MECHANICS OF MATERIALS

Fall 2023

ME 323- 005

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Axial stress
Axial strain

Lecture 3: Shear Stress
Shear Strain

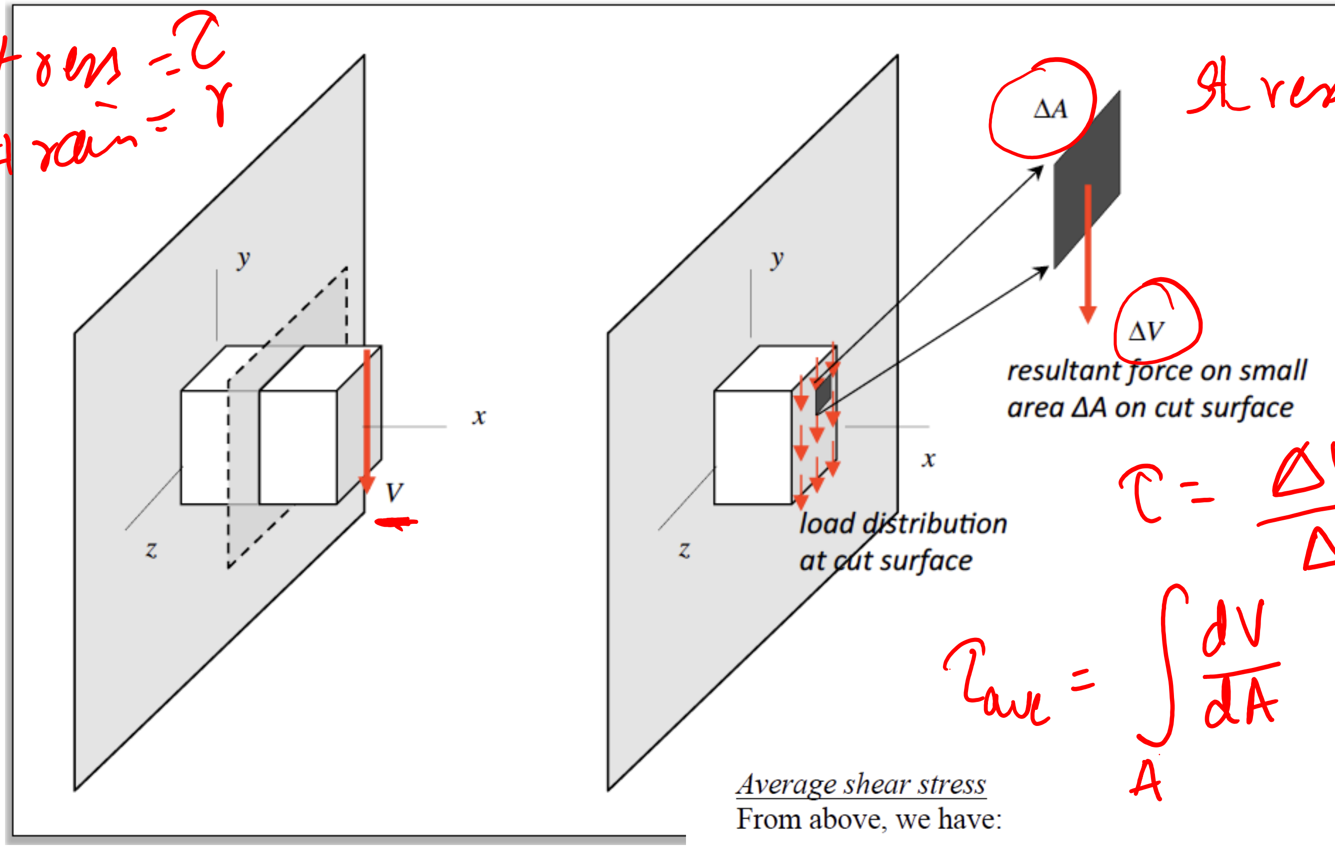
PURDUE UNIVERSITY | School of Mechanical Engineering

Shear Stress

Axial stress = σ
Axial strain = ϵ

Shear stress = τ
shear strain = γ

Stress = $\frac{\text{force}}{\text{area}}$



$$\tau = \frac{\Delta V}{\Delta A}$$

$$\tau_{ave} = \int_A \frac{dV}{dA}$$

$$\tau = \text{shear stress} = \lim_{\Delta A \to 0}\left(\frac{\Delta V}{\Delta A}\right) = \frac{dV}{dA} \quad \Rightarrow \quad dV = \tau dA$$

Average shear stress

From above, we have:

$$V = \int_{area} dV = \int_{area} \tau\, dA = \text{resultant shear force at the surface}$$

Using the definition of the average value of a function over an area:

$$\tau_{ave} = \frac{1}{A}\int_{area} \tau\, dA$$

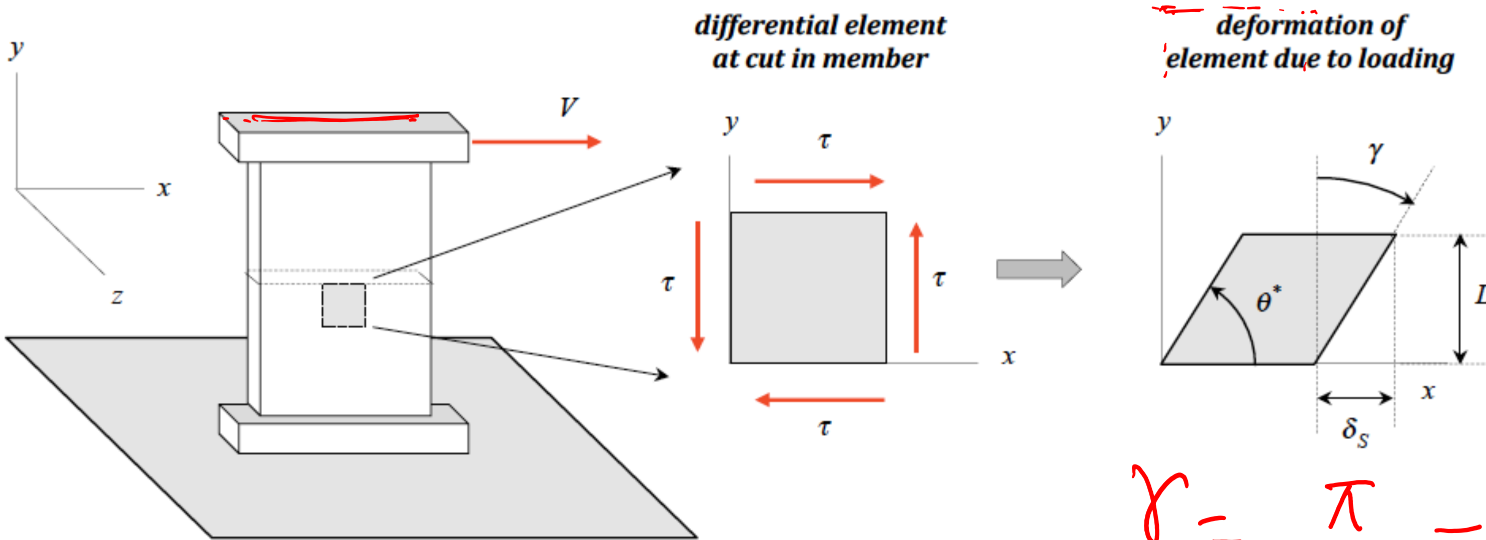
we see that:

$$\boxed{\tau_{ave} = \frac{V}{A}}$$

Shear Strain

Hooke's Law $\sigma = E\varepsilon$

strain = $\dfrac{\text{change in dimens}}{\text{initial dimens}}$



$$\gamma = \frac{\pi}{2} - \theta$$

$$\tan\gamma = \frac{\delta_s}{L_s}$$

Shear strain deformations produce skewing in a rectangular-shaped element: the angle between adjacent sides changes from $\pi/2$ to θ^*. Here, we will define the shear strain as γ representing this change in angle:

$$\gamma = shear\ strain = \frac{\pi}{2} - \theta^*$$

where, from the figure, we have:

$$tan\,\gamma = \frac{\delta_s}{L_s}.$$

For small strains, $tan\,\gamma \approx \gamma$, and therefore,

$$\gamma = \frac{\pi}{2} - \theta^* \approx \frac{\delta_s}{L_s}$$

$$\tau = G\gamma$$

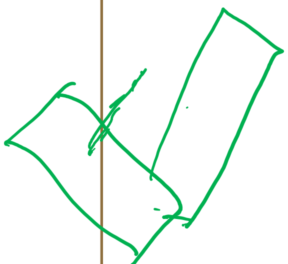
G = shear modulus

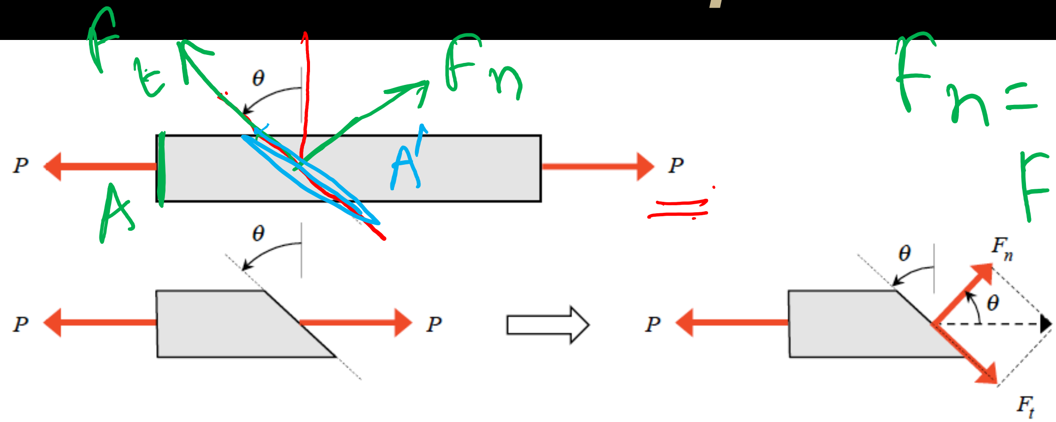
$$G = \frac{E}{2(1+\nu)}$$

$$\boxed{E = 2G(1+\nu)}$$

where E is the Young's modulus and ν is the Poisson's ratio for the material.

Normal Stress and Shear Components





$F_n = P\cos\theta$

$F_t = P\sin\theta$

$A' = \dfrac{A}{\cos\theta}$

As a result of this cut, we see two changes. First, the axial load now has both normal and tangential components on the cut face of:

$$F_n = P\cos\theta$$

$$F_t = P\sin\theta$$

$$\tau = \frac{F_t}{A'} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P\sin\theta\cdot\cos\theta}{A}$$

respectively. Second, the area of the cross section over which these resultant components act is larger:

$$A_c = \frac{A}{\cos\theta}$$

With this, the components of stress normal and tangent to the cut are now written as:

$$\sigma = \frac{F_n}{A_c} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}\left(1+\cos 2\theta\right)$$

$$\tau = \frac{F_t}{A_c} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\cos\theta\sin\theta = \frac{P}{2A}\sin 2\theta$$

From this, we see that the orientation of the cut through the member influences the values of normal and shear components of stress. Furthermore, we see that an axial loading can produce both normal and shear components of stress. (Note that the maximum shear stress occurs for a cut at $\theta = 45°$, where $\tau = P/2A$.)

Example 3.1 from Lecture Book

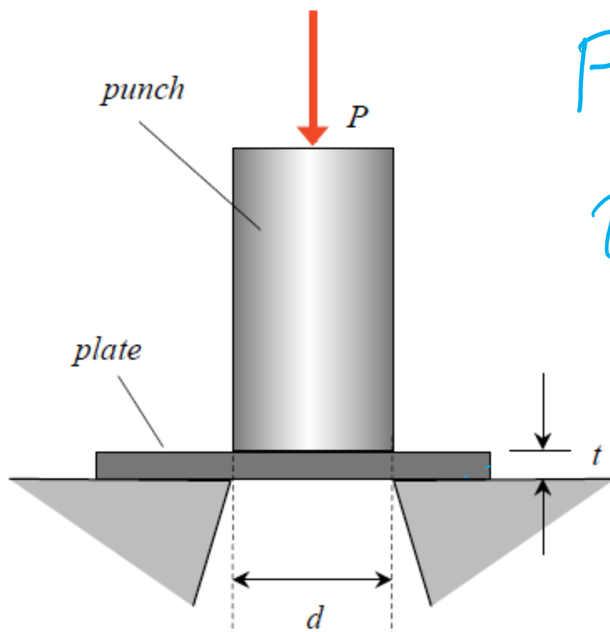
A hydraulic punch press that can apply a maximum punch force of P_{max} is used to punch circular holes in an aluminum plate of thickness t. If the average punching shear resistance of this plate is τ, what is the maximum diameter d of the hole that can be punched?

thickness t

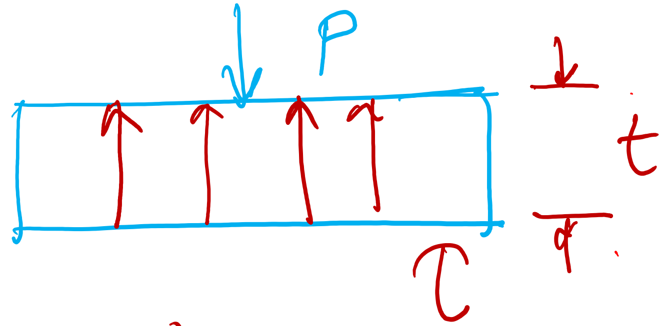
P $\quad$ P_{max}

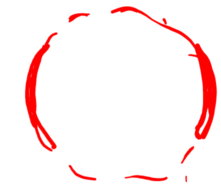
$\tau \rightarrow$ shear stress

$d_{max} = ?$



diameter of hole = d





area over which shear stress τ = A

$$P = \tau \underline{\underline{A}} = [(\pi d)t]\,\tau$$

$$P = (\pi d t)\,\tau$$

$$\underline{\underline{d}} = \frac{P}{\pi t \tau} \longrightarrow$$

circumference of hole = $\underline{\underline{2\pi(\text{radius})}} = \underline{\underline{\pi d}}$

area of hole = $\underline{\underline{\pi d t}}$

$$d = \frac{P}{\pi t \tau}$$

$$\boxed{d_{max} = \frac{P_{max}}{\pi t \tau}}$$

Example 3.2 from Lecture Book

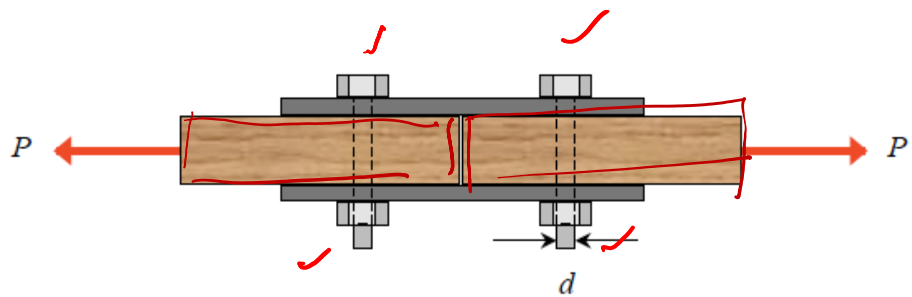
Two boards are spliced together using four bolts (each of diameter d) running through the boards and two steel splice plates, as shown below. The average direct shear stress experienced by each bolt is known to be τ. What is the value of the tensile load P carried by the spliced boards?

bolt diameter = d

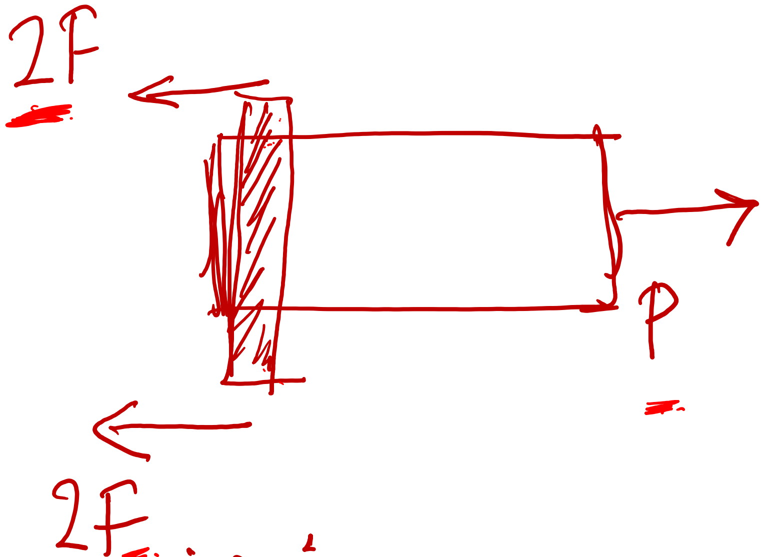
τ

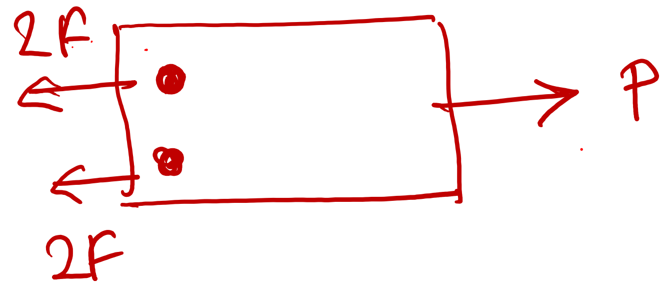
P = ?

TOP view



EDGE view





at equilibrium,

$$\sum F_x = P - 4F = 0 \Rightarrow P = 4F$$

F = resultant shear force.

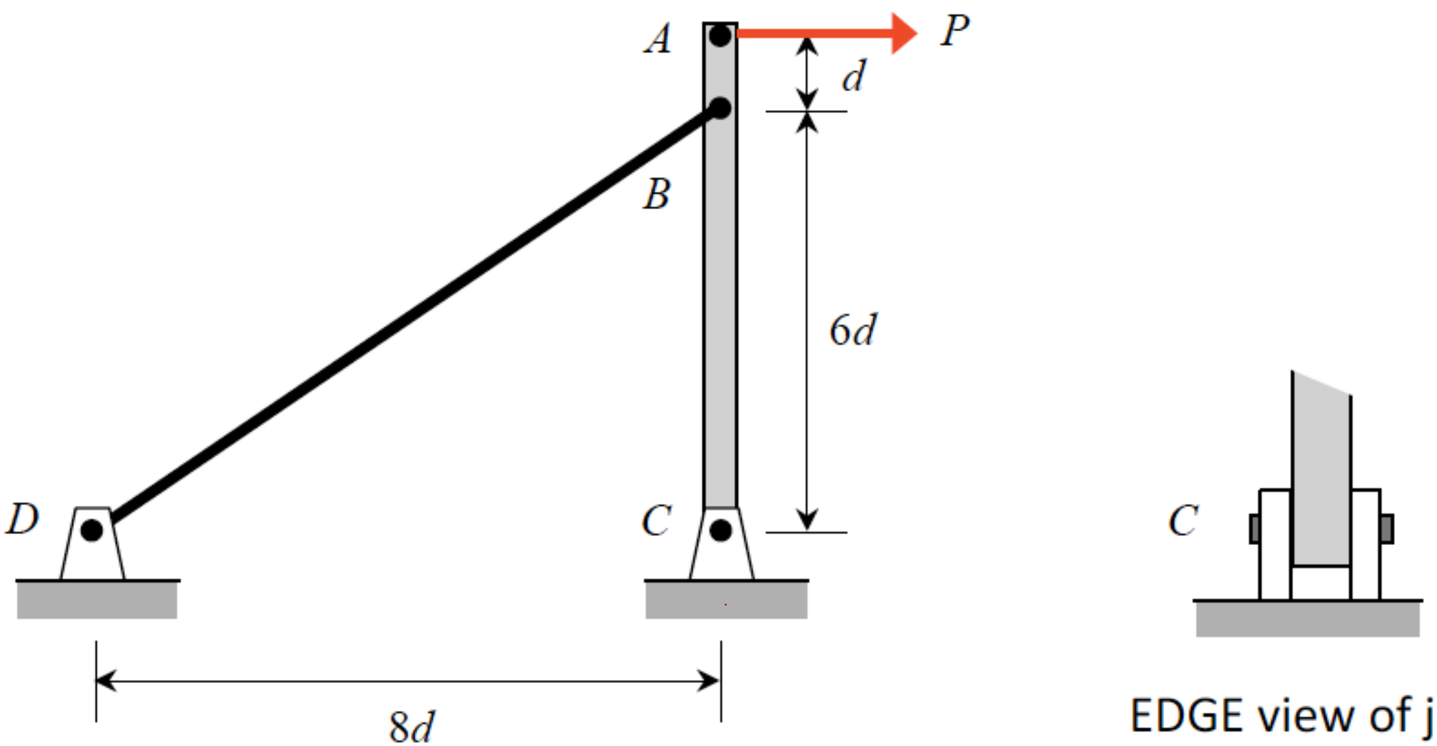
$$= \tau\left[\pi\left(\frac{d}{2}\right)\right]^2 = \tau\pi d^2/4.$$

$$P = 4\left(\frac{\tau\pi d^2}{4}\right) = \tau\pi d^2$$

$$\boxed{P = \pi \tau d^2}$$

Example 3.3 from Lecture Book

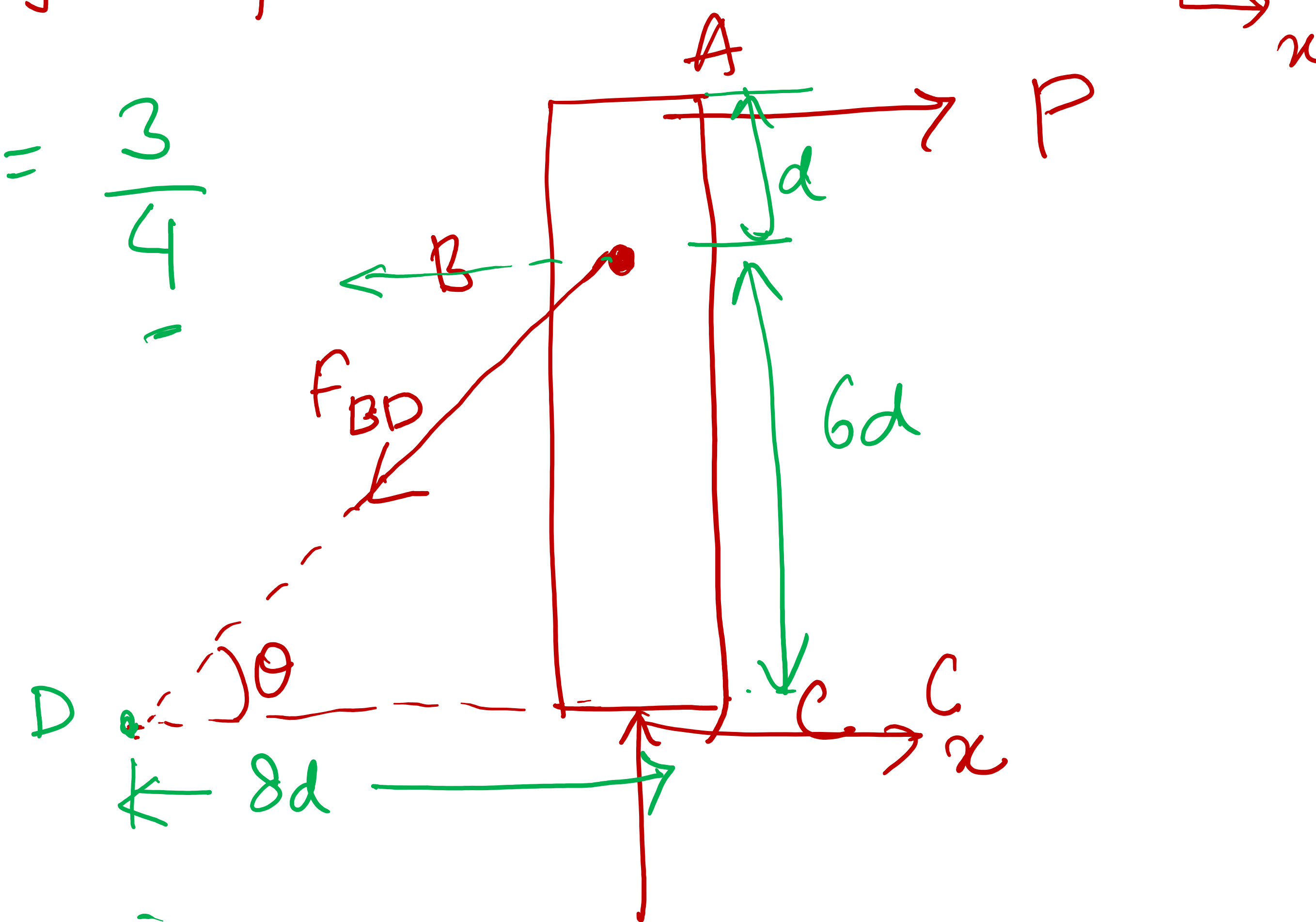
The pin at C has a diameter of D. Determine the average shear stress in the pin at C. Neglect the weight of link AC.



EDGE view of joint C

free body diagram ABC

$\tan\theta = \frac{6d}{8d} = \frac{3}{4}$



moment equation at B — Cy

$$(\Sigma M)_B = (C_x)(6d) - P(d) = 0$$

$$C_x = P/6$$

Sum forces —

$$\Sigma F_x = -F_{BD}\cos\theta + C_x + P = 0$$

$$F_{BD} = \frac{C_x + P}{\cos\theta} = \frac{P/6 + P}{\cos\theta}$$

$$F_{BD} = \frac{7P}{6\cos\theta}$$

$$\Sigma F_y = -F_{BD}\sin\theta + C_y = 0 \Rightarrow C_y = F_{BD}\sin\theta$$

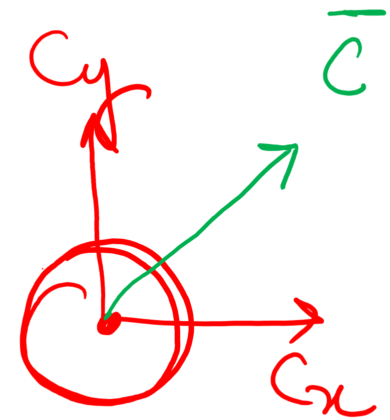
$$C_y = \left(\frac{7P}{6\cos\theta}\right)\sin\theta = \frac{7P}{6}\tan\theta$$

$$C_y = \frac{7P}{6}\left(\frac{3}{4}\right) = \frac{7P}{8}$$

$$C_x = P/6$$

$$C_y = 7P/8$$

$$F_{BD} = \frac{7P}{6\cos\theta}$$



$$\bar{C} = \sqrt{C_x^2 + C_y^2}$$

$$= \left[\left(\frac{P}{6}\right)^2 + \left(\frac{7P}{8}\right)^2\right]^{1/2} = 0.891\,P$$

= magnitude of total shear force at C.

Shear force carried by each face (double-sided connection)

$$\frac{|\bar{C}|}{2} = \frac{0.891 P}{2}$$

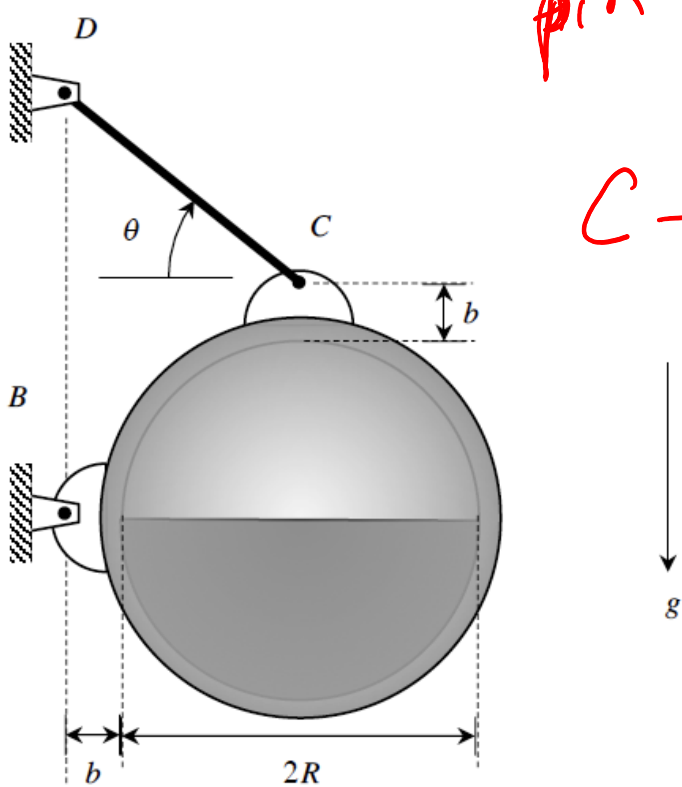
$$\tau_{ave} \text{ on each face} = \frac{\bar{C}/2}{\text{area}}$$

$$\text{area} = \pi \left(\frac{D}{2}\right)^2$$

$$\tau_{ave} = \frac{0.891 P/2}{\pi (D/2)^2} = \frac{0.445 P}{\pi D^2/4}$$

Example 3.5 from Lecture Book

A spherical tank (having an inner radius of R) is half-filled with a liquid that has a mass density of ρ. The tank is supported by a pin joint at B and by a cable at C (at the top of the tank). The pin joint at B is a double-sided connection, whereas the cable is attached to the tank at C with a single-sided connection. The pins at B and C are identical, each having a diameter of D. Determine the shear stress in pins B and C. Assume that the weight of the tank is negligible compared to the weight of the liquid that it contains.



pin joint B → double-sided

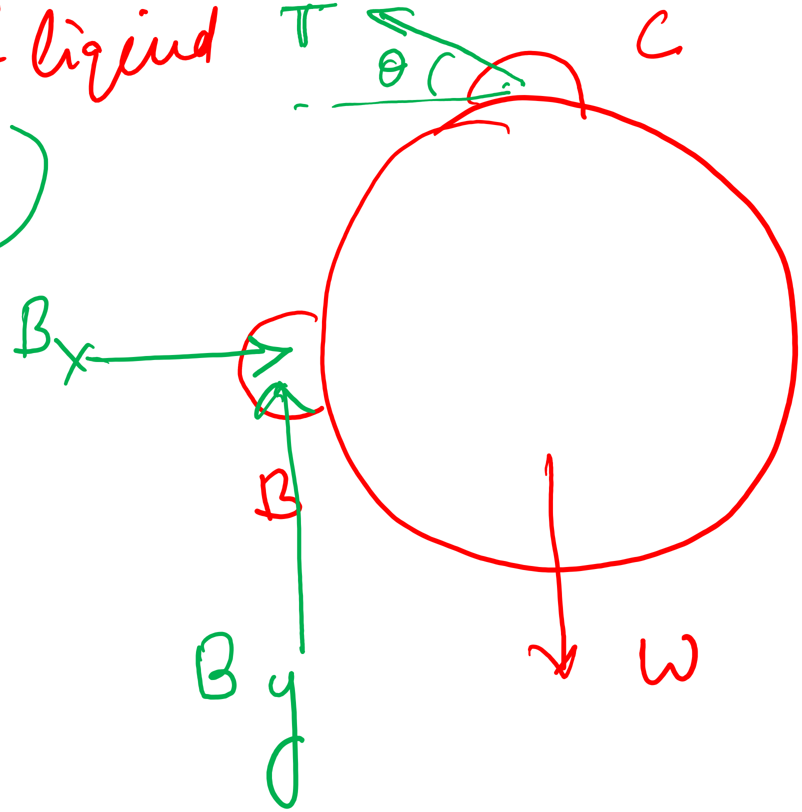
C → single sided.

pin diameter = D

$W \equiv$ weight of liquid

$$W = \rho\left(\frac{\frac{4}{3}\pi R^3}{2}\right)$$

$$W = \frac{2}{3}\rho\pi R^3$$



at equilibrium

$$\Sigma F_x = B_x - T\cos\theta = 0 \Rightarrow B_x = T\cos\theta \quad \text{—— (1)}$$

$$\Sigma F_y = B_y - W + T\sin\theta = 0$$

$$B_y = W - T\sin\theta \quad \text{—— (2)}$$

$$(\Sigma M)_B = -w(R+b) + (T\cos\theta)(R+b) + (T\sin\theta)(R+b) = 0$$

$$T = \frac{w\,\cancel{(R+b)}}{\cancel{(R+b)}\,[\cos\theta + \sin\theta]} = \frac{w}{\cos\theta + \sin\theta}$$

$$T = \frac{w}{\cos\theta + \sin\theta}$$

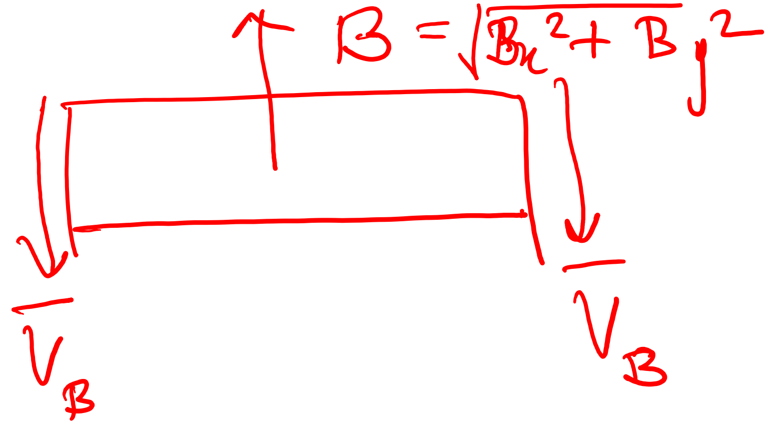
$$B_x = T\cos\theta = \frac{w\cos\theta}{(\cos\theta + \sin\theta)}$$

$$B_y = w - T\sin\theta = \left[\frac{w\cos\theta}{\cos\theta + \sin\theta}\right]$$

B → double-sided, C → single sided

Pin B

$$\Sigma F_B = B - 2\bar{V}_B = 0$$



$$\bar{V}_B = \frac{B}{2} = \frac{\sqrt{B_x^2 + B_y^2}}{2}$$

$$\tau_{ave} = \frac{\bar{V}_B}{A} = \frac{\bar{V}_B}{\pi \left(D/2\right)^2} = \frac{4}{\pi}\frac{\sqrt{B_x^2 + B_y^2}}{2D^2}$$

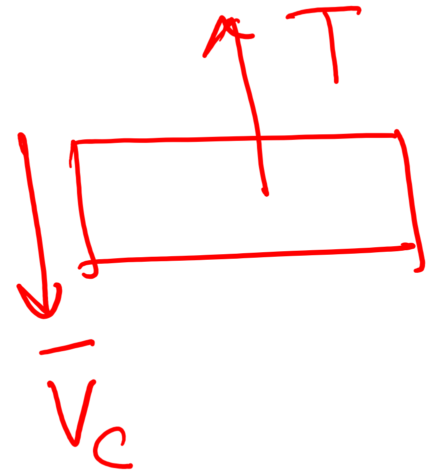
$$= \frac{2}{\pi D^2}\left[2\left(\frac{w\cos\theta}{\cos\theta + \sin\theta}\right)^2\right]^{1/2}$$

$$\tau_{ave} = \left[\frac{2\sqrt{2}\; w\cos\theta}{\pi D^2 (\cos\theta + \sin\theta)}\right]$$

$$\tau_{ave}\Big|_B = \frac{2\sqrt{2}\,\omega\cos\theta}{\pi D^2(\cos\theta+\sin\theta)}$$

for pin C —

$$\Sigma F = T - \bar{V}_c = 0$$



$$\bar{V}_c = T$$

$$\tau_{ave,c} = \frac{\bar{V}_c}{A} = \frac{T}{\pi\left(D/2\right)^2}$$

$$\tau_c = \frac{\omega/(\cos\theta+\sin\theta)}{\pi D^2/4}$$

$$\boxed{\tau_c = \frac{4W}{\pi D^2(\cos\theta + \sin\theta)}}$$

$$\boxed{\tau_B = \frac{2\sqrt{2}\, W \cos\theta}{\pi D^2(\cos\theta + \sin\theta)}}$$

Example 3.6 from Lecture Book

A prismatic bar has a equilateral triangle cross section, with the sides of the equilateral triangle being b. Determine the normal and shear components of stress on a cross section of the bar with the cross section oriented at an angle of θ from the y axis, as shown in the figure below. At what angle θ is the shear stress a maximum, and what is the value of the maximum shear stress? $\tau_{max} = ?$

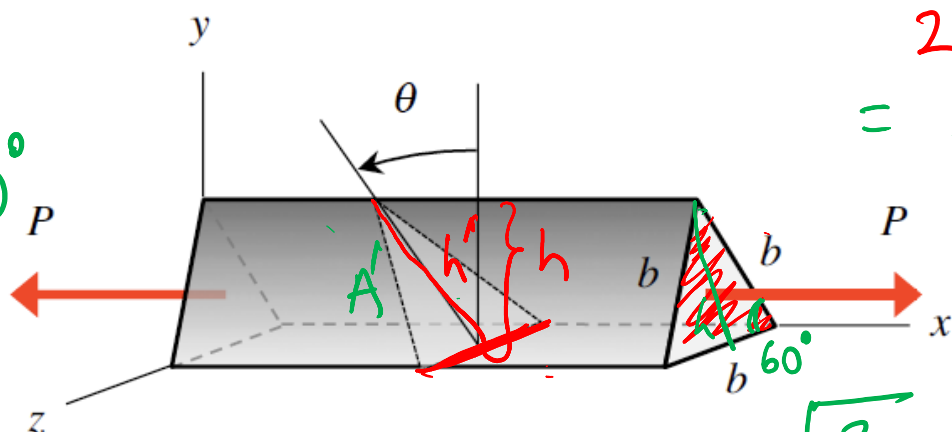
$$\tan 60^\circ = \frac{h}{b/2}$$

$$h = \frac{b}{2}\tan 60^\circ$$

$$\boxed{h = \frac{\sqrt{3}}{2} b}$$

$$\text{area} = \frac{1}{2}(b)(h)$$

$$= \frac{1}{2}(b)\left(\frac{\sqrt{3}}{2} b\right)$$



$$\text{area} = \frac{\sqrt{3}}{4} b^2$$

$$\cos\theta = \frac{h}{h'} \Rightarrow h' = \frac{h}{\cos\theta} \checkmark$$

$$\text{area of } A' = \frac{1}{2}(b)\left(\frac{h}{\cos\theta}\right)$$

$$A' = \frac{1}{2}(b)(h') = \frac{1}{2}(b)\left(\frac{h}{\cos\theta}\right)$$

$$h = b\sin\theta = \frac{\sqrt{3}}{2}b$$

$$A' = \frac{1}{2}(b)\left(\frac{\sqrt{3}\,b}{2\cos\theta}\right) = \frac{\sqrt{3}\,b^2}{4\cos\theta}$$

$$\sigma = \frac{P\cos\theta}{A'} = \frac{P\cos\theta}{\sqrt{3}\,b^2/4\cos\theta}$$

$$\boxed{\sigma = \frac{4P\cos^2\theta}{b^2\sqrt{3}}}$$

$$\tau = \frac{P\sin\theta}{\frac{\sqrt{3}}{4}\left(\frac{b^2}{\cos\theta}\right)} = \frac{4P\sin\theta\cdot\cos\theta}{b^2\sqrt{3}}$$

$$\boxed{\tau = \frac{4P\sin\theta\cdot\cos\theta}{b^2\sqrt{3}}}$$

$\theta = ?$

$$\tau_{max} \Rightarrow \frac{\partial\tau}{\partial\theta} = 0$$

$$\frac{d}{d\theta}\left[\frac{4P\sin\theta\cdot\cos\theta}{b^2\sqrt{3}}\right] = 0 \Rightarrow \frac{4P}{b^2\sqrt{3}}\frac{d}{d\theta}\left[\sin\theta\cdot\cos\theta\right] = 0$$

$$\cos\theta\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\cos\theta) = 0$$

$$(\cos\theta)(\cos\theta) + \sin\theta(-\sin\theta) = 0$$

$$\cos^2\theta = \sin^2\theta \Rightarrow \theta = 45^\circ$$

$$\tau_{max} = \frac{4P\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}{b^2\sqrt{3}} = \frac{2P}{b^2\sqrt{3}}$$

$$\boxed{\tau_{max} = \frac{2P}{b^2\sqrt{3}}}$$

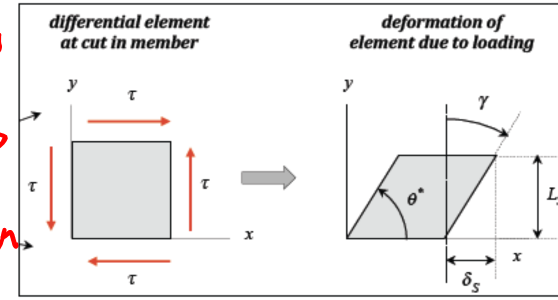
Summary

- *SHEAR STRAIN AND STRESS:*

$$\gamma = \frac{\delta_s}{L_s}$$

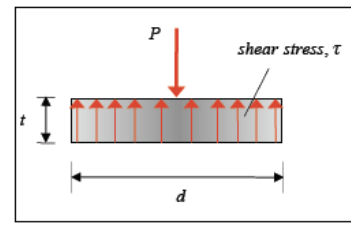
$$\tau = G\gamma \quad ; \quad G = \frac{E}{2(1+\nu)}$$

τ = shear stress
G = shear modulus
γ = shear strain



- *APPLICATIONS:*
  - Punching a circular hole:

$$\tau = \frac{P}{A} \quad ; \quad A = \pi d t \quad \left(for\ a\ circular\ hole\right)$$

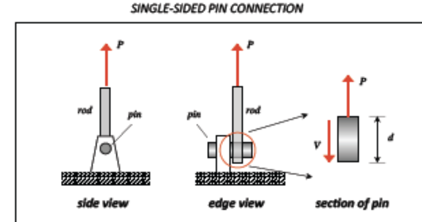


FBD of sheet metal slug under punch

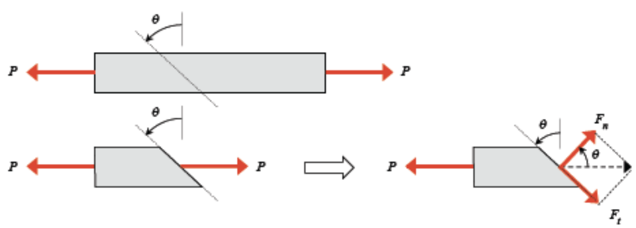
  - Shear stress in pin:

$$\tau = \frac{V}{A} \quad ; \quad A = \pi\left(d/2\right)^2 \quad \left(for\ a\ circular\ pin\right)$$

SINGLE-SIDED PIN CONNECTION



- *YES, THERE IS SHEAR STRESS IN AXIAL LOADING!*



$$\sigma = \frac{F_n}{A_c} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta = \frac{P}{2A}\left(1+\cos 2\theta\right)$$

$$\tau = \frac{F_t}{A_c} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P}{A}\cos\theta\sin\theta = \frac{P}{2A}\sin 2\theta$$

THANK YOU