# *A.1 Summary of axially-loaded members*

**Summary topics:**

a) Summary of fundamental results

b) Problem solving method

c) Sign conventions

d) Force/elongation and torque/angle of twist equations

e) Compatibility equations

f) Determinate and indeterminate problems

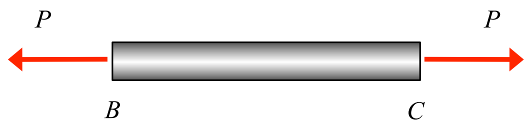
***Lecture material***

Up to this point in the course, we have focused on straight structural members whose loading has been aligned with the member axis, either through an axial force on a rod or an axial torque on a shaft. Although each type of problem is unique in its own way, the approach that we use in solving the stress analysis problem is essentially the same for both. In this section of the notes, we will review this approach, giving an emphasis on the similarities in its application to each problem.

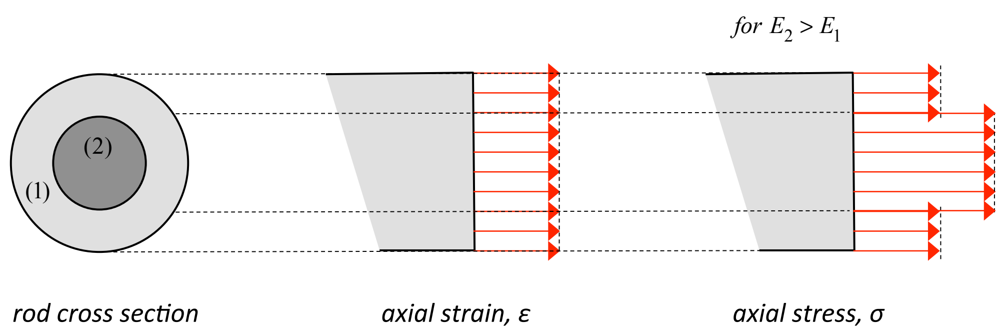
*a) Summary of fundamental results*

Before reviewing the solution methods, let us first review some fundamental results from each problem.

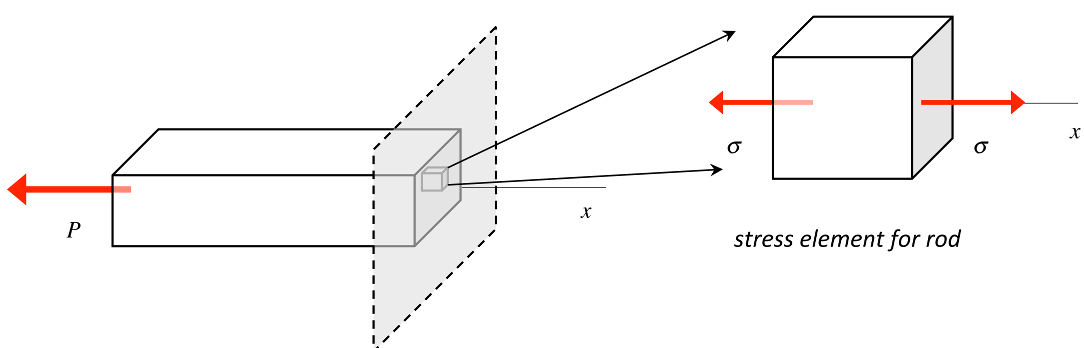
*Axially-loaded rod*



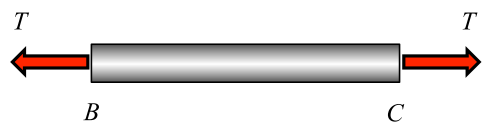

The axial strain $\varepsilon$ is *constant* across a given cross section of the rod. Since the axial stress is given by $\sigma = E\varepsilon$ the stress is also constant across a given cross section if the Young's modulus is constant; however, if E varies across a cross section, then the axial stress will vary also. Say we consider a rod made up of two materials: material (1) shaped as a tube and material (2) as an inner core of the tube, with Young's moduli of $E_2 > E_1$. Since the two materials experience the same strain, we conclude that the axial stresses are related by $\sigma_2 > \sigma_1$.



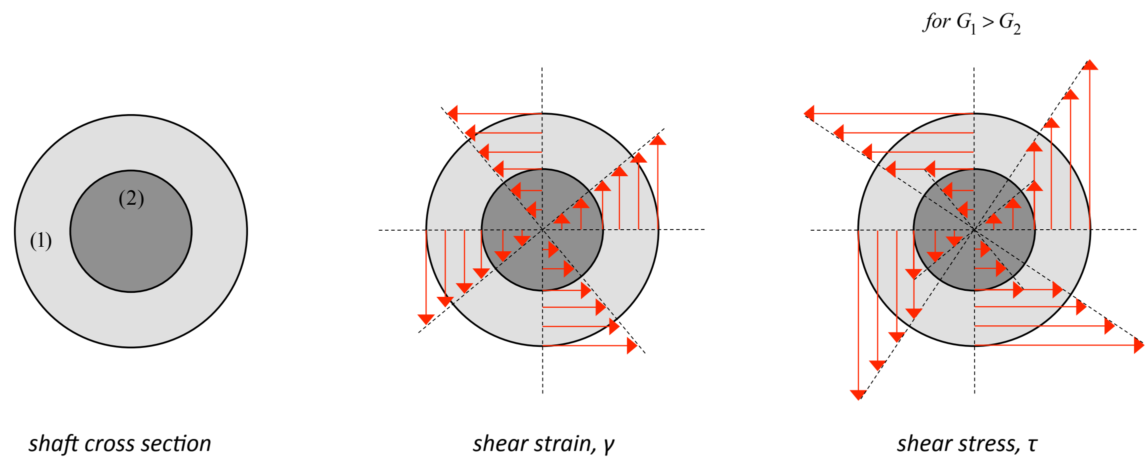

A stress element on a cut made perpendicular to the member axis is as shown below with a normal component of stress on the $\pm x$ faces.



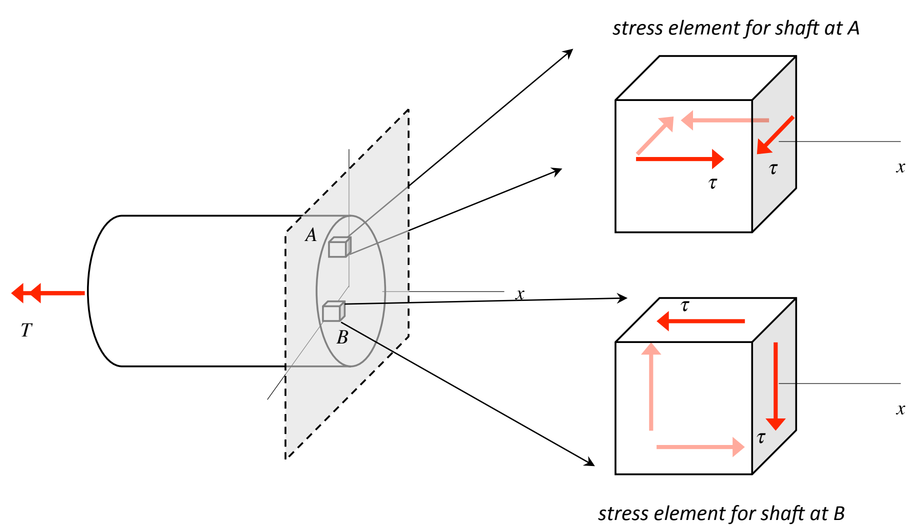
*Shaft with axial torque*




The angle of twist across the cross section of the shaft is a constant; therefore, the shear strain $\gamma$ varies linearly with radial position on the cross since $\gamma = \rho\, d\phi / dx$. Since the shear stress is given by $\tau = G\gamma = G\rho\, d\phi / dx$ the shear stress also varies linearly across a given cross section if the shear modulus is constant; however, if G varies across a cross section, then the axial stress will vary from this linear distribution. Say we consider a shaft made up of two materials: material (1) shaped as a tube and material (2) as an inner core of the tube, with shear moduli of $G_1 > G_2$. Throughout each material, the shear stress will vary linearly with radial position; however, the shear stress will have a "jump up" in value where material (1) interfaces with material (2).




Stress elements on a cut made perpendicular to the member axis are as shown.



### *b) Problem solving method*

1) Draw free body diagram(s) (FBDs) and write down equilibrium equations from FBDs.
2) Write down force/elongation, or torque/angle of twist, equations.
3) Write down compatibility relations in terms of the axial forces/torques.
4) Solve the equilibrium and compatibility equations.

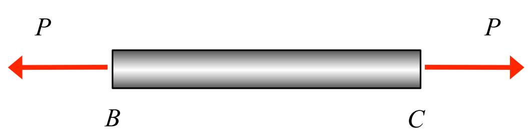
It is recommended that you follow these steps on *all* problems; usage of these will greatly simplify your problem-solving thought process. The procedure is applicable to both determinate and indeterminate problems; it is only in Step 4) where you see a difference in terms of which equations from equilibrium and compatibility must be solved simultaneously for your solution.

### *c) Sign conventions*

Sign conventions follow throughout all of the analysis, including: axial forces/axial torques, normal stresses, shear stresses, axial strains, shear strains, axial elongations and angles of twist. For all problems, regardless of complexity, it is important to note and remember these sign conventions. *Do not change sign conventions for different elements in the structure; this will lead to confusion for both you and the grader.* It is also recommended (but not required) that you assign positive directions to all unknown forces and torques in your FBDs.

*Axial forces, normal stress and axial elongation*

Axial forces are defined as POSITIVE when they put the loaded element in TENSION, as shown in the figure below. Normal stresses are defined as positive for those resulting from tensile axial loads, just as axial strains are positive when the element is in tension. Positive elongations are defined as those that result from a positive load on the element.




*Axial torques, shear stress and angles of twist*

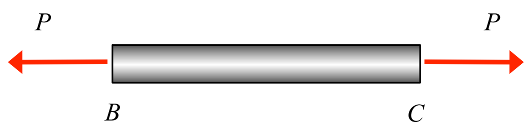
Axial torques are defined as POSITIVE when the applied torques point OUTWARD on the face of the element, as shown in the figure below. Shear stresses are defined as positive for those resulting from positive torques, just as shear strains are positive corresponding to positive torques. Positive angles of twist are defined as those that result from a positive torque on the element.

## d) Force/elongation and torque/angle of twist equations equations

The force/elongation equation for an element experiencing an axial load P and a temperature change $\Delta T$ is:

$$e = \frac{PL}{EA} + \alpha \Delta T L$$

where L is the length of the element, A is the cross-sectional area, E is the Young's modulus, and $\alpha$ is the thermal expansion coefficient. Recall that usage of this equation is based on the sign conventions reviewed earlier.




The torque/angle of twist equation for an element experiencing an axial torque is:

$$\phi = \frac{TL}{GI_P}$$

where L is the length of the element, $I_P$ is the polar area moment, and G is the shear modulus. Recall that usage of this equation is based on the sign conventions reviewed earlier.


## e) Compatibility equations

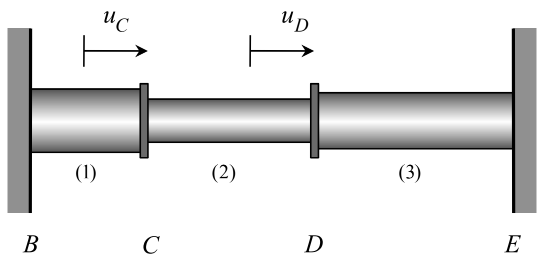
The compatibility equations provide information on the geometry of deformation in the problem. Typically, we use the compatibility equations to relate the deformations of different elements in the structure. Note that both the *magnitudes* AND the *signs* of the deformations are important in writing down the compatibility equations. It is important to recall the sign conventions discussed earlier before writing down the compatibility equations. The following is a listing of the compatibility equations for some typical connections of axial elements. Do not attempt to memorize these, rather, focus on the logic behind these relationships.

*Collinear elements – axial forces applied*

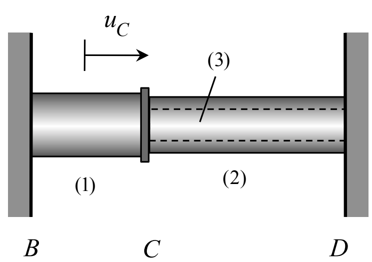
$$u_C = e_1$$

$$u_D = u_C + e_2 = e_1 + e_2$$

$$u_E = u_D + e_3 = e_1 + e_2 + e_3 = 0$$

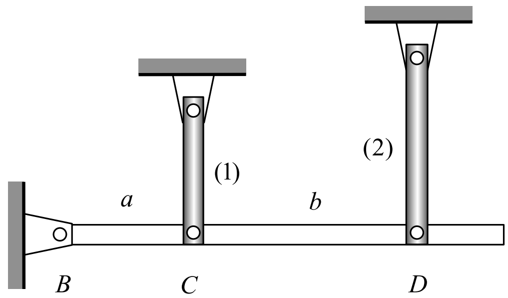



$$u_C = e_1$$
$$u_D = u_C + e_2 = e_1 + e_2 = 0 \quad \Rightarrow \quad e_1 = -e_2$$
$$e_2 = e_3$$

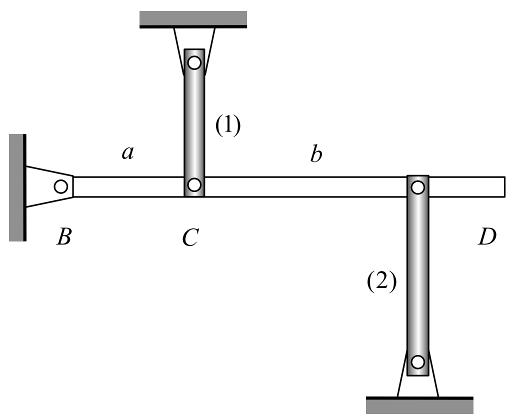


Elements connected to common rigid bar

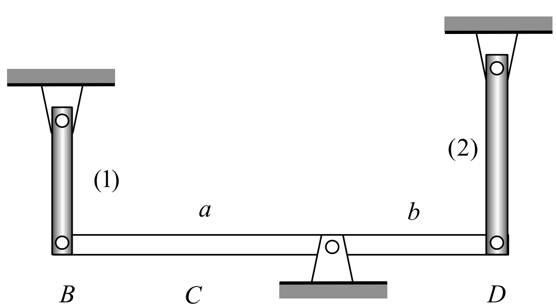
$$\theta = \textit{angle of rigid bar rotation} = \frac{e_1}{a} = \frac{e_2}{a+b} \quad \Rightarrow \quad e_1 = \left(\frac{a}{a+b}\right) e_2$$



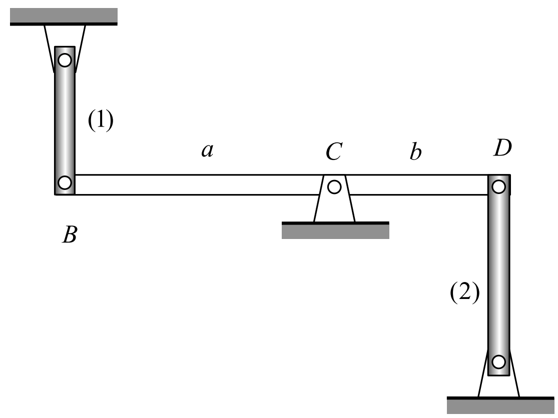

$$\theta = \textit{angle of rigid bar rotation} = \frac{e_1}{a} = -\frac{e_2}{a+b} \quad \Rightarrow \quad e_1 = -\left(\frac{a}{a+b}\right) e_2$$




$$\theta = \textit{angle of rigid bar rotation} = \frac{e_1}{a} = -\frac{e_2}{b} \quad \Rightarrow \quad e_1 = -\left(\frac{a}{b}\right) e_2$$



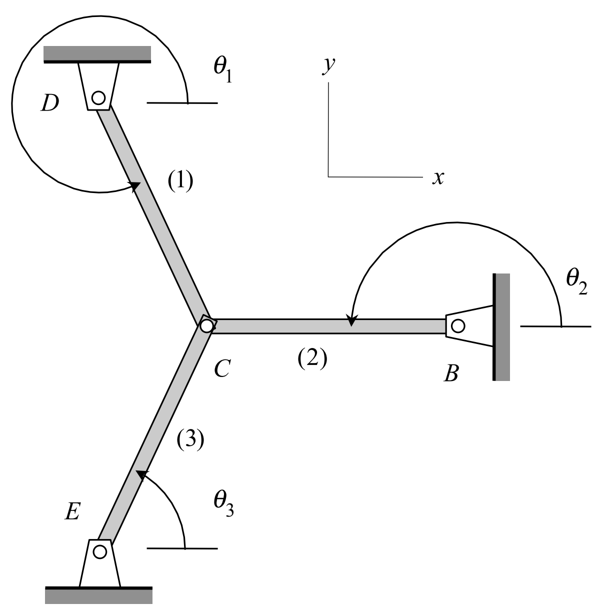
$$\theta = angle\ of\ rigid\ bar\ rotation = \frac{e_1}{a} = \frac{e_2}{b} \Rightarrow e_1 = \left(\frac{a}{b}\right)e_2$$




*Planar truss – axial deformations*

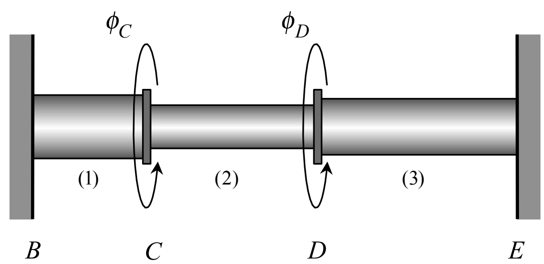
Recall that all element angles $\theta_j$ must be measured CCW at the BASE of the jth element, as shown in the figure.

$$e_1 = u_C cos\theta_1 + v_C sin\theta_1$$
$$e_2 = u_C cos\theta_2 + v_C sin\theta_2$$
$$e_3 = u_C cos\theta_3 + v_C sin\theta_3$$

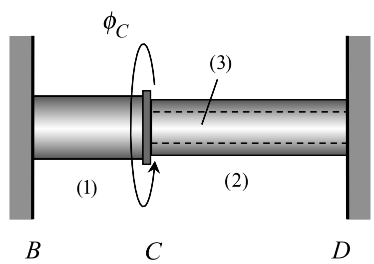


*Collinear elements – axial torques applied*

$$\phi_C = \phi_1$$
$$\phi_D = \phi_C + \phi_2 = \phi_1 + \phi_2$$
$$\phi_E = \phi_D + \phi_3 = \phi_1 + \phi_2 + \phi_3 = 0$$




$$\phi_C = \phi_1$$
$$\phi_D = \phi_C + \phi_2 = \phi_1 + \phi_2 = 0 \quad \Rightarrow \quad \phi_1 = -\phi_2$$
$$\phi_2 = \phi_3$$

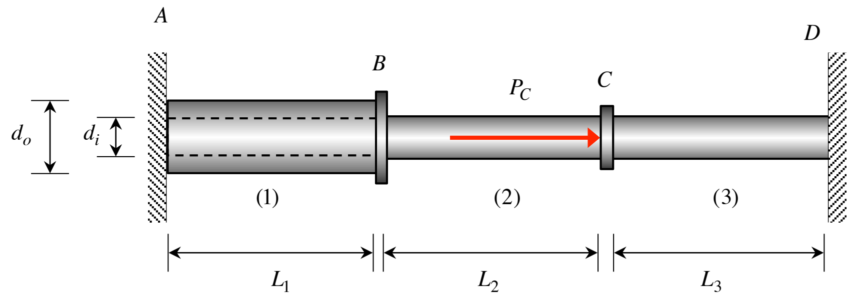



## f) Solving determinate and indeterminate problems

Recall that for determinate problems (problems for which you are able to solve for the external reactions directly from the equilibrium equations), solving for internal forces or torques can be found directly from equilibrium relations. For indeterminate problems, one must simultaneously solve the equilibrium and compatibility equations. Other than that, the solution processes for determinate and indeterminate problems are identical. However, note that the reaction forces/torques for indeterminate problems typically depend on material and cross-sectional properties, whereas for determinate problems they do not.

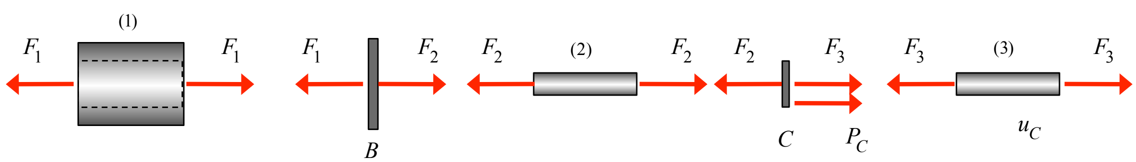
## *Problem A.1*

A steel pipe with an outer diameter $d_o$ and inner diameter $d_i$, and a solid aluminum-alloy rod of diameter $d$ form a three-segment system that undergoes axial deformation due to a single load $P_C$ acting on a collar at point C, as shown in the figure. Calculate the axial stresses in the three segments, and determine the displacements at connectors B and C.




*SOLUTION*

***Equilibrium***




$$B: \sum F_x = -F_1 + F_2 = 0 \quad \Rightarrow \quad F_1 = F_2 \tag{1}$$

$$C: \sum F_x = -F_2 + F_3 + F_C = 0 \quad \Rightarrow \quad F_2 = F_3 + F_C \tag{2}$$

***Force/elongation***

$$e_1 = \frac{F_1 L_1}{E_1 A_1} \quad ; \quad e_2 = \frac{F_2 L_2}{E_2 A_2} \quad ; \; e_3 = \frac{F_3 L_3}{E_3 A_3} \tag{3}$$

where $A_1 = \pi\left(d_o^2 - d_i^2\right)/4$ and $A_2 = A_3 = \pi d^2/4$.

***Compatibility***

$$u_B = e_1 \quad ; \quad u_C = u_B + e_2 = e_1 + e_2 \tag{4}$$

$$u_D = u_C + e_3 = e_1 + e_2 + e_3 = 0 \quad \Rightarrow \quad \frac{F_1 L_1}{E_1 A_1} + \frac{F_2 L_2}{E_2 A_2} + \frac{F_3 L_3}{E_3 A_3} = 0 \tag{5}$$

***Solve***

Combining (1), (2) and (5):

$$\left(F_3 + F_C\right)\left(\frac{L_1}{E_1 A_1} + \frac{L_2}{E_2 A_2}\right) + \frac{F_3 L_3}{E_3 A_3} = 0 \quad \Rightarrow \quad F_3 = -\frac{L_1/E_1 A_1 + L_2/E_2 A_2}{L_1/E_1 A_1 + L_2/E_2 A_2 + L_3/E_3 A_3} F_C$$

$$F_1 = F_2 = F_3 + F_C = \left[1 - \frac{L_1/E_1 A_1 + L_2/E_2 A_2}{L_1/E_1 A_1 + L_2/E_2 A_2 + L_3/E_3 A_3}\right] F_C$$

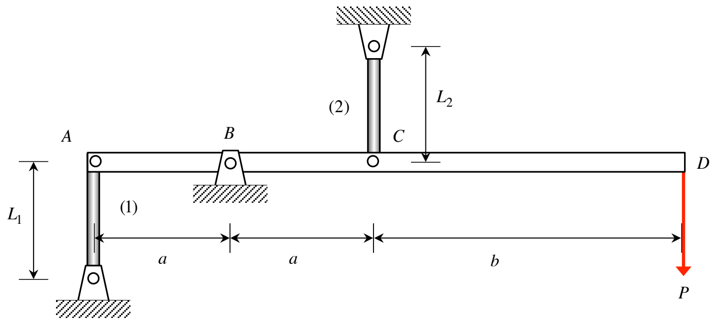
$$\sigma_1 = \frac{F_1}{A_1} = \left[1 - \frac{L_1/E_1A_1 + L_2/E_2A_2}{L_1/E_1A_1 + L_2/E_2A_2 + L_3/E_3A_3}\right]\frac{F_C}{A_1} \quad (TENSION)$$

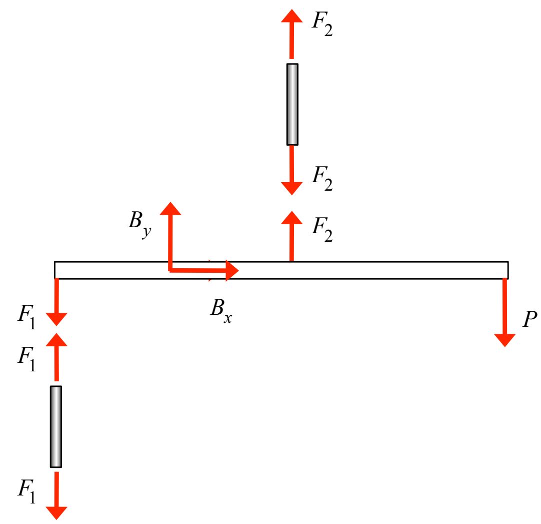
$$\sigma_2 = \frac{F_2}{A_2} = \left[1 - \frac{L_1/E_1A_1 + L_2/E_2A_2}{L_1/E_1A_1 + L_2/E_2A_2 + L_3/E_3A_3}\right]\frac{F_C}{A_2} \quad (TENSION)$$

$$\sigma_3 = \frac{F_3}{A_3} = -\left[\frac{L_1/E_1A_1 + L_2/E_2A_2}{L_1/E_1A_1 + L_2/E_2A_2 + L_3/E_3A_3}\right]\frac{F_C}{A_3} \quad (COMPRESSION)$$

### *Problem A.2*

A rigid beam AD is supported by a smooth pin at B and by vertical rods atached to the beam at points A and C. Neglect the weight of the beam and assum that the rods are stress-free when P = 0. Solve for the load carried the rods, the axial stress in each rod and the elongation of rod (1). Assume small rotations of the beam.




***Equilibrium***

$$(1)\quad \sum M_B = F_1 a + F_2 a - P(a+b) = 0$$

***Force/elongation***

$$(2)\quad e_1 = \frac{F_1 L_1}{E_1 A_1} = \frac{F_1 L}{E A_1}$$

$$(3)\quad e_2 = \frac{F_2 L_2}{E_2 A_2} = \frac{F_2 L}{E A_2}$$

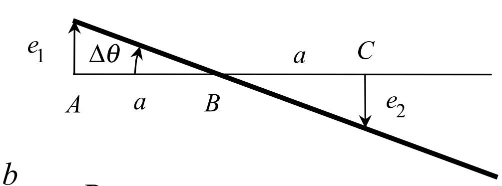
***Compatibility*** (take note of signs)

$$(4)\quad \Delta\theta = \frac{e_1}{a} = \frac{e_2}{a} \quad \Rightarrow \quad e_1 = e_2$$

***Solve***

Combining (2)-(4):

$$(5)\quad \frac{F_1 L}{E A_1} = \frac{F_2 L}{E A_2} \quad \Rightarrow \quad F_1 = \frac{A_1}{A_2} F_2$$




Combining (1) and (5):

$$\frac{A_1}{A_2} F_2 a + F_2 a - P(a+b) = 0 \quad \Rightarrow \quad F_2 = \frac{a+b}{a\left(1 + A_1 / A_2\right)} P$$

$$F_1 = \frac{A_1}{A_2} F_2 = \frac{(a+b)\left(A_1 / A_2\right)}{a\left(1 + A_1 / A_2\right)} P$$

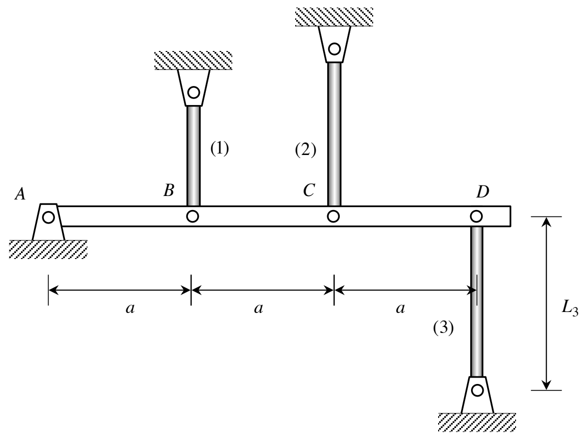
$$\sigma_1 = \frac{F_1}{A_1} = \frac{(a+b)\left(A_1 / A_2\right)}{a\left(1 + A_1 / A_2\right)} \frac{P}{A_1} \quad (TENSION)$$

$$\sigma_2 = \frac{F_2}{A_2} = \frac{a+b}{a\left(1+A_1/A_2\right)}\frac{P}{A_2} \quad \left(TENSION\right)$$

$$e_1 = \frac{F_1 L}{EA_1} = \frac{\left(a+b\right)\left(A_1/A_2\right)}{a\left(1+A_1/A_2\right)}\frac{PL}{EA_1} \quad \left(EXTENSION\right)$$

## *Problem A.3*

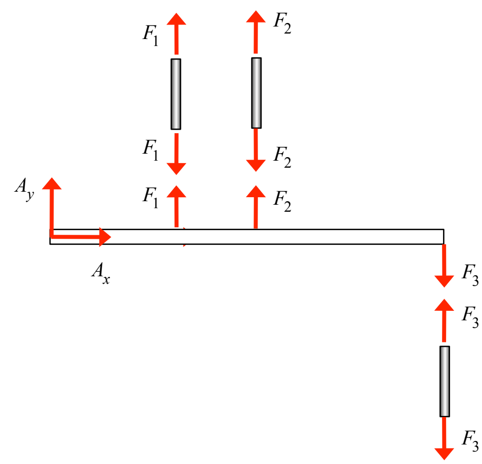
The structure shown consists of a rigid beam AD supported by three rods and a pin joing. Determine the loads carried by the rods when rod (2) is *increased* by an amount of $\Delta T$ and the temperature of the other rods is held constant.




*SOLUTION*

***Equilibrium***

$$(1)\quad \sum M_A = F_1(a) + F_2(2a) - F_3(3a) = 0$$




***Force/elongation***

$$(2)\quad e_1 = \frac{F_1 L_1}{E_1 A_1} = \frac{F_1 L_1}{EA}$$

$$(3)\quad e_2 = \frac{F_2 L_2}{E_2 A_2} + \alpha \Delta T L_2 = \left(\frac{F_2}{EA} + \alpha \Delta T\right) L_2$$

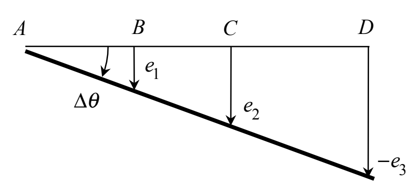
$$(4)\quad e_3 = \frac{F_3 L_3}{E_3 A_3} = \frac{F_3 L_3}{EA}$$

***Compatibility*** (take notes of signs)

$$\Delta\theta = \frac{e_1}{a} = \frac{e_2}{2a} = -\frac{e_3}{3a} \quad \Rightarrow$$

$$(5)\quad e_2 = 2e_1$$

$$(6)\quad e_3 = -3e_1$$




***Solve***

Combining (3)-(5):

$$(7)\quad \left(\frac{F_2}{EA} + \alpha \Delta T\right) L_2 = 2\frac{F_1 L_1}{EA} \quad \Rightarrow \quad F_2 = 2\frac{L_1}{L_2} F_1 - EA\alpha\Delta T$$

Combining (2), (4) and (6):

$$(8)\quad \frac{F_3 L_3}{EA} = -3\frac{F_1 L_1}{EA} \;\Rightarrow\; F_3 = -3\frac{L_1}{L_3}F_1$$

Combining (1), (7) and (8) gives:

$$F_1(a) + \left(2\frac{L_1}{L_2}F_1 - EA\alpha\Delta T\right)(2a) - \left(-3\frac{L_1}{L_3}F_1\right)(3a) = 0 \;\Rightarrow$$

$$F_1 = \frac{2EA\alpha\Delta T}{1 + 4L_1/L_2 + 9L_1/L_3} > 0 \;\; (TENSION)$$

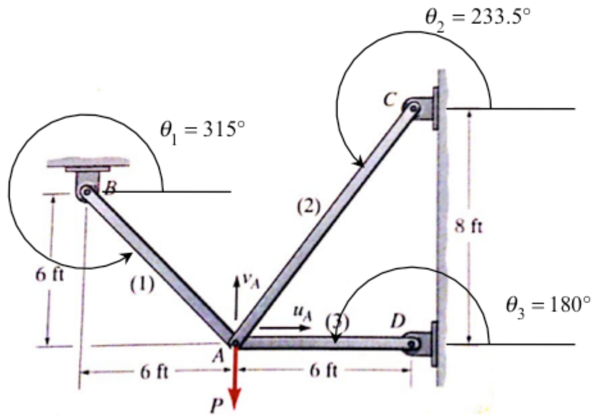
and:

$$F_2 = 2\frac{L_1}{L_2}F_1 - EA\alpha\Delta T = -\left[1 - \frac{4L_1/L_2}{1 + 4L_1/L_2 + 9L_1/L_3}\right]EA\alpha\Delta T < 0 \;\; (COMPRESSION)$$

$$F_3 = -3\frac{L_1}{L_3}F_1 = -\frac{L_1}{L_3}\left[\frac{6EA\alpha\Delta T}{1 + 4L_1/L_2 + 9L_1/L_3}\right] < 0 \;\; (COMPRESSION)$$

### *Problem A.4*

The truss shown is made up of members having the same Young's modulus E and with the same cross-sectional area A. Determine the horizontal and vertical displacements of joint A, along with the loads carried by the three truss members.




*SOLUTION*

***Equilibrium***

$$\sum F_y = F_1 sin45° + F_2 sin53.13° - P = 0$$

$$\sum F_y = -F_1 cos45° + F_2 cos53.13° + F_3 = 0$$

or,

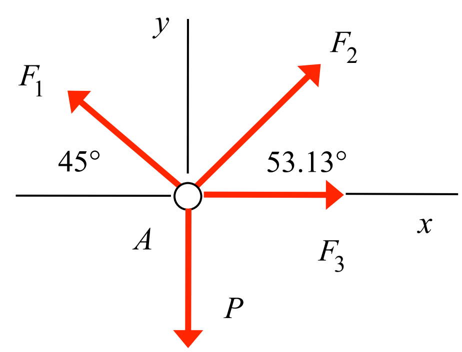
$$F_1 / \sqrt{2} + 0.8F_2 = P$$

$$F_3 = F_1 / \sqrt{2} - 0.6F_2$$

***Force/elongation***

$$e_1 = \frac{F_1 L_1}{E_1 A_1} = \frac{F_1 L_1}{EA} \quad ; \quad e_2 = \frac{F_2 L_2}{E_2 A_2} = \frac{F_2 L_2}{EA} \quad ; \; e_3 = \frac{F_3 L_3}{E_3 A_3} = \frac{F_3 L_3}{EA}$$

***Compatibility*** (see angles in figure above)




$$e_1 = u_A cos\theta_1 + v_A sin\theta_1 = \frac{u_A}{\sqrt{2}} - \frac{v_A}{\sqrt{2}}$$

$$e_2 = u_A cos\theta_2 + v_A sin\theta_2 = -0.6u_A - 0.8v_A$$

$$e_3 = u_A cos\theta_3 + v_A sin\theta_3 = -u_A$$

***Solve***

Combining above:

$$\frac{F_1 L_1}{EA} = \frac{u_A}{\sqrt{2}} - \frac{v_A}{\sqrt{2}} \quad \Rightarrow \quad F_1 = \frac{EA}{\sqrt{2} L_1}\left(u_A - v_A\right)$$

$$\frac{F_2 L_2}{EA} = -0.6u_A - 0.8v_A \quad \Rightarrow \quad F_2 = -\frac{EA}{L_2}\left(0.6u_A + 0.8v_A\right)$$

$$\frac{F_3 L_3}{EA} = -u_A \quad \Rightarrow \quad F_3 = -\frac{EA}{L_3} u_A$$

Therefore,

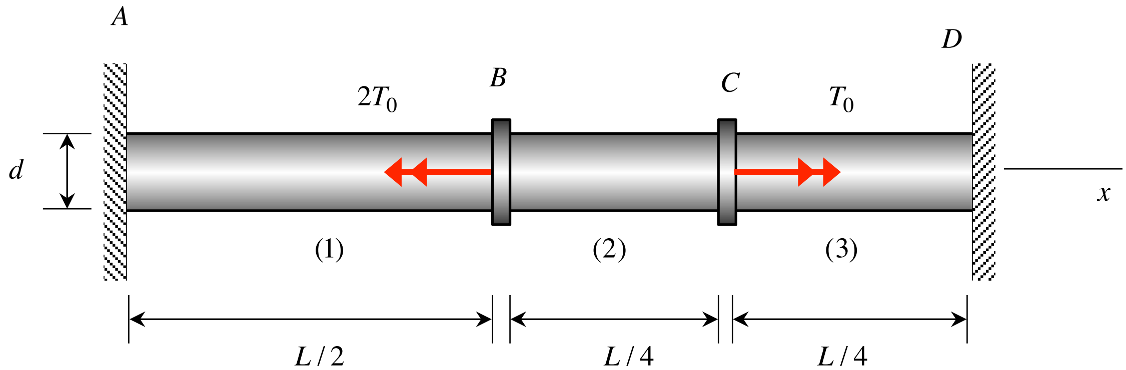
$$\frac{EA}{2L_1}\left(u_A - v_A\right) - \frac{EA}{L_2}\left(0.48u_A + 0.64v_A\right) = P \quad \Rightarrow \quad \left(\frac{0.5}{L_1} - \frac{0.48}{L_2}\right)u_A - \left(\frac{0.5}{L_1} + \frac{0.64}{L_2}\right)v_A = \frac{P}{EA}$$

$$-\frac{EA}{L_3}u_A = \frac{EA}{2L_1}\left(u_A - v_A\right) + \frac{EA}{L_2}\left(0.36u_A + 0.48v_A\right) \quad \Rightarrow \quad \left(\frac{0.5}{L_1} + \frac{0.36}{L_2} + \frac{1}{L_3}\right)u_A + \left(-\frac{0.5}{L_1} + \frac{0.48}{L_2}\right)v_A = 0$$

Solve the above two equations for $u_A$ and $v_A$.

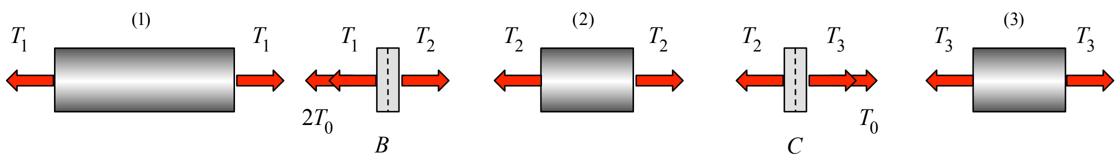
### *Problem A.5*

A uniform shaft with fixed ends A and D is subjected to external torques at B and C. Determine the maximum shear stress in each of the three segments of the shaft and the angle of twist at B.




*SOLUTION*

***Equilibrium***




$$B: \quad \sum M_x = -T_1 - 2T_0 + T_2 = 0 \quad \Rightarrow \quad T_2 = T_1 + 2T_0 \tag{1}$$

$$C: \quad \sum M_x = -T_2 + T_3 + T_0 = 0 \quad \Rightarrow \quad T_3 = T_2 - T_0 = T_1 + T_0 \tag{2}$$

***Torque/angle of twist***

$$\phi_1 = \frac{T_1 L_1}{G_1 I_{P1}} = \frac{T_1 L}{2GI_P} \quad ; \quad \phi_2 = \frac{T_2 L_2}{G_2 I_{P2}} = \frac{T_2 L}{4GI_P} \quad ; \phi_3 = \frac{T_3 L_3}{G_3 I_{P3}} = \frac{T_3 L}{4GI_P} \tag{3}$$

where $I_P = \pi\left(d/2\right)^4/2 = \pi d^4/32$.

***Compatibility***

$$\phi_B = \phi_1 \quad ; \quad \phi_C = \phi_B + \phi_2 = \phi_1 + \phi_2 \tag{4}$$

$$\phi_D = \phi_C + \phi_3 = \phi_1 + \phi_2 + \phi_3 = 0 \quad \Rightarrow \quad \frac{T_1 L}{2GI_P} + \frac{T_2 L}{4GI_P} + \frac{T_3 L}{4GI_P} = 0 \quad \Rightarrow$$

$$2T_1 + T_2 + T_3 = 0 \tag{5}$$

***Solve***

Combining (1), (2) and (5):

$$0 = 2T_1 + T_2 + T_3 = 2T_1 + \left(T_1 + 2T_0\right) + \left(T_1 + T_0\right) = 4T_1 + 3T_0 \quad \Rightarrow \quad T_1 = -0.75T_0$$

$$T_2 = T_1 + 2T_0 = -0.75T_0 + 2T_0 = 1.25T_0$$

$$T_3 = T_1 + T_0 = -0.75T_0 + T_0 = 0.25T_0$$

and,

$$\left(\tau_1\right)_{max} = \frac{T_1\left(d/2\right)}{I_P} = -0.75\frac{T_0 d/2}{\pi d^4/32} = -\frac{12}{\pi}\frac{T_0}{d^3}$$

$$\left(\tau_2\right)_{max} = \frac{T_2\left(d/2\right)}{I_P} = 1.25\frac{T_0 d/2}{\pi d^4/32} = \frac{20}{\pi}\frac{T_0}{d^3}$$

$$\left(\tau_3\right)_{max} = \frac{T_3\left(d/2\right)}{I_P} = 0.25\frac{T_0 d/2}{\pi d^4/32} = \frac{4}{\pi}\frac{T_0}{d^3}$$

and:

$$\phi_B = \phi_1 = \frac{T_1 L}{2GI_P} = \frac{\left(-0.75T_0\right)L}{2G\left(\pi d^4/32\right)} = -\frac{12T_0 L}{\pi G d^4}$$