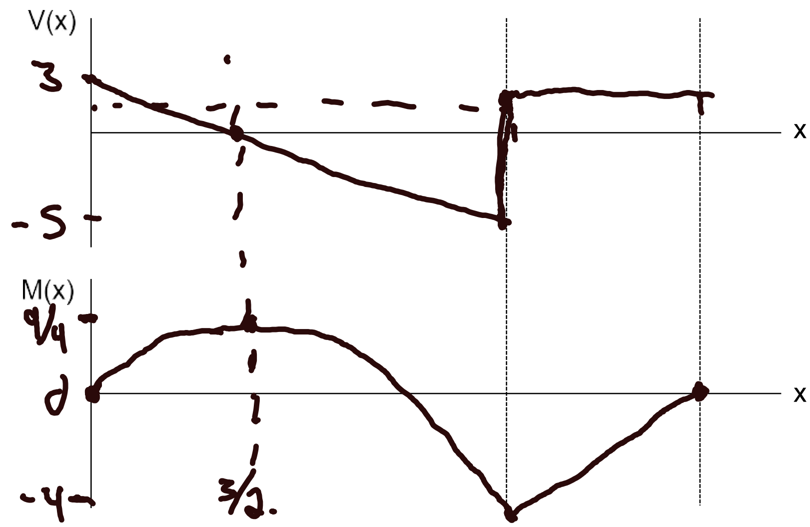
Section AB

$p(x) = -2$

$$V(x) = V(0) + \int_0^x p(x)dx = 3 - 2x.$$

$$M(x) = M(0) + \int_0^x V(x)\,dx$$

$$M(x) = 0 + \int_0^x (3-2x)dx = 3x - x^2$$



$V(4) = 3-8 = -5$

$M(4) = 12-16 = -4$

$0 = 3-2x \Rightarrow x = \frac{3}{2}$

$$M\left(\tfrac{3}{2}\right) = 3\left(\tfrac{3}{2}\right) - \left(\tfrac{3}{2}\right)^2 = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$$

Section BC.

$p(x) = 0$

$V(x) = V(4^+) \qquad V(4^+) = V(4^-) + 7 = 2.$

$$M(x) = M(4) + \int_4^x V(x)dx.$$

$$M(x) = -4 + [2x]_4^x = -4-8+2x = -12+2x$$

Example 9.4

Draw the shear force and bending moment diagrams in the plot axes below for the loaded beam shown.
`