If you develop  a new manufacturing process that increases the precision of parts, how would the Factor of Safety be adjusted:

- Increase
- Decrease
- Stay the same

# *5. Stress and strain: generalized concepts*

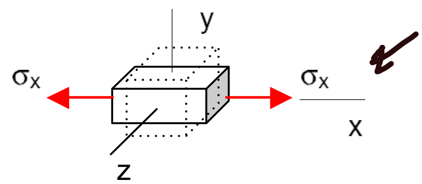
**Objectives**:
To study the relationships between stress and strain due to a three-dimensional loading of a body.

**Background**:

- Stress-strain relationship for uni-axial loading (loading in the x-direction):

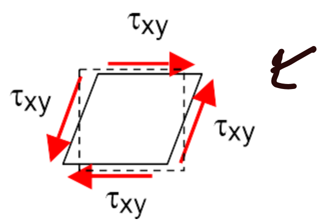
$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$




- Stress-strain relationship for pure shear loading (loading in y-direction on face perpendicular to x-axis):

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$




**Lecture topics:**

a) Resolution of internal forces into normal and tangential (shear) components
b) Thermal strains
c) Generalized Hooke's law for normal stresses/strains
d) Generalized Hooke's law for shear stresses/strains

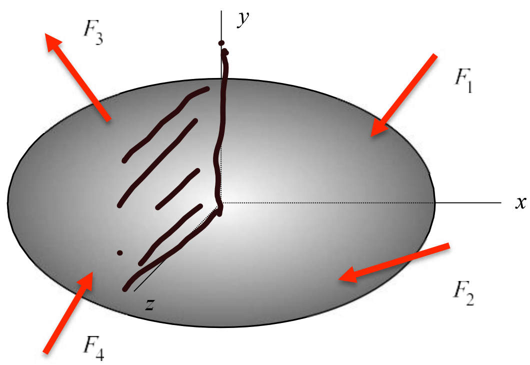
Learning objective:

- Determine the strains in all three directions when an object is subjected to stresses in multiple directions.

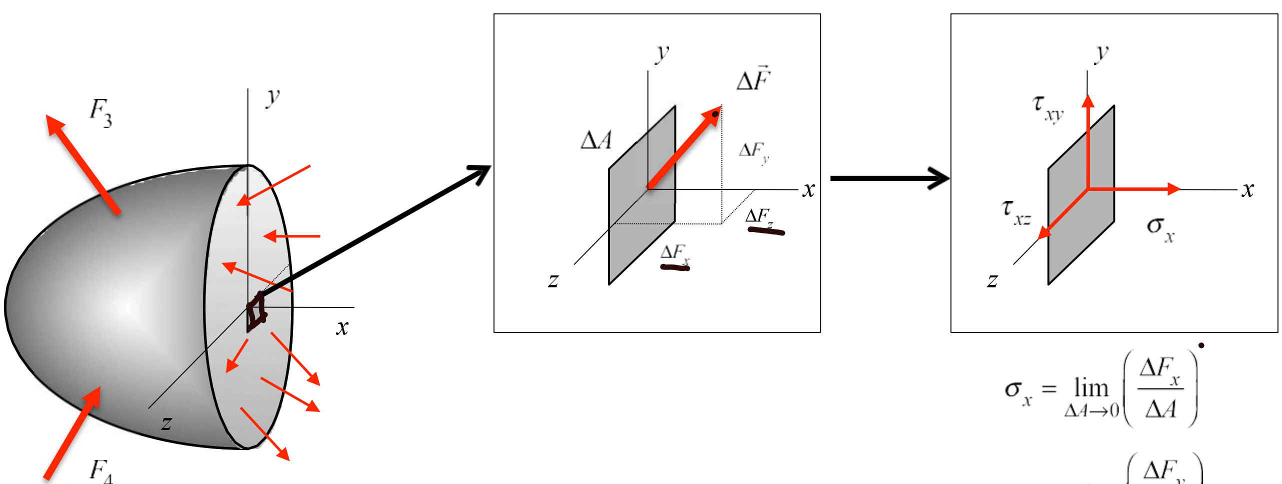
## Lecture Notes

### a) *Resolution of internal force into normal and tangential (shear) components*

General 3-D loading on a component:




Making cut through body parallel to yz-plane:

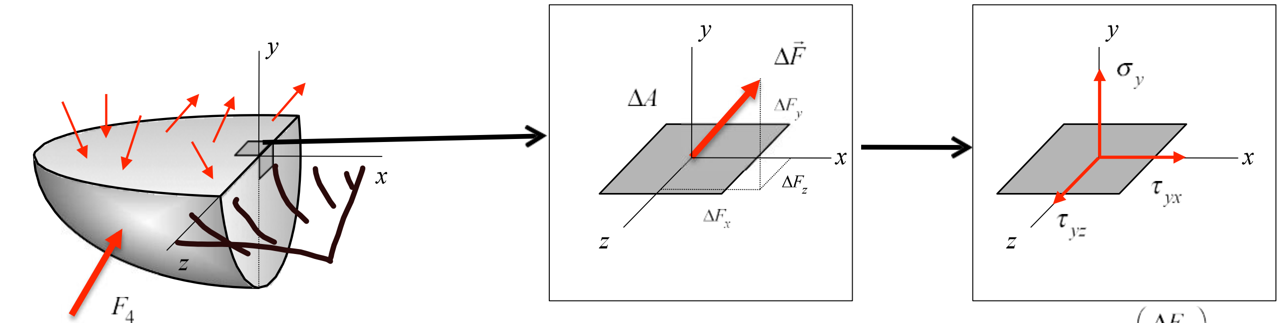



$$\sigma_x = \lim_{\Delta A \to 0} \left( \frac{\Delta F_x}{\Delta A} \right)$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \left( \frac{\Delta F_y}{\Delta A} \right)$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \left( \frac{\Delta F_z}{\Delta A} \right)$$

Making cut through body parallel to xz-plane:



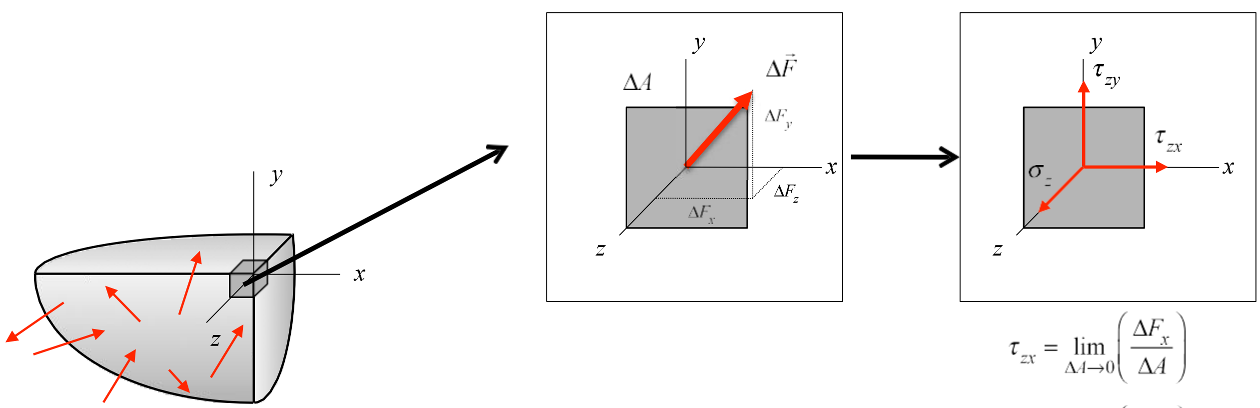

$$\tau_{yx} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_x}{\Delta A}\right)$$

$$\sigma_y = \lim_{\Delta A \to 0}\left(\frac{\Delta F_y}{\Delta A}\right)$$

$y$

$$\tau_{yz} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_z}{\Delta A}\right)$$

Making cut through body parallel to xy-plane:




$$\tau_{zx} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_x}{\Delta A}\right)$$

$$\tau_{zy} = \lim_{\Delta A \to 0}\left(\frac{\Delta F_y}{\Delta A}\right)$$

$$\sigma_z = \lim_{\Delta A \to 0}\left(\frac{\Delta F_z}{\Delta A}\right)$$

Summary

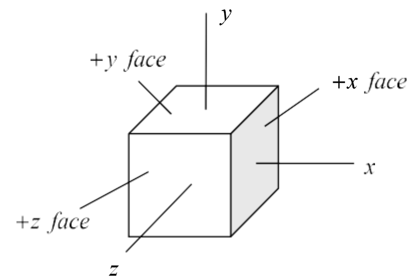
In the above, we have described the components of stress at a point in a body by the projection of forces onto three mutually-perpendicular surfaces. From this, we introduce the concept of a stress element (cube) as a cube of infinitesimal dimensions centered on the point with the cube being oriented with these three planes.

*Naming convention*

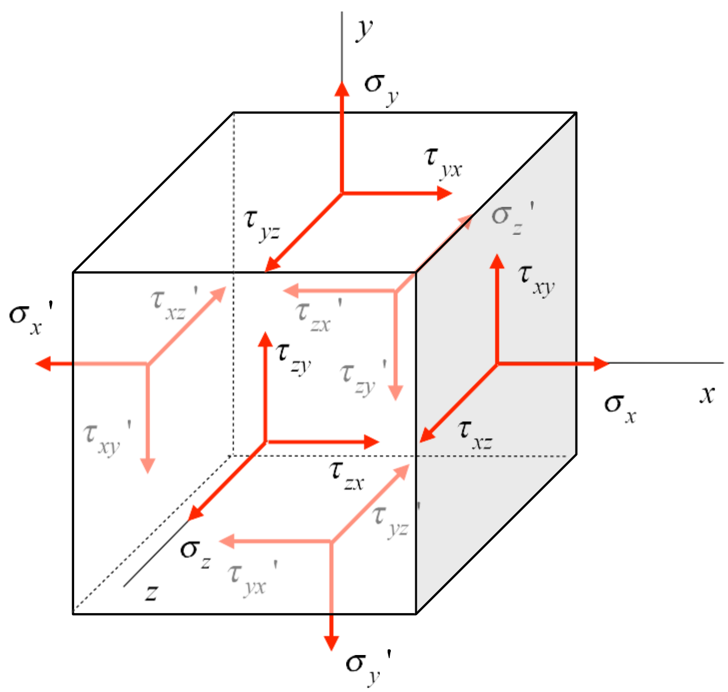
- $\sigma_i$ is the normal stress on face "*i*". $\sigma_x$, $\sigma_{-x}$
- $\tau_{ij}$ is the shear stress in the "j" direction on face "*i*".

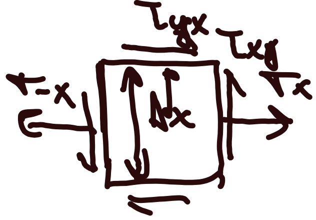
*Sign convention – signs for components of stress on a stress element*

- A normal stress $\sigma_i$ is positive (negative) if it points outward (inward) on face "*i*" of the cube, for $i = x, y, z$. Note that a positive (negative) normal stress corresponds to tension (compression).
- A shear stress $\tau_{ij}$ is positive if it points in the positive (negative) *j*-direction on the positive (negative) *i*-face of the stress cube. Otherwise, the shear stress is negative.




The three components of normal stress and the six components of shear stress shown on the body on the previous pages all correspond to positive quantities. These components on the positive faces, along with those on the negative faces, are shown on the stress cube below, where a "primed" quantities (e.g., $\sigma'_x$ and $\tau'_{xy}$) are those components on the negative faces.






$$\Sigma M = \tau_{xy}\left(\tfrac{1}{2}\Delta x\right) - \tau_{yx}\left(\tfrac{1}{2}\Delta y\right) = 0$$

$$\tau_{xy} = \tau_{yx}.$$

$$\Sigma F_x = A\sigma_x - A\sigma_{-x} = 0$$

$$\sigma_x = \sigma_{-x}$$

In total, we have six components of normal stress:

$$\left(\sigma_x, \sigma_y, \sigma_z, \sigma'_x, \sigma'_y, \sigma'_z\right)$$

and 12 components of shear stress:

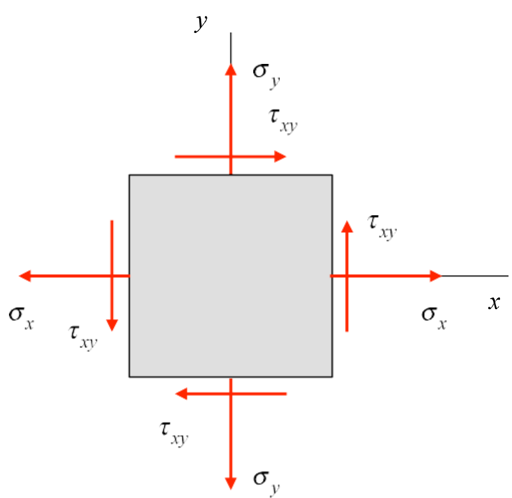
$$\left(\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}, \tau'_{xy}, \tau'_{xz}, \tau'_{yx}, \tau'_{yz}, \tau'_{zx}, \tau'_{zy}\right)$$

on the stress element.

Using static equilibrium equations for the cube, the following relations can be derived relating these stresses:

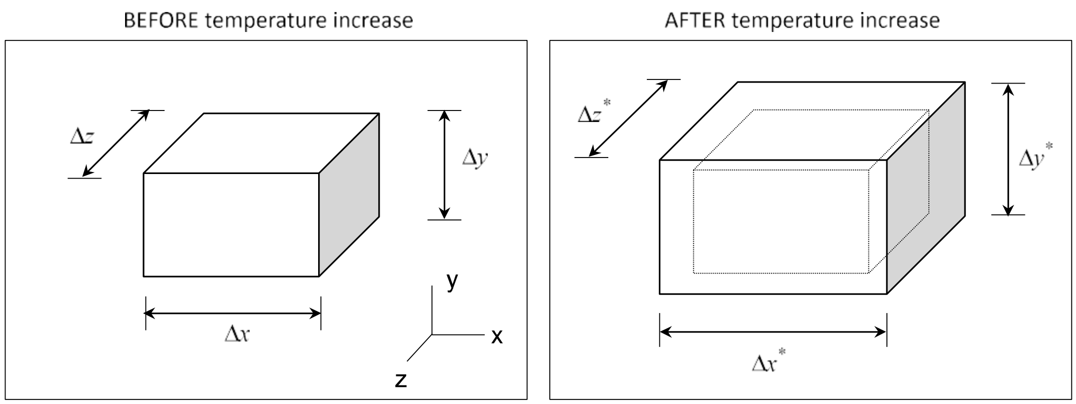
$$\left.\begin{array}{lll} \sigma'_x = \sigma_x & \sigma'_y = \sigma_y & \sigma'_z = \sigma_z \\ \tau'_{xy} = \tau_{xy} = \tau'_{yx} = \tau_{yx} & \tau'_{yz} = \tau_{yz} = \tau'_{zy} = \tau_{zy} & \tau'_{zx} = \tau_{zx} = \tau'_{xz} = \tau_{xz} \end{array}\right\} \text{6 stress components.}$$

In summary, there are, in total, three unique normal stresses $\left(\sigma_x, \sigma_y, \sigma_z\right)$ and three unique shear stresses acting on the cube $\left(\tau_{xy}, \tau_{xz}, \tau_{yz}\right)$. For example, looking down the z-axis we see the following stresses acting on the x and y faces of the cube:



### *b) Thermal strains*

As a result of a uniform increase in temperature $\Delta T$, most engineering materials will experience a uniform extensional strain in all three directions. This extensional strain is proportional to the temperature increase $\Delta T$. Consider the cube shown below that is given a uniform temperature increase:




The thermal strains induced by the temperature increase are found from the usual definitions:

$$\varepsilon_{x,T} = \lim_{\Delta x \to 0} \left( \frac{\Delta x^* - \Delta x}{\Delta x} \right)$$

$$\varepsilon_{y,T} = \lim_{\Delta y \to 0} \left( \frac{\Delta y^* - \Delta y}{\Delta y} \right)$$

$$\varepsilon_{z,T} = \lim_{\Delta z \to 0} \left( \frac{\Delta z^* - \Delta z}{\Delta z} \right)$$

Since this thermal strain is uniform and is proportional to $\Delta T$, we can write these as:

$$\varepsilon_{x,T} = \varepsilon_{y,T} = \varepsilon_{z,T} = \alpha \Delta T$$

where $\alpha$ is the coefficient of thermal expansion (having units of 1/°F, or 1/°C).

Note that temperature changes produce only extensional strains (no shear strains).

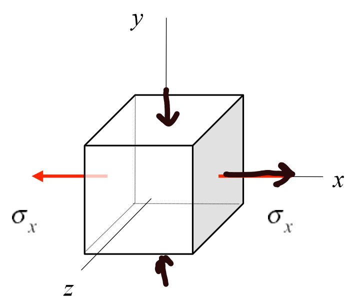
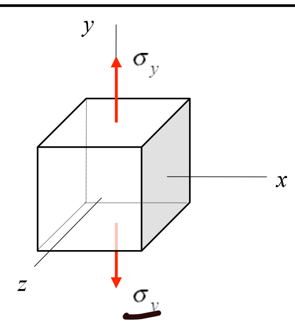
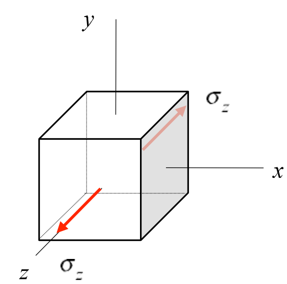
## c) *Generalized Hooke's law for normal stresses/strains*

Recall that for uni-axial loading along the x-axis, the normal strains in the $x$, $y$ and $z$ directions in the body were found to be:

$$\varepsilon_x = \sigma_x / E.$$
$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$$

where $E$ and $\nu$ are the Young's modulus and Poisson's ratio for the material. For a 3-D loading of a body, we have three normal stress components $\sigma_x$, $\sigma_y$ and $\sigma_z$ acting simultaneously. For this case, we will consider the strains due to each normal component of stress individually and add these together using linear superposition (along with the thermal strains) to determine the resulting three components of strain $\varepsilon_x$, $\varepsilon_y$ and $\varepsilon_z$.

Consider the individual contributions of the three components of stress shown below:

| *Strains due to mechanical loading in the x-direction* | *Strains due to mechanical loading in the y-direction* | *Strains due to mechanical loading in the z-direction* |
|---|---|---|
|   |   |   |
| $\varepsilon_x = \sigma_x / E$ | $\varepsilon_x = -\nu\varepsilon_y = -\nu\sigma_y / E$ | $\varepsilon_x = -\nu\varepsilon_z = -\nu\sigma_z / E$ |
| $\varepsilon_y = -\nu\varepsilon_x = -\nu\sigma_x / E$ | $\varepsilon_y = \sigma_y / E$ | $\varepsilon_y = -\nu\varepsilon_z = -\nu\sigma_z / E$ |
| $\varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$ | $\varepsilon_z = -\nu\varepsilon_y = -\nu\sigma_y / E$ | $\varepsilon_z = \sigma_z / E$ |

The total strain in each direction is found through superposition of the individual strains along with the thermal strains. Adding together these components (across each row of the above table) gives:

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha\Delta T$$


The above are known as the generalized Hooke's law equations for normal stresses/strains due to 3-D loadings on a body.

*Observation:*

Note from the preceding equations that thermal strains can exist in the absence of stresses; that is, if $\sigma_x = \sigma_y = \sigma_z = 0$, we still have: $\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha \Delta T$. Later on in the course, we will observe that stresses from thermal loadings can develop only in the presence of mechanical forces. In particular, a body that is heated in the absence of displacement constraints will be stress-free.

### *(e) Generalized Hooke's law for shear stresses/strains*

It can be shown that the three components of shear stress, $\left(\tau_{xy}, \tau_{xz}, \tau_{yz}\right)$ are related to the corresponding shear strains $\left(\gamma_{xy}, \gamma_{xz}, \gamma_{yz}\right)$ by the following equations:

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{xz} &= \frac{\tau_{xz}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \end{aligned}\right\} \text{Not affected by temp.}$$

where $G$ is the shear modulus of the material.

***Example 5.3***

Determine the state of strain that corresponds to the following 3-D state of stress at a certain point in a steel machine component:

$$\sigma_x = 60\ MPa \quad ; \quad \sigma_y = 20\ MPa \quad ; \quad \sigma_z = 30\ MPa$$

$$\tau_{xy} = 20\ MPa \quad ; \quad \tau_{xz} = 15\ MPa \quad ; \quad \tau_{yz} = 10\ MPa$$

Use $E = 210\ GPa$ and $\nu = 0.3$ for steel.

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) = \frac{1}{210\times10^9}\left(60 - 0.3(20+30)\right)\times10^6 = \frac{45}{210}\times10^{-3}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right) = \frac{1}{210\times10^9}\left(20 - 0.3(60+30)\right)\times10^6 = -\frac{7}{210}\times10^{-3}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad G = \frac{E}{2(1+\nu)} = \frac{210\times10^9}{2(1.3)} = 80.8\ GPa$$

$$\gamma_{xy} = \frac{20\times10^6}{80.8\times10^9}$$

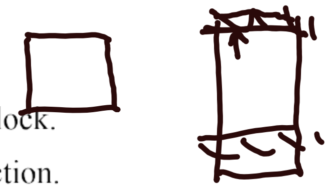
### *Example 5.8*

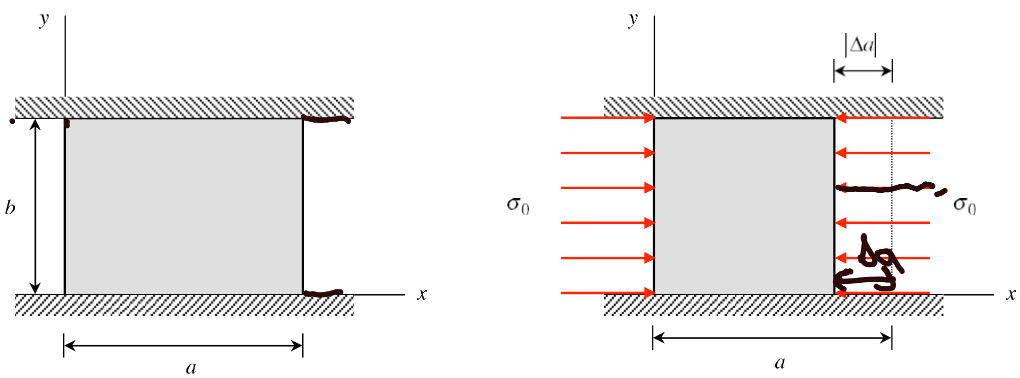
A block of linearly elastic material ($E$, $\nu$) is compressed between two rigid, perfectly smooth surfaces by an applied stress $\sigma_x = -\sigma_0$. ~~The only~~ non-zero stress is the stress $\sigma_y$ induced by the restraining surfaces at $y = 0$ and $y = b$.

(a) Determine the value of the restraining stresses $\sigma_y$.

(b) Determine $\Delta a$, the change in the $x$ dimension of the block.

(c) Determine the change $\Delta t$ in the thickness in the $z$ direction.






$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \cancelto{0}{\sigma_z}\right)\right) + \alpha\cancelto{0}{\Delta T}$$

$$\cancelto{0}{\varepsilon_y} = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \cancelto{0}{\sigma_z}\right)\right) + \alpha\cancelto{0}{\Delta T}$$

$$\varepsilon_z = \frac{1}{E}\left(\cancelto{0}{\sigma_z} - \nu\left(\underline{\sigma_x + \sigma_y}\right)\right) + \alpha\cancelto{0}{\Delta T}$$

$\sigma_x = -\sigma_0$ $\quad$ $\varepsilon_x = ?$

$\sigma_y = ?$ $\quad$ $\varepsilon_y = 0$

$\sigma_z = 0$ $\quad$ $\varepsilon_z = ?$

(a) $0 = \frac{1}{E}\left(\sigma_y + \nu\sigma_0\right) \Rightarrow \sigma_y = -\nu\sigma_0$

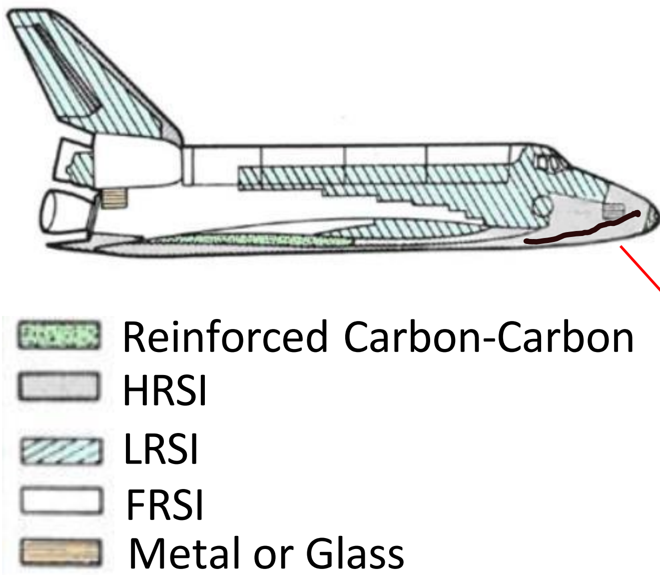
(b) $\varepsilon_x = \frac{1}{E}\left(-\sigma_0 - \nu\left(-\nu\sigma_0\right)\right)$

$\frac{\Delta a}{a_0} = -\frac{1}{E}\sigma_0\left(1-\nu^2\right) \Rightarrow \Delta a = -\frac{a}{E}\sigma_0\left(1-\nu^2\right)$

(c) $\varepsilon_z = \frac{1}{E}\left(-\nu\left(-\sigma_0 - \nu\sigma_0\right)\right)$

$\frac{\Delta t}{t_0} = \frac{\sigma_0}{E}\left(\nu + \nu^2\right)$

# Shuttle Reentry

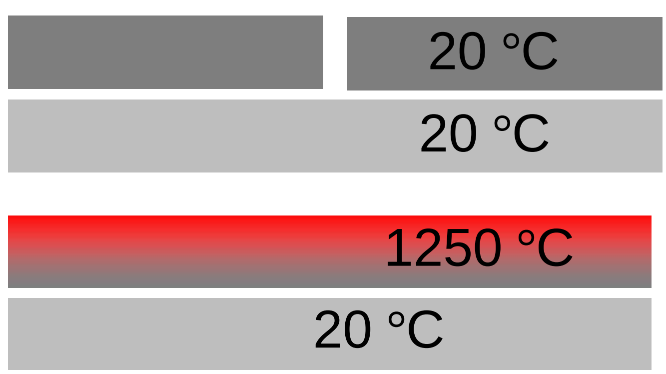


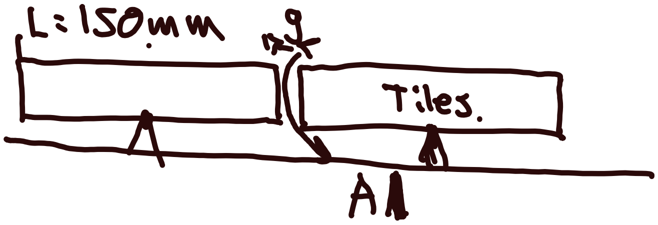

During re-entry, the space shuttle locally experiences temperatures up to 1500 °C. The heat-protection tiles on the surface of the shuttle are designed to have a small thermal expansion coefficient, but gaps are still left between tiles to allow thermal expansion.

a) At 20 °C, the tiles have a thickness of 50 mm and widths of 150 mm. During reentry (heat from 20 °C to 1250 °C), what spacing between tiles is needed to prevent the tiles from coming into contact? The temperature of the aluminum remains at 20 °C during reentry.

b) If the shuttle overheats and the temperature increases to 1350 , what magnitude of stresses will develop in the tiles?

| | |
|---|---|
| $E_{tile}$ | 680 MPa |
| $E_{Al}$ | 70 GPa |
| $\alpha_{tile}$ | $1.66 \times 10^{-6}$ |
| $\alpha_{Al}$ | $20.3 \times 10^{-6}$ |
| $\upsilon_{tile}$ | 0.1 |
| $\upsilon_{Al}$ | 0.35 |






$L = 150\,mm$

$g = \Delta L$

$$\varepsilon_x = \frac{1}{E}\left(\cancel{\sigma_x}^{0} - \nu\left(\cancel{\sigma_y}^{0} + \cancel{\sigma_z}^{0}\right)\right) + \alpha \Delta T$$

$$\varepsilon_x = \alpha \Delta T$$

$$\frac{g}{L_0} = \alpha \Delta T$$

$$g = 150\,\alpha \Delta T$$

$$g = 0.31\,mm$$

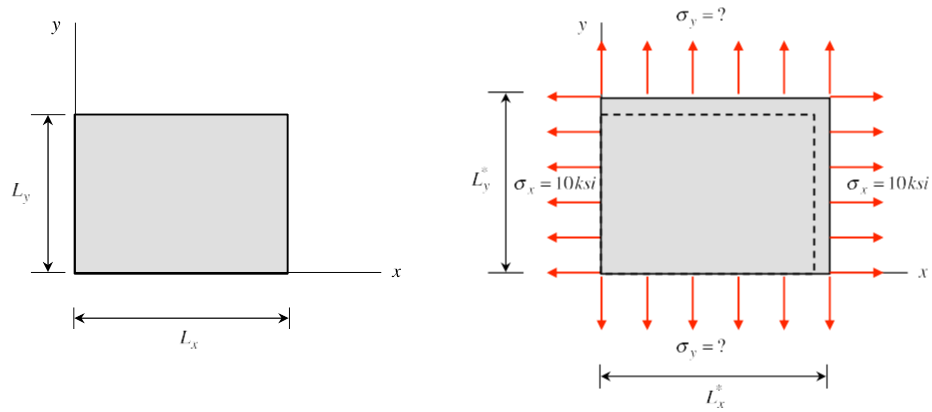
Al $\Rightarrow$ 3.75 mm.

***Example 5.9***

A thin, rectangular plate is subjected to a uniform biaxial state of stress $(\sigma_x, \sigma_y)$. All other components of stresses are zero. The initial dimensions of the place at $L_x = 4$ in. and $L_y = 2$ in., but after the loading is applied, the dimensions are $L_x^* = 4.00176$ in., and $L_y^* = 2.00344$ in. If it is known that $\sigma_x = 10$ ksi and E = $10\times10^3$ ksi:

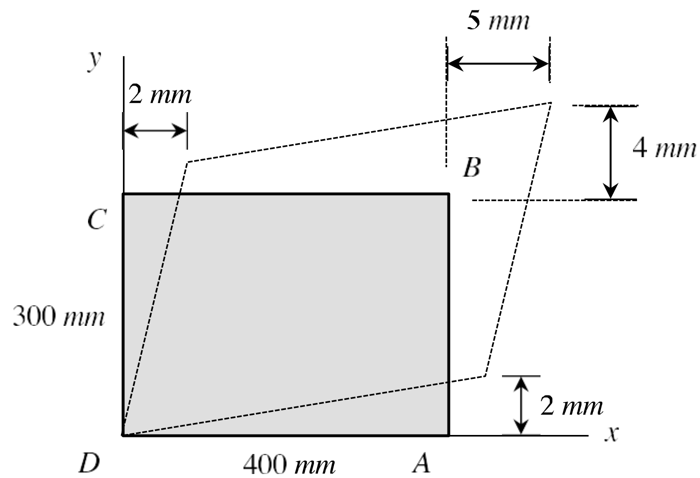
(a) What is the value of the Poisson's ratio?

(b) What is the value of $\sigma_y$?



### *Example 5.4*

Recall the general definition of strain. Find the normal and shear strains along $x$ and $y$ at point D.



## *Example 5.6*

When thin sheets of material, like the top "skin" of the airplane wing in the following figure is subjected to stresses, they are said to be in a state of plane stress, with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. For the case that $\Delta T = 0$, show that the Hooke's may can be written as

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_x + \upsilon\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_y + \upsilon\varepsilon_x\right)$$