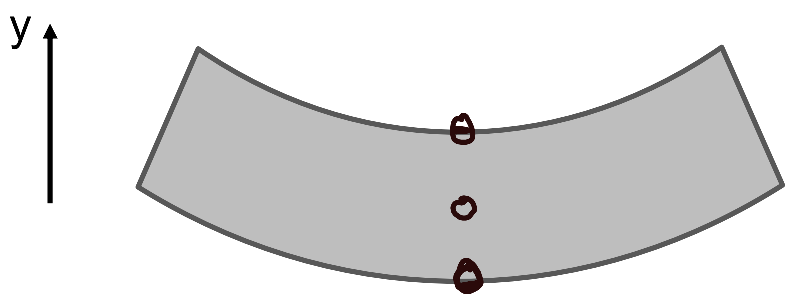
For a rectangular beam cross section under positive bending, the most positive value of the normal stress is:

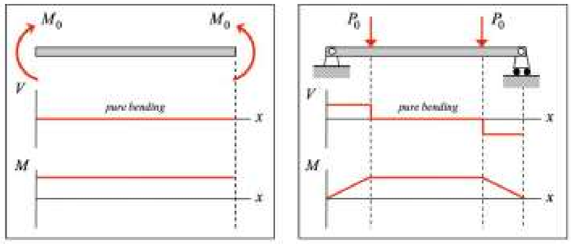
- At the top of the beam (+ y)
- At the center of the beam
- At the bottom of the beam (- y)



$\sigma = -\frac{My}{I}$

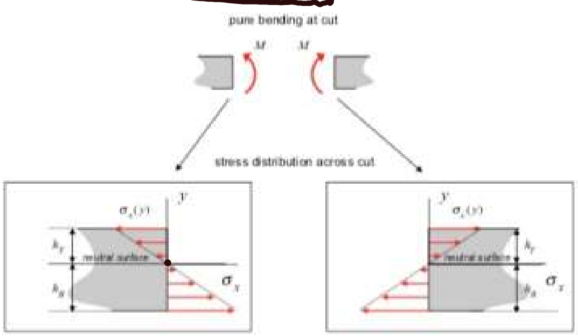
Summary: flexural stresses in pure bending in beams

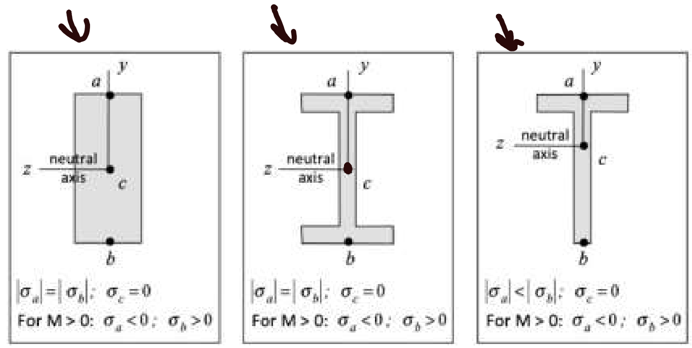
- *Pure bending*: locations on a beam for which the shear force is zero. Examples:



- *Flexural s*tresses in pure bending:

$$\boxed{\sigma = -\frac{My}{I}}$$ (linearly-varying on the cross-section with y measured from *neutral surface*)

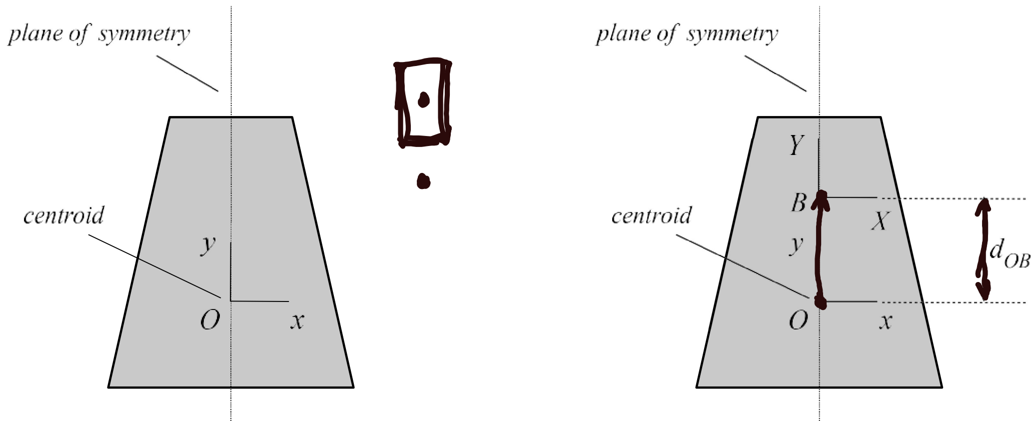




- Where on the cross-section is the flexural stress the greatest?

Second area moment of a cross section

Consider the beam cross section shown below left that is symmetrical about the y-axis but with no symmetry assumptions about the x-axis, where the origin of the x-y axis, O, is placed at the centroid of the cross section.



In the preceding derivation of the stress distribution across a cross section:

$$\sigma_x = -\frac{M\,y}{I_O} \tag{6}$$

we saw that this relationship depends on the "second area moment" I_O for the cross section:

$$I_O = \int_A y^2\, dA$$

where y is measured from the centroid of the cross section. Note that this parameter depends solely on the shape of the cross section and does not depend on either the material properties of the beam or the strain in the beam.

Tabulated expressions for the centroidal second area moments for a number of common beam cross sections are provided on the following pages.

For reasons that we will discuss later on, we often times need to know the second area moment about points on the plane of symmetry but not at the centroid of the cross section. Consider point B shown in the figure above right that is located at a distance d_{OA} from the centroid O on the plane of symmetry. Suppose we place a set of *X-Y* coordinate axes with its origin at A such that $X = x$ and $Y = y - d_{OB}$ Therefore, the second area moment about point B is found from:

$$I_B = \int_A Y^2\, dA = \int_A \left(y - d_{OB}\right)^2 dA$$

$$= \int_A \left(y^2 - 2d_{OB}y + d_{OB}^2\right) dA$$

$$= \int_A y^2\, dA - 2d_{OB}\int_A y\, dA + d_{OB}^2 \int_A dA$$

$$= I_O - 2d_{OB}\bar{y}A + Ad_{OB}^2$$

where A is the area of the cross section and $\bar{y}$ is the y-position of the centroid of the area. Since the origin O for the x-y axes is located at the centroid of the cross section, we have $\bar{y} = 0$. Therefore,

$$\boxed{I_B = I_O + Ad_{OB}^2} \rightarrow \text{tool.} \qquad (7)$$

Equation (7) is the "*parallel axis theorem*" for second areas of moments. In words, in order to determine the second area moment about an arbitrary point B on the plane of symmetry, simply add Ad_{OB}^2 to the centroidal second area moment I_O, where d_{OB} is the distance between O and B.

In general, one needs to perform an integration over the cross section of the beam in order to evaluate this integral representation for I_O. We have seen this process in the earlier examples. However, for certain cross sections, we can use results from simple shapes to construct the overall second area moment for the cross section. To this end, we will need to use the above parallel axis theorem. This process is demonstrated in the following examples.

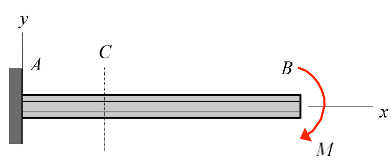
Example 10.5

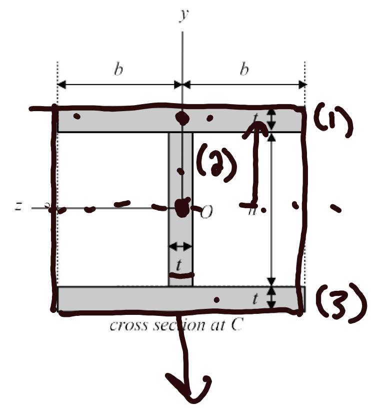
The cantilevered beam shown below is loaded in pure bending. The beam has a cross section at location C on the beam as shown below right. The origin O is located on the neutral axis of the beam.

a) Determine the second area moment I_{Oz} corresponding to the neutral axis of the beam.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) normal stress occurring on the cross-sectional face at C.





a) $I_o = (I_1)_o + (I_2)_o + (I_3)_o$

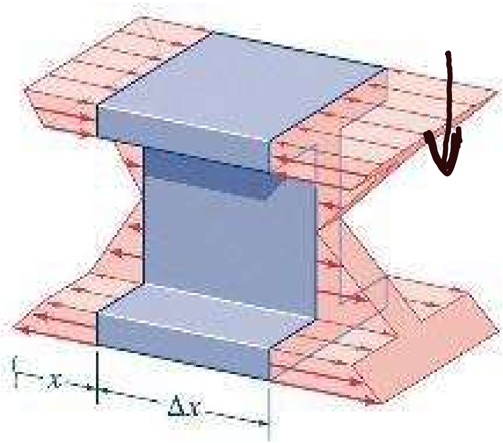
$(I_2)_o = \frac{th^3}{12}$

$(I_1)_o = \frac{2bt^3}{12} + A d_{OB}^2$ $\qquad d_{OB} = \frac{h}{2} + \frac{t}{2}$ $\qquad A = 2bt.$

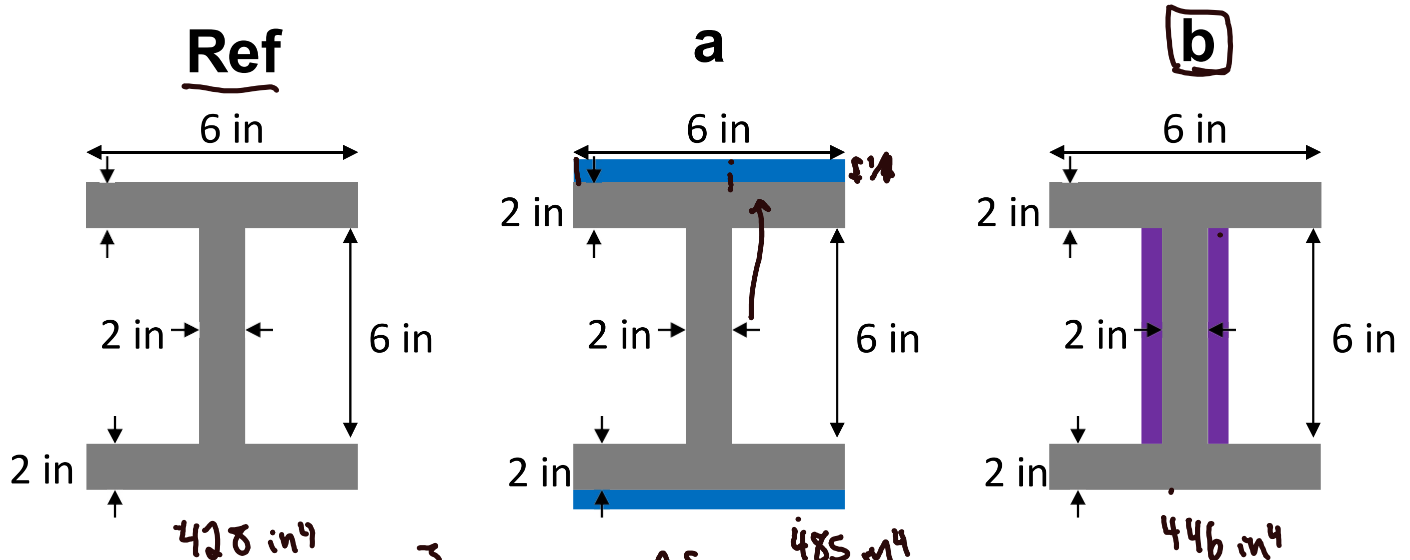
$(I_3)_o = (I_1)_o = \frac{2bt^3}{12} + 2bt\left(\frac{h}{2} + \frac{t}{2}\right)^2$

b) $\sigma = -\frac{My}{I_o}$ $\qquad$ c) $\sigma_{max} = \frac{-M\left(\frac{h}{2}+t\right)}{I_o}.$

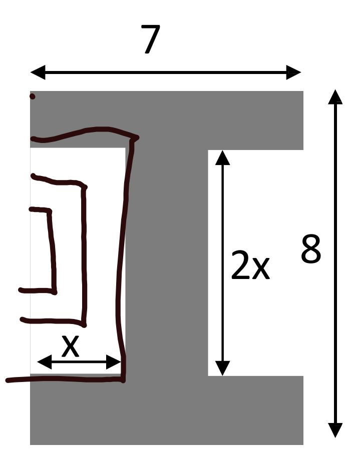
Bending of I-Beams

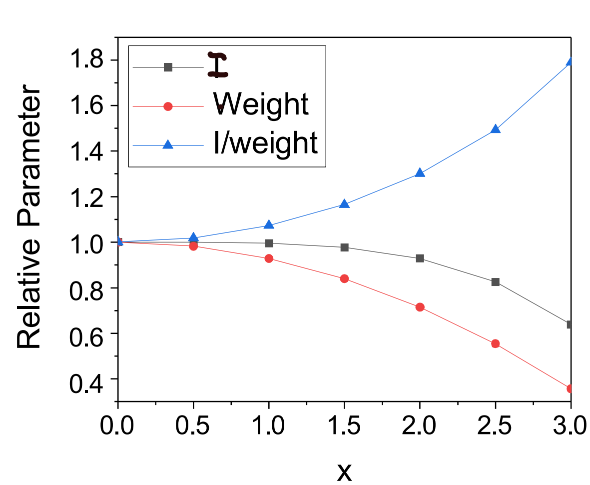


How can we increase the strength of the beam?



$$I = \frac{2(6)^3}{12} + 2\left[\frac{6(2)^3}{12} + 6*2*(4)^2\right]$$





Example 10.4

The beam shown below is loaded in pure bending. The beam has a cross section at location C on the beam as shown below right. The origin O is located on the neutral axis of the beam.

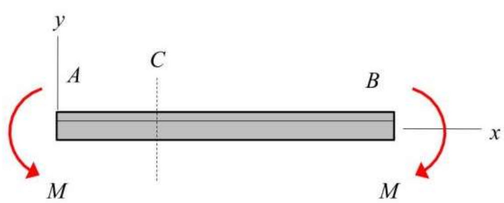
a) Determine the location of the centroid for the cross of this beam; i.e., what is the distance d?

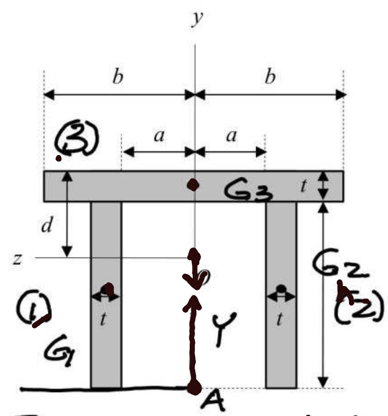
b) Determine the second area moment I_{Oz} corresponding to the neutral axis of the beam.

c) Determine the distribution of normal stress on the cross section of the beam as a function of y.

d) Determine the maximum (magnitude) normal stress occurring on the cross-sectional face at C.

Use the following dimensions: $M = 2000\ N \cdot m$, $t = 20\ mm$, $b = 80\ mm$, $a = 40\ mm$ and $h = 80\ mm$.





cross section at C

$d = h + t - \bar{y}$

a) $\bar{y}(A_1 + A_2 + A_3) = A_1 \bar{y}_1 + A_2 \bar{y}_2 + A_3 \bar{y}_3$

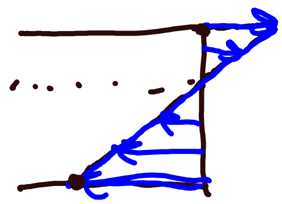
$A_1 = A_2 = th$ $\qquad \bar{y}_1 = h/2 = \bar{y}_2$

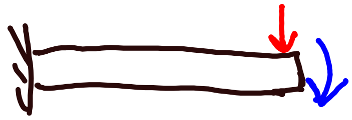
$A_3 = 2bt$ $\qquad \bar{y}_3 = h + t/2.$

$$\bar{y} = \frac{h^2 + 2b(h + t/2)}{2(h+b)}$$

b). $I_0 = (I_1)_0 + (I_2)_0 + (I_3)_0$

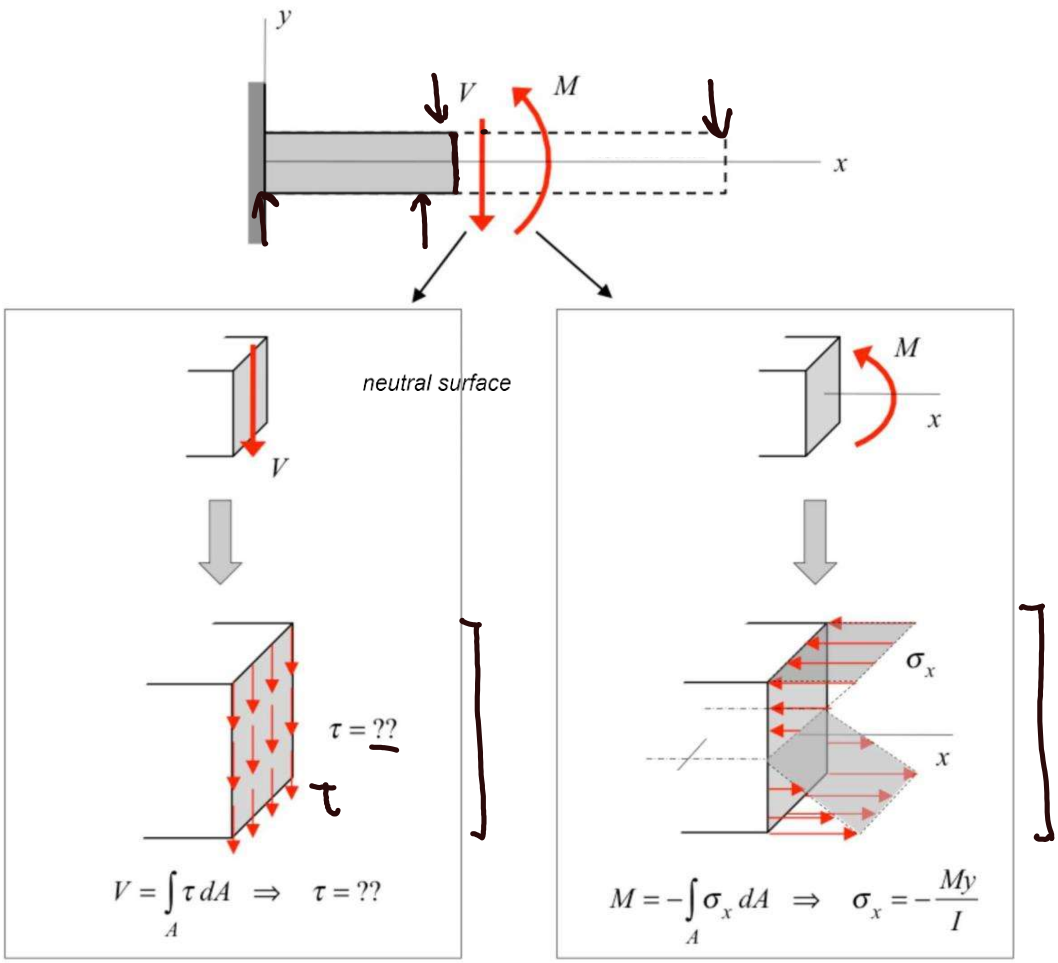
$$(I_1)_0 = I_1 + A_1\left(\frac{h}{2} + t - d\right)^2 = I_1 + A_1\left(-\frac{h}{2} + \bar{y}\right)^2$$





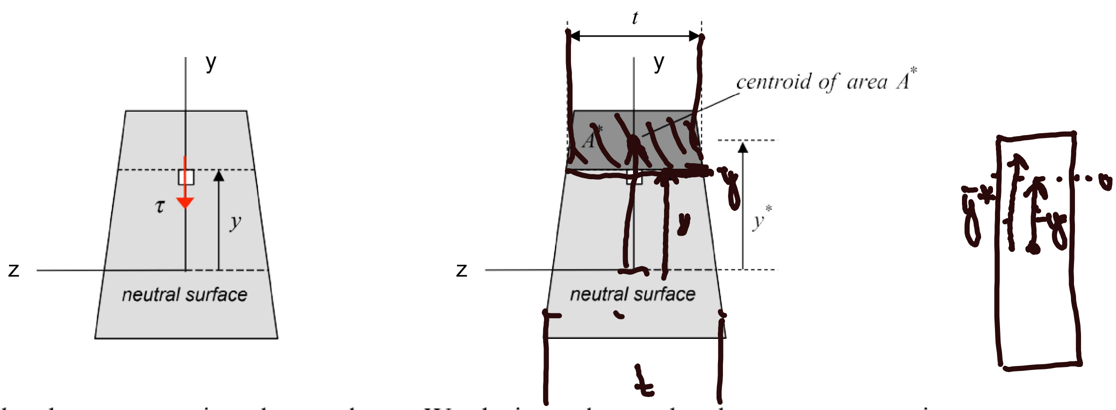
c) *Stresses due general transverse force and bending-couple loading of beams*

Earlier in the chapter, we considered the normal stress distribution within the cross section of a beam experiencing pure bending (i.e., in the absence of a shear force resultant on the cross-sectional cut). Here we will now consider the more general case of having both shear force and bending moment couples on the cross-sectional cut, as demonstrated by the figure below.



We have seen that the normal stresses due to the bending moment M are linearly distributed over the cross section, with maximum magnitudes of normal stress occuring on the outer fibers of the beam and with zero normal stress at the neutral axis (the neutral axis passing through the centroid of the cross section).

With the shear force V now added to the cross-sectional cut, we now need to determine the shear stress distribution on the cross section. With our earlier assumptions of symmetry of the beam cross section about the *xy*-plane, we know that the distribution of the shear force will be constant through the depth of the beam (*z*-direction). For the case of direct shear (zero bending moment), the shear stress was also constant in the *y*-direction, making shear force constant throughout the cross section a constant. However, the presence of the bending moment induces a redistribution of shear stresses in the *y*-direction.



Consider the cross section shown above. We desire to know the shear stress τ acting on a stress element at a distance of y from the neutral surface. This shear stress along the axis of symmetry (the y-axis) can be expressed as:

$$\tau = \frac{VQ}{It} = \frac{V A^*(y)\,\bar{y}^*(y)}{I\,t(y)} \qquad (8)$$

where:

V = shear force at cross section

$Q = A^* \bar{y}^*$

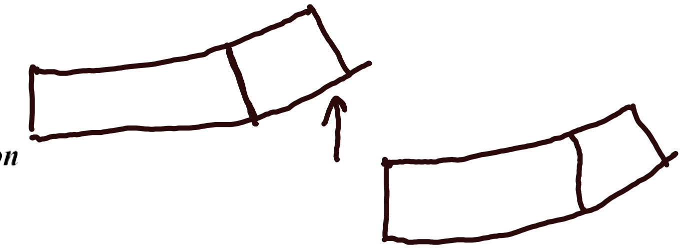
A^* = cross-sectional area above the element

$\bar{y}^*$ = centroid of the area above the element

I = centroidal second area moment for the entire cross section

t = depth dimension of the beam at the location of the stress element of interest

The derivation of equation (8) will be presented on the following pages.



Derivation of the shear stress distribution equation

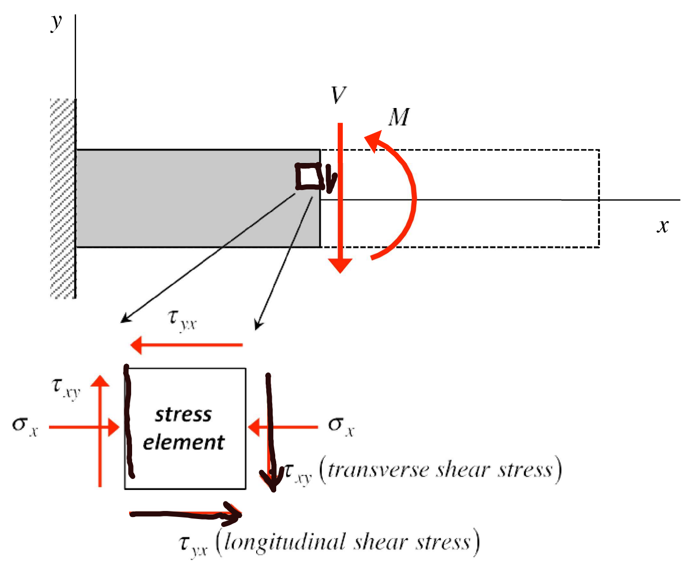
Background:

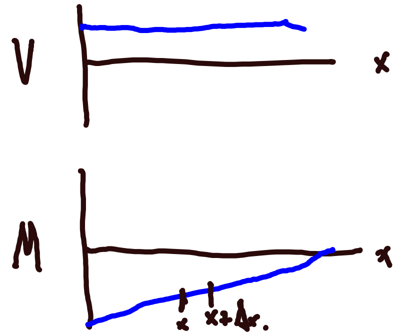
a) Recall that in the derivation of the equation for the normal stress distribution for pure bending:

$$\sigma_x = -\frac{M\,y}{I} \tag{6}$$

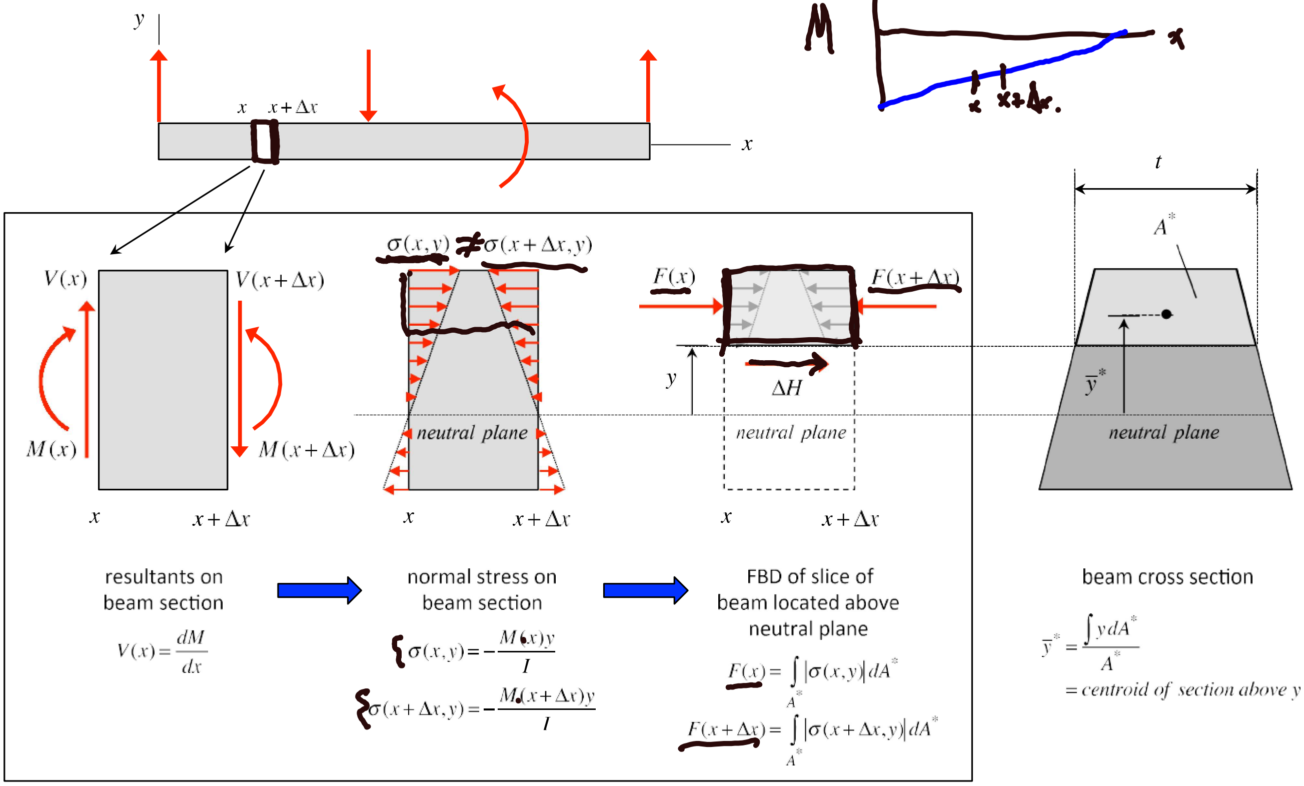
we assumed that plane sections of the cross section remain plane, and that they remain perpendicular to the deformed axis of the beam. For the more general situation in which a shear force V acts along with the bending moment M, a component of shear stress will exist. As we have seen earlier, the resulting shear strains correspond to a change in angle of the stress element. This angle change is somewhat in contradiction with the pure bending assumption of the cross section remaining perpendicular to the deformed beam axis. For our derivation, *we will assume that the shear strain effects will be slight and that, even in the presence of shear stress, the distribution of flexural stress on a given cross section is unaffected by the deformation due to shear* and that *equation (6) is still valid for computing the normal stresses on the cross section.*

b) Suppose we consider a stress element on the side of a beam with a non-zero shear force resultant on the face of the cut. Our goal here is to determine the *transverse* shear stress component τ_{xy} that corresponds to the shear force resultant V. Note, however, that since $\tau_{yx} = \tau_{xy}$, the transverse shear stress component τ_{xy} is the same as the longitudinal shear stress component τ_{yx}. Stated in different words, we can determine the transverse shear stress by calculating the longitudinal shear stress. This will be the process that we will use here in deriving equation (8).





Derivation of shear stress equation:



Consider the aribitrarily-loaded beam shown above. Here we isolate a section of the beam between locations x and $x+\Delta x$, with the resultant shear forces and bending moments acting on this section, as shown above left. The resultant bending moments $M(x)$ and $M(x+\Delta x)$ produce normal stresses of $\sigma(x)$ and $\sigma(x+\Delta x)$ on the left and right faces of the beam section, respectively. Suppose we further isolate a slice of this beam section found *above* a given value of y on the beam cross section. As shown in the above figure, the resultants of the normal components of stress on the left and right faces are given by $F(x)$ and $F(x+\Delta x)$, respectively. A resultant longitudinal shear force ΔH also acts on the lower surface of the slice at y. From static equilibrium of the slice we have:

$$\sum F_x = F(x) - F(x+\Delta x) + \Delta H = 0 \quad \Rightarrow \quad \Delta H = F(x+\Delta x) - F(x)$$

The shear stress corresponding to this resultant shear force is found from the usual definition of stress in terms of the force resultant as:

$$\tau = \lim_{\Delta x \to 0}\left(\frac{\Delta H}{t\Delta x}\right) = \frac{1}{t}\lim_{\Delta x \to 0}\left(\frac{F(x+\Delta x)-F(x)}{\Delta x}\right) = \frac{1}{t}\frac{dF}{dx} \qquad (9)$$

$\sigma(y) = -\frac{My}{I}$

From the above we have:

$$F(x) = \frac{M(x)}{I}\int_{A^*} y\,dA^* = \frac{\bar{y}^* A^*}{I} M(x) \qquad (10)$$

where A^* and $\bar{y}^*$ are the area and the centroid of the area of the cross section above y. Combining equations (9) and (10) gives:

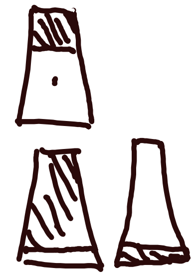
$$\tau = \frac{A^* \bar{y}^*}{It} \frac{dM}{dx} \qquad (11)$$

Finally, recall that from equilibrium analysis that $V = dM / dx$. Therefore, (11) becomes:

$$\boxed{\tau = \frac{VA^* \bar{y}^*}{It}} \qquad (8)$$

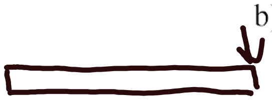
use this.

<u>*Comments on the usage of the shear stress equation*</u>

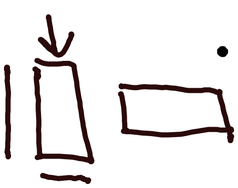


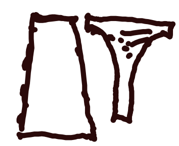
a) Note that this derivation was based on considering a slice of the beam section ABOVE the location y; hence, we ended up with $A^* \bar{y}^*$ representing the area *above* y. Alternately, we could have easily kept a slice of the section BELOW position y. In that case $A^* \bar{y}^*$ in the equation would then represent that area *below* y. We will get the same magnitude for the shear stress using the area below y as if we consider the area above y.

b) There are limitations on the usage of this shear stress equation, as listed below.



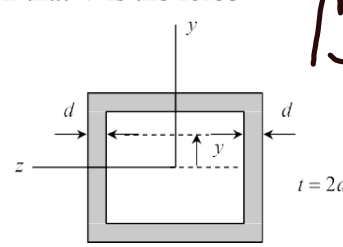
- Effect of load distribution: The assumptions of plane sections remaining plane and perpedicular to the neutral surface are valid for beams that are long compared to their depth. This assumption limits the influence of shear deformations in the beam and, hence, limits the error in the flexural stresses.





- Effect of cross section shape: The shear stress equation derived is particularly accurate for beams that are thin in the depth dimension ("t") and for which this dimension t does not vary rapidly with y. For thin-walled beams, the shear stress equation is valid for sections of the cross section that are aligned with the y-axis, and most accurately so near the neutral plane.

c) Other remarks on the shear stress equation:

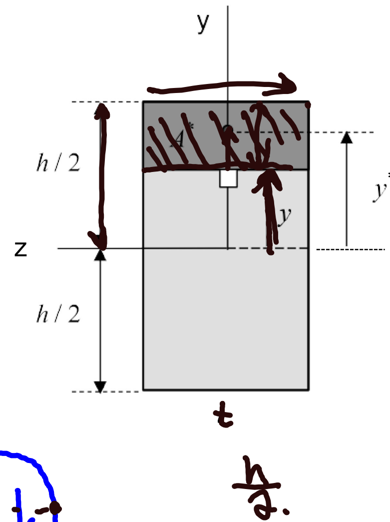


A*→0

- The sign of τ is the same as the sign on V. Also, recall that V is the force resultant of the shear stress: $V = \int \tau \, dA$
- I is the second area moment of the cross-section (independent of the location y).
- t is the net thickness of the beam at the location y.
- Regardless of the cross section, $\tau = 0$ at the top and bottom fibers of the beam.
- If the beam cross section is symmetric about the neutral axis, the maximum shear stress occurs at the neutral axis.

Example – shear stress distribution in a rectangular cross section

As an example, consider a rectangular cross section beam of dimensions of thickness h and depth t. From before, we know that the centroidal second area moment for a rectangular beam of these dimensions is $I = th^3/12$. For a stress element at y, we have:



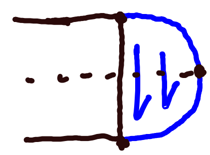
$$A^* = \left(\frac{h}{2} - y\right)t$$

$$\bar{y}^* = \frac{1}{2}\left(\frac{h}{2} + y\right) \qquad \bar{y}^* = y + \frac{1}{2}\left(\frac{h}{2} - y\right)$$

Combining the above gives:

$$\tau = \frac{V\left[(h/2 - y)t\right]\left[(h/2 + y)/2\right]}{\left(th^3/12\right)t}$$

$$\tau(y) = \frac{6}{h^3 t}\left(\frac{h^2}{4} - y^2\right)V = \frac{6}{Ah^2}\left(\frac{h^2}{4} - y^2\right)V$$



From this result, we observe the following for the shear stress distribution across a cut of a rectangular cross section beam experiences a shear force V:

- The stress distribution is quadratic with location y of the stress element.
- The shear stress is zero at the outer fibers of the beam ($y = \pm h/2$), as expected since these fibers experience no horizontal loads.
- The shear stress is a maximum at the neutral axis ($y = 0$). This maximum shear stress is given by:

$$\boxed{\tau_{max} = \frac{3V}{2A}}$$

- Recall that the average shear stress across the cut is given by $\tau_{ave} = V/A$, which would be the shear stress on the cut in the absence of a bending moment. From this we see that the bending moment produces a 50% increase in the maximum shear stress for a rectangular cross sectioned beam.