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6. Axial deformation

Objectives:

To study the relationships between applied axial loads and axial deformations in structural components.

Background:

- Stress-strain relationship for uni-axial loading for isotropic, linearly elastic material (loading in the x-direction):

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$

Lecture topics:

a) General theory of axial deformation

b) Axial deformation examples
   i) uniform axial deformations
   ii) non-homogeneous cross sections
   iii) axially varying cross sections
   iv) axially varying loadings

c) Stress analysis of statically indeterminate structures with externally applied loads

d) Deformation and stress analysis of one-node planar trusses

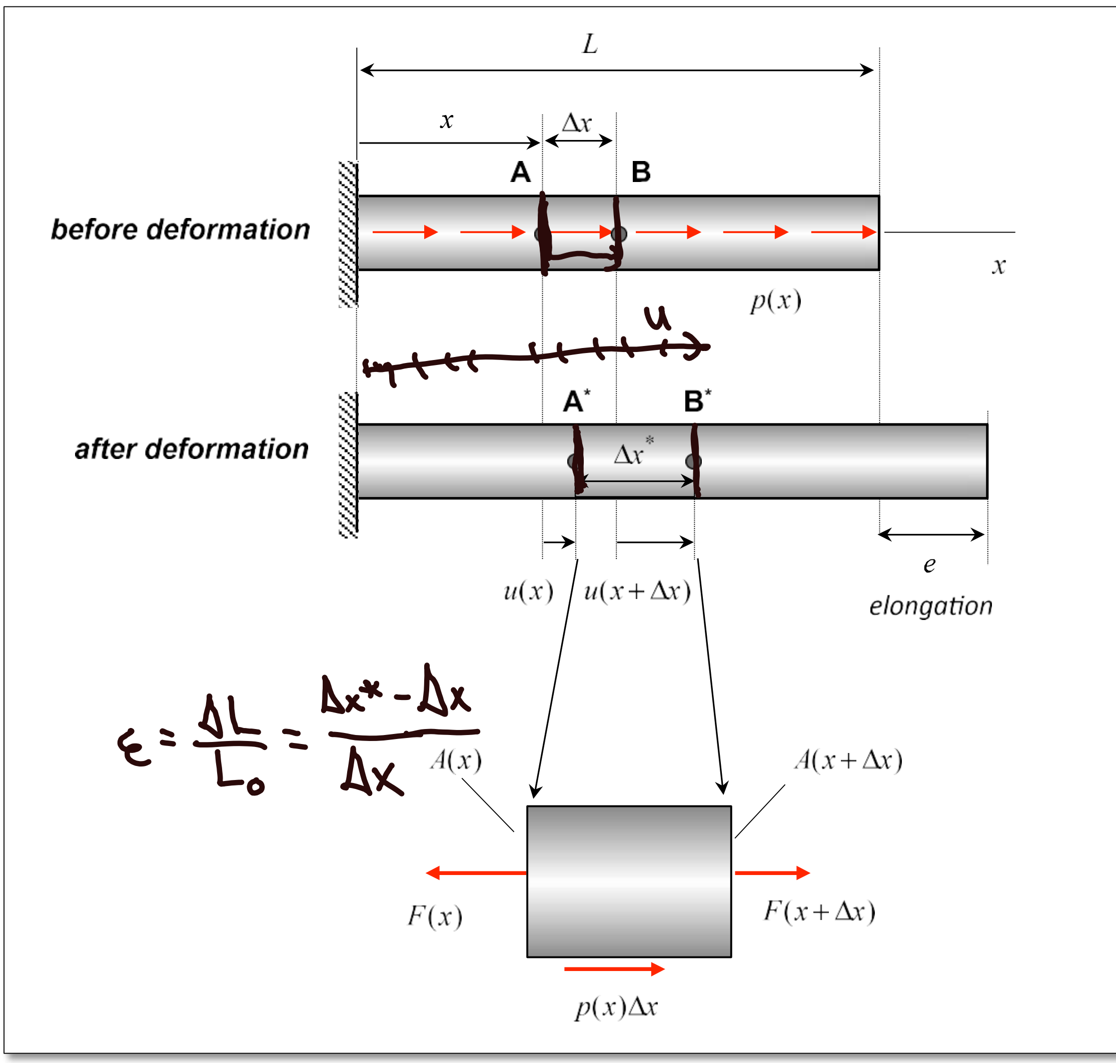
Learning Objective:

- Given the loading conditions on an axial member, determine the elongation of the member.

Lecture Notes

a) *General theory of axial deformation*

- Loading p(x) (*force/length*) applied along *x*-axis on a structural component.
- Two points A and B (originally at locations x and x+Δx), respectively, move to points A^* and B^* through displacements u(x) and u(x+Δx), respectively.
- Before deformation, A and B are separated by a distance of Δx.
- After deformation, A^* and B^* are separated by a distance of Δx^*, where $\Delta x^* - \Delta x = u(x+\Delta x) - u(x)$.



$$\varepsilon = \frac{\Delta L}{L_0} = \frac{\Delta x^* - \Delta x}{\Delta x}$$

$$\Sigma F = -F(x) + F(x+\Delta x) + p(x)\Delta x = 0$$

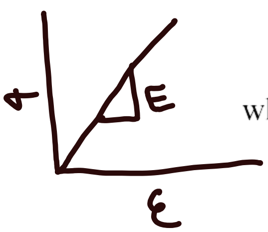
Axial strain (geometric relationship)

$$\varepsilon(x) = \lim_{\Delta x \to 0} \frac{\Delta x^* - \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad ; \quad \textit{see preceding figure} \tag{1}$$

$$= \frac{du}{dx} \quad ; \quad \textit{definition of a derivative}$$

Hooke's Law (stress-strain relationship)



$$\sigma(x) = E(x)\varepsilon(x) \quad \Rightarrow \quad \varepsilon(x) = \frac{\sigma(x)}{E(x)} = \frac{F(x)}{A(x)E(x)} \tag{2}$$

where:

$$F(x) = \int_{area} \sigma(x)\, dA = A(x)\sigma(x)$$

Resultant axial force corresponding to axial load/length p(x) (force balance relationship)

$$\sum F_x = F(x+\Delta x) - F(x) + p(x)\Delta x = 0 \quad \Rightarrow \quad \frac{F(x+\Delta x) - F(x)}{\Delta x} = -p(x)$$

Therefore,

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{dF}{dx} = -p(x)$$

Integrating gives:

$$F(x) = F(0) - \int_0^x p(x)\, dx \tag{3}$$

Axial elongation (force-displacement relationship)

$$\varepsilon(x) = \frac{du}{dx} \quad \Rightarrow \quad u(x) = u(0) + \int_0^x \varepsilon(x)\, dx \tag{4}$$

Therefore,

$$e = u(L) - u(0) = \int_0^L \varepsilon(x)\, dx = \textit{total elongation of member}$$

$$= \int_0^L \frac{F(x)}{A(x)E(x)}\, dx \tag{5}$$

Summary

Using equation (3), we can determine the resultant axial force *F(x)* at any location x along the member through integration:

$$F(x) = F(0) - \int_0^x p(x)\,dx$$

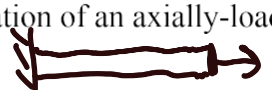
This result is then used in equation (5) to determine the total axial elongation *e* of the member through integration:

$$e = \int_0^L \frac{F(x)}{A(x)E(x)}\,dx$$

b) Axial deformation examples

In the following, we will consider examples of special situations in using the above equations for determining the total elongation of an axially-loaded member:

i) *Uniform axial deformations*:



For this special case, we consider examples for which the applied axial loading $p(x)$ is zero, along with E and A being constant across a cross section and constant along the length of the member. For this special case, equation (6) reduces to:

$$e = \int_0^L \frac{F(x)}{A(x)E(x)} dx = \frac{F}{AE}\int_0^L dx = \frac{FL}{AE} \qquad (5a)$$

$e = \frac{FL}{EA}$

This is a simple, easy-to-use equation for deformation analysis. However, recall the assumptions made in order to arrive at this simple equation.

ii) *Non-homogeneous cross sections*:

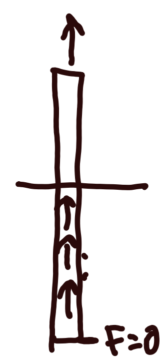
Here the cross section of the rod is not homogeneous in its material makeup (that is, the elastic modulus is not a constant across the rod cross section). The load shared by the different materials is not the same; however, the axial strain seen by each material is the same. We will see examples of this in this section of the course material.

iii) *Non-constant cross section along the length of the member*:

Here the cross sectional area Young's modulus E is assumed to the constant throughout the member and the axial load F is assumed to be constant along its length. However, the cross-section area $A(x)$ varies with position x. Evaluation of the total member elongation requires the following integration:

$$e = \int_0^L \frac{F(x)}{E(x)A(x)} dx = \frac{F}{E}\int_0^L \frac{dx}{A(x)} \qquad (5b)$$

iv) *Non-uniform loading along the member*:



Here the cross sectional area Young's modulus E is assumed to the constant throughout the member and the cross sectional area A is assumed to be constant along its length. However, the force/length loading $p(x)$ varies with position x. Evaluation of the total member elongation requires the following integration:

$$e = \int_0^L \frac{F(x)}{E(x)A(x)} dx = \frac{1}{EA}\int_0^L F(x)\, dx \qquad (5c)$$

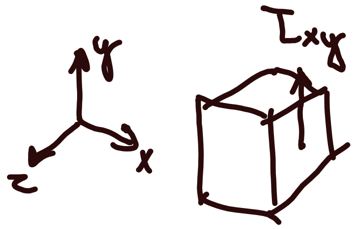
where:

$$F(x) = F(0) - \int_0^x p(x)\, dx$$

Lecture 5 Review Questions:

How many independent stresses can exist on a general stress element?

- 4
- 6
- 8
- 18
- Depends on the geometry



$\sigma_x, \sigma_y, \sigma_z$

$\tau_{xy}, \tau_{xz}, \tau_{yz}$.

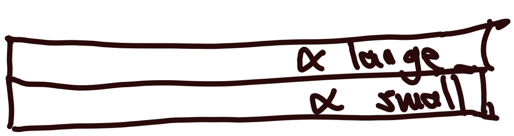
The two subscripts on the shear stress represent: τ_{ij}

- Direction along which the force acts, direction perpendicular to the force
- ~~Direction of primary force, direction of secondary force~~
- Plane on which force acts, direction along which the force acts

For a structure with no mechanical constraints, the shear strain increases with temperature. γ

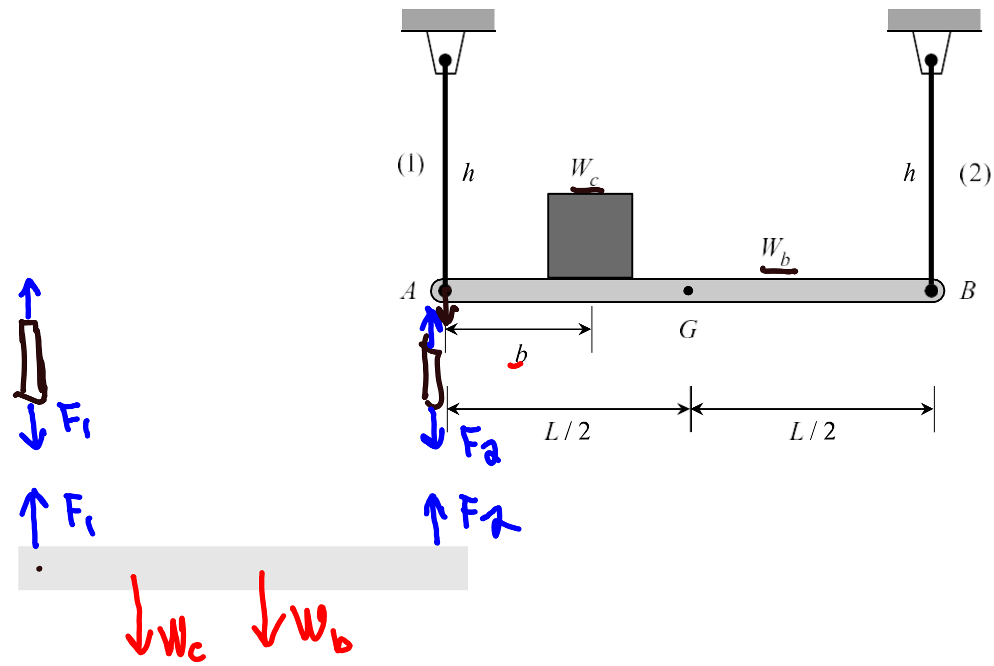
- True
- False

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) + \alpha \Delta T \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$



Example 6.1

Rigid beam AB (weighing $W_b = 200\ lb$) supports a crate weighing $W_c = 1000\ lb$. In turn, the beam is supported by rods (1) and (2) with lengths of $h = 6\ ft$, Young's moduli $E_1 = E_2 = E = 30 \times 10^3\ ksi$, and diameters $d_1 = d_2 = 0.4\ in$, respectively. What are the downward displacements u_A and u_B of ends A and B, respectively, of the beam? Assume small rotations in the beam.



$$(\Sigma M)_A = -W_c(b) - W_b\left(\frac{L}{2}\right) + F_2(L) = 0$$

$$F_{(2)} = \frac{b}{L} W_c + W_b\left(\frac{1}{2}\right)$$

$$\Sigma F_y = F_1 + F_2 - W_c - W_b = 0$$

$$F_{(1)} = W_c + W_b - \left(\frac{b}{L} W_c + \frac{1}{2} W_b\right)$$

$$F_{(1)} = \left(1 - \frac{b}{L}\right) W_c + \frac{1}{2} W_b$$

$$v_A = \underline{e_{(1)}} = \frac{F_{(1)} L_{(1)}}{E_{(1)} A_{(1)}} =$$

$$v_B = e_{(2)} = \frac{F_{(2)} L_{(2)}}{E_{(2)} A_{(2)}}.$$

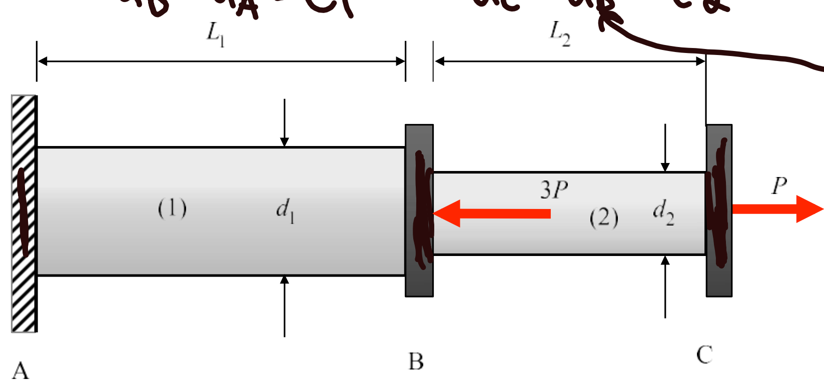
Example 6.6

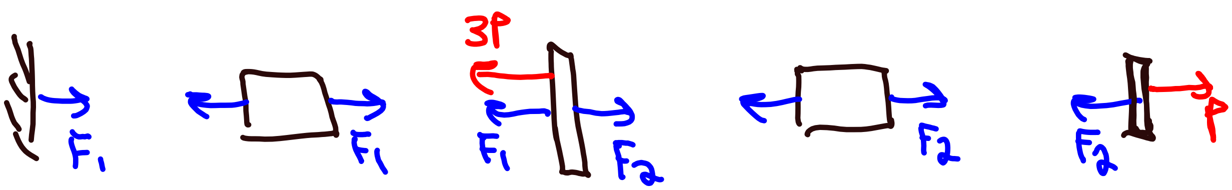
A rod is constructed from elements (1) and (2), with these elements being made up of materials having Young's moduli of E_1 and E_2, respectively. Elements (1) and (2) have lengths of $L_1 = 2L$ and $L_2 = L$, and diameters of $d_1 = 2d$ and $d_2 = d$, respectively. Elements (1) and (2) are joined by a rigid connector at B, with a rigid connector being attached to element (2) at C. The rod is loaded on connectors B and C, as shown in the figure below. Determine the displacement of C.

$u_B - u_A = e_1$ $\qquad$ $u_C - u_B = e_2$ $\qquad$ $u_A = 0$

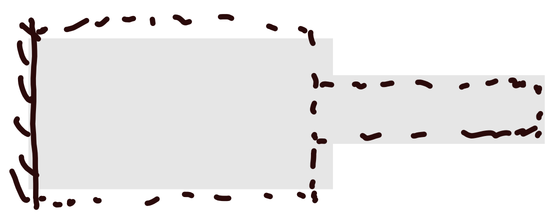
$u_B = e_1$

$u_C = e_1 + e_2.$





$$\left.\begin{aligned}(\Sigma F_x)_C &= P - F_2 = 0 \Rightarrow F_2 = P \\ (\Sigma F_x)_B &= F_2 - F_1 - 3P = 0 \Rightarrow F_{(1)} = -2P\end{aligned}\right\}$$



$$e_2 = \frac{F_2 L_2}{E_2 A_2} = \frac{PL}{E_2(\pi (d/2)^2)} \qquad e_1 = \frac{F_1 L_1}{E_1 A_1} = \frac{-2P(2L)}{E_1(\pi (2d/2)^2)}$$

$$\left. e_2 = \frac{4PL}{E_2 \pi d^2} \right\} \qquad e_1 = \frac{-4PL}{E_1 \pi d^2}$$

$$u_c = e_1 + e_2$$

$$= \frac{-4PL}{\pi E_1 d^2} + \frac{4PL}{\pi E_2 d^2}$$

$$u_c = \frac{4PL}{\pi d^2}\left(\frac{1}{E_2} - \frac{1}{E_1}\right)$$

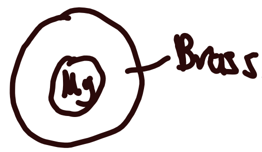
Example 6.3

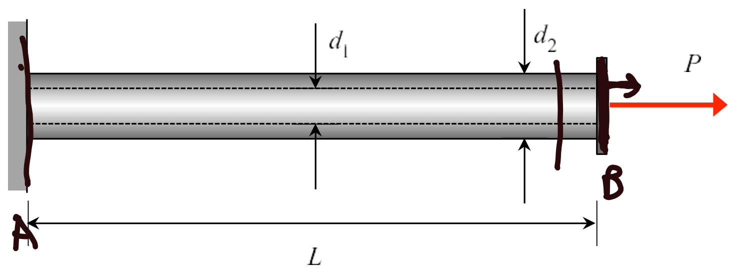
A magnesium-alloy rod ($E_1 = 8\times10^3 ksi$), having a diameter of $d_1 = 1\ in$, is encased in a brass tube ($E_2 = 16\times10^3 ksi$), having an outer diameter of $d_2 = 2\ in$. The rod and tube both have a length of $L = 30\ in$. An axial load $P = 20\ kips$ is applied to the free end, as shown below.

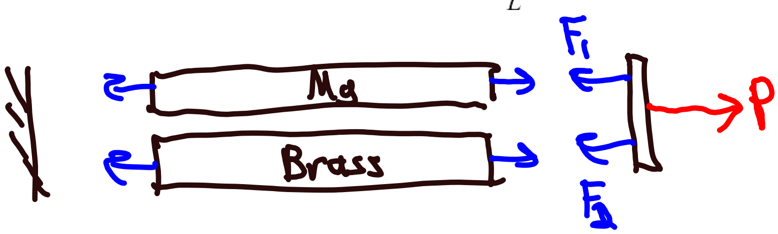
a) Determine the normal stresses σ_1 and σ_2 in the two materials.

b) Determine the elongation of the bimetallic rod.

$u_B - u_A = e_1 \qquad u_B - u_A = e_2$







$$\left.\begin{array}{l} \Sigma F_x = P - F_1 - F_2 = 0 \\ (1) \quad F_1 + F_2 = P \end{array}\right\} \begin{array}{l} 1 \text{ eqn} \\ 2 \text{ unknowns.} \end{array}$$

Elongation:

$$e_1 = \frac{F_1 L_1}{E_1 A_1} \qquad e_2 = \frac{F_2 L_2}{E_2 A_2}.$$

Compatibility:

$$e_1 = e_2. \qquad \frac{F_1 L_1}{E_1 A_1} = \frac{F_2 L_2}{E_2 A_2}. \quad (2)$$

$$F_1 = F_2\left(\frac{L_2}{E_2A_2}\right)\left(\frac{E_1A_1}{L_1}\right)$$

$$F_2\left(\frac{L_2}{E_2A_2}\right)\left(\frac{E_1A_1}{L_1}\right) + F_2 = P$$

$$F_2\left(1 + \frac{E_1A_1}{E_2A_2}\right) = P$$

$$\sigma_{U,concrete} = 0.33\ ksi$$

$$\sigma_{U,steel} = 60\ ksi$$

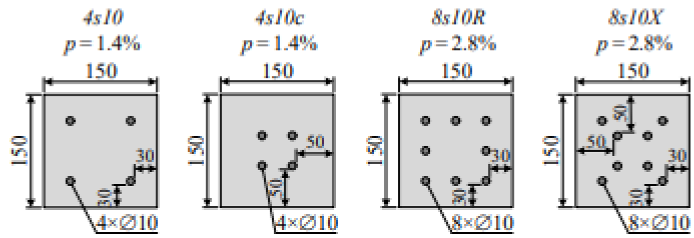
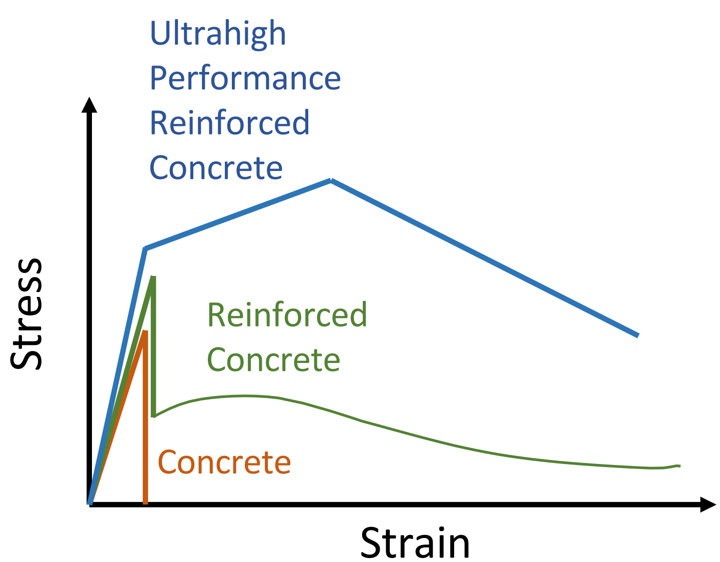


Figure 5. Cross-sections of the test specimens



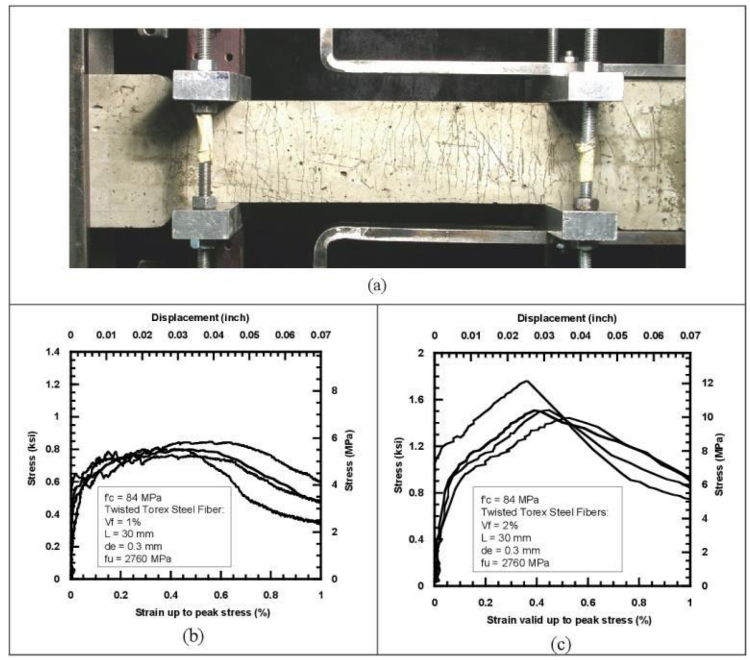
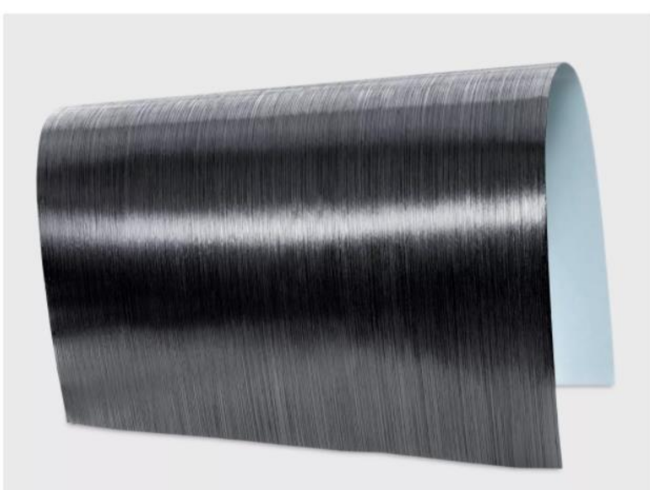
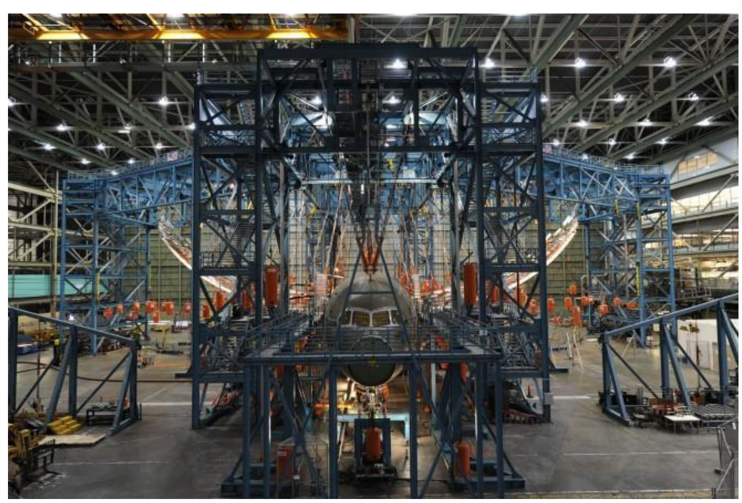


Figure 6: Typical stress-strain curves in tension of HPFRC composite for two series of tests

A. E. Naaman. *Proceedings of the 2nd International RILEM Conference*, pp. 17-26. 2011



https://www.sglcarbon.com/en/markets-solutions/material/sigrapreg-pre-impregnated-materials/



https://www.wired.com/2010/03/boeing-787-passes-incredible-wing-flex-test/

Example 6.4

A tapered rod having a length of L and made up of a material with a Young's modulus of E is loaded with an axial load P, as shown below. The cross sectional area of the rod varies linearly from A_0 at $x = 0$ to A_1 at $x = L$. Determine the total axial elongation of the road as a result of the axial load P.