Lecture topics:

a) Simple examples of plane stress.

b) Stress transformation equations for plane stress.

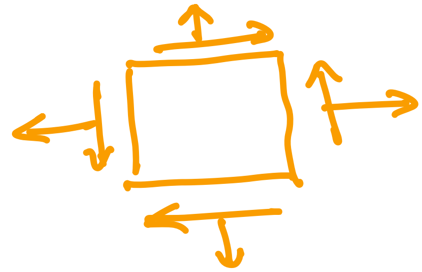
c) Principal normal stresses and maximum shear stress.

d) Mohr's circle.

e) Absolute maximum shear stress for plane stress.

Plane stress →

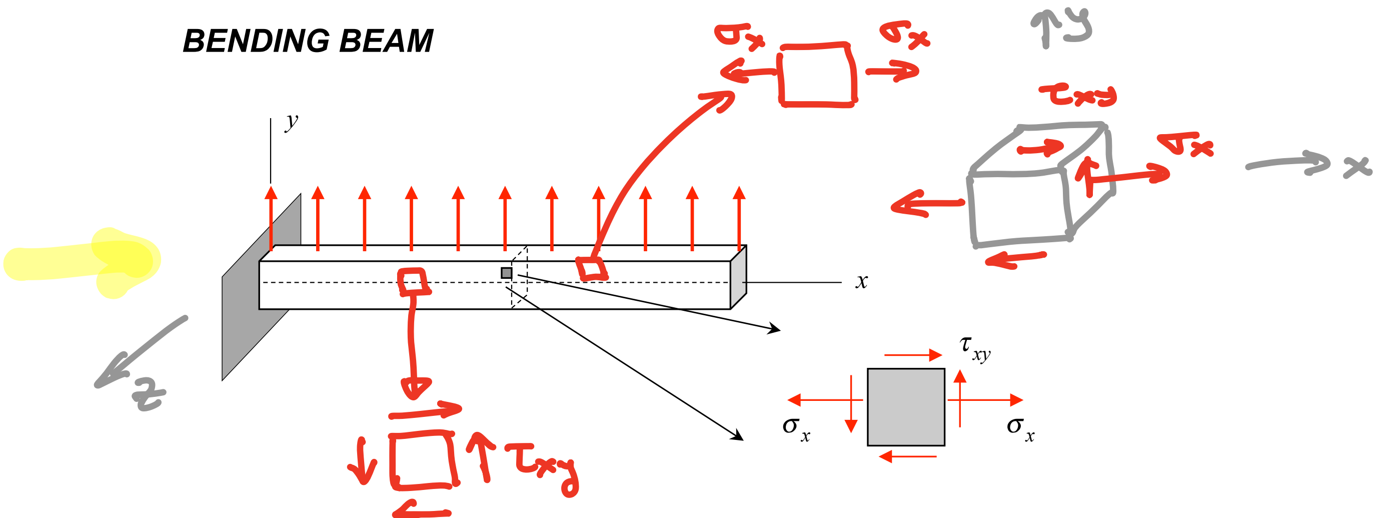
No out of plane stress



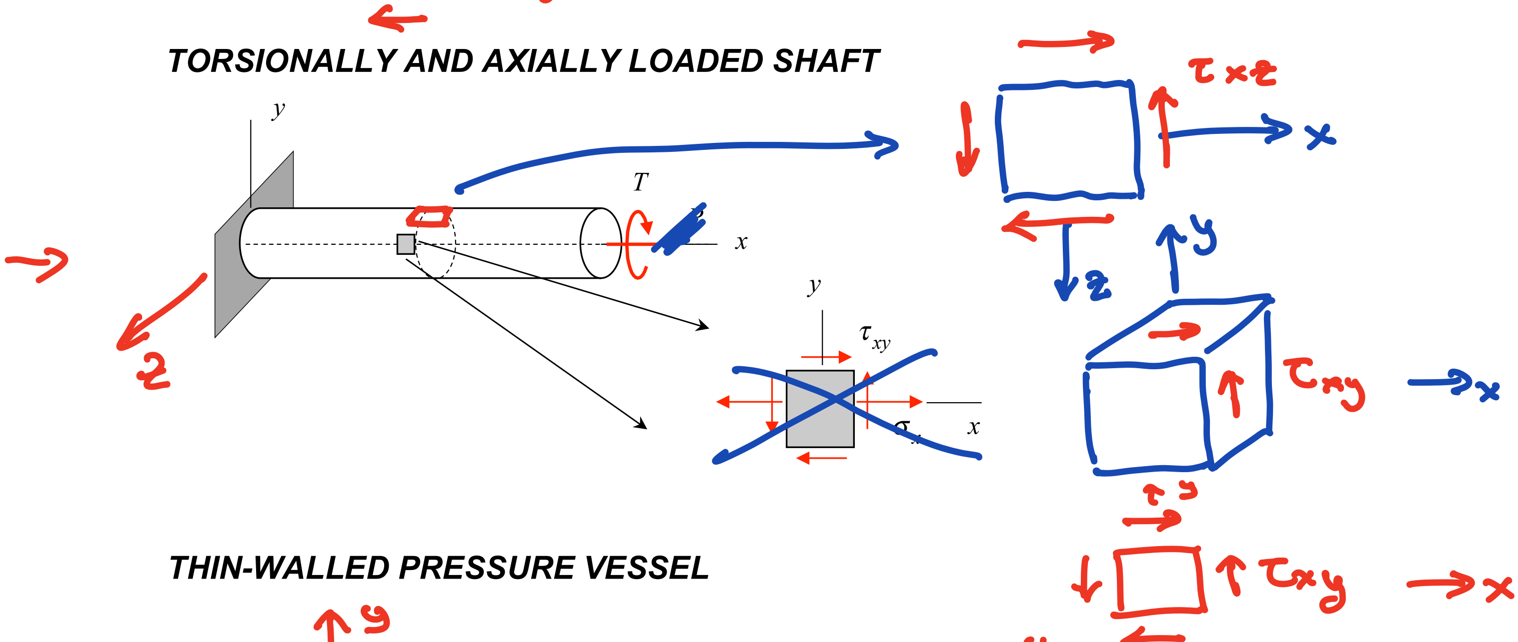
Lecture notes

a) *Simple examples of plane stress*

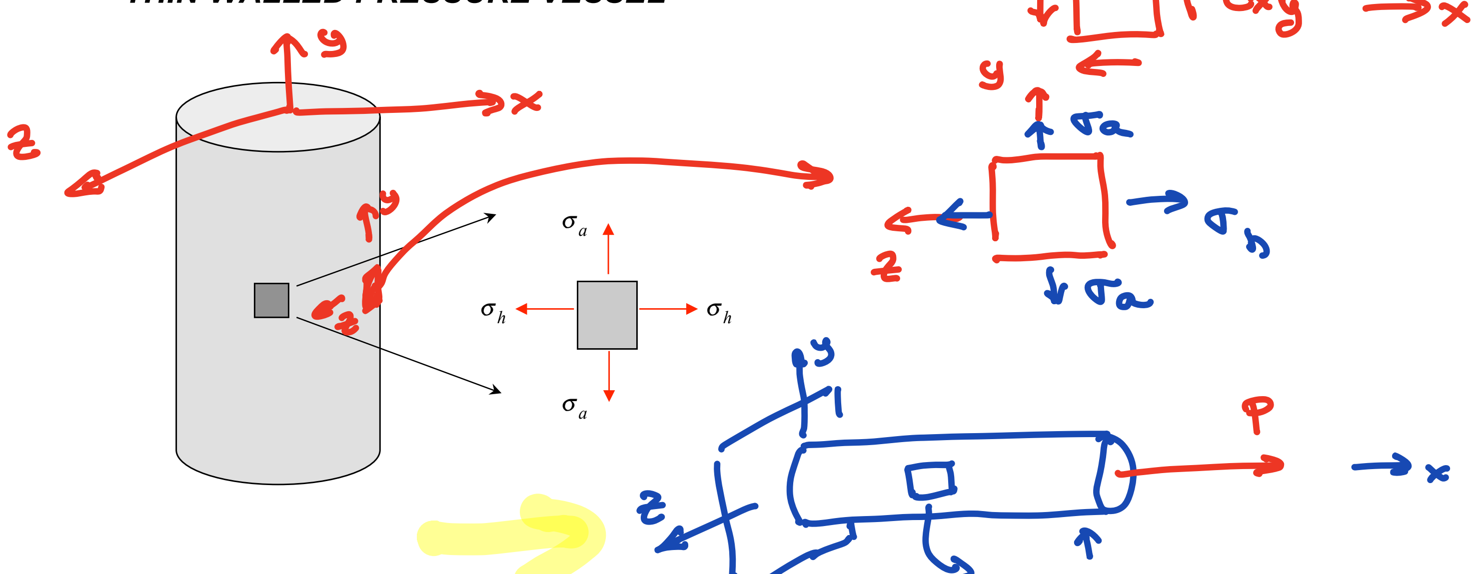
BENDING BEAM

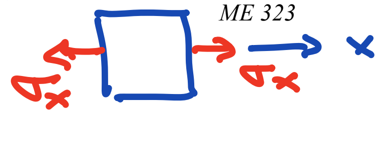


TORSIONALLY AND AXIALLY LOADED SHAFT



THIN-WALLED PRESSURE VESSEL

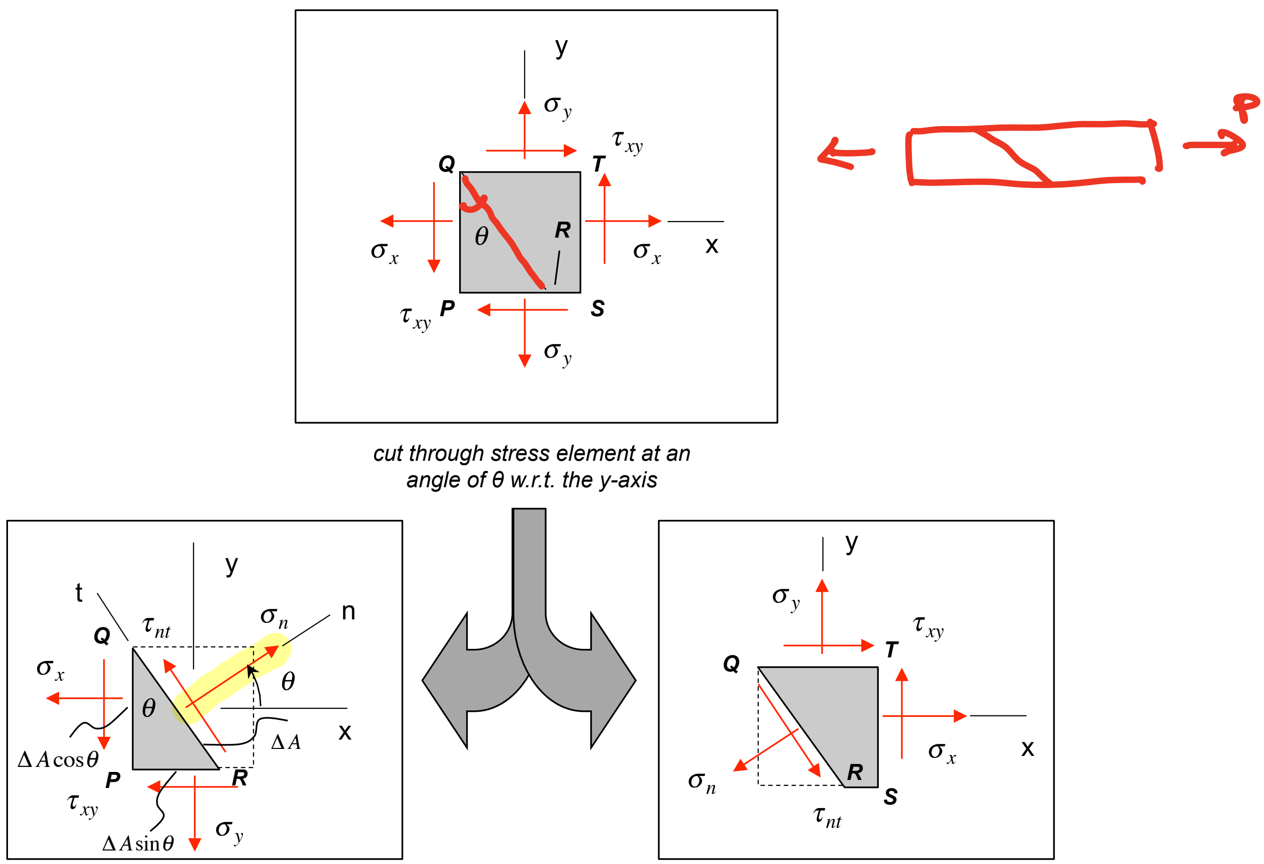




b) Stress transformation for plane stress

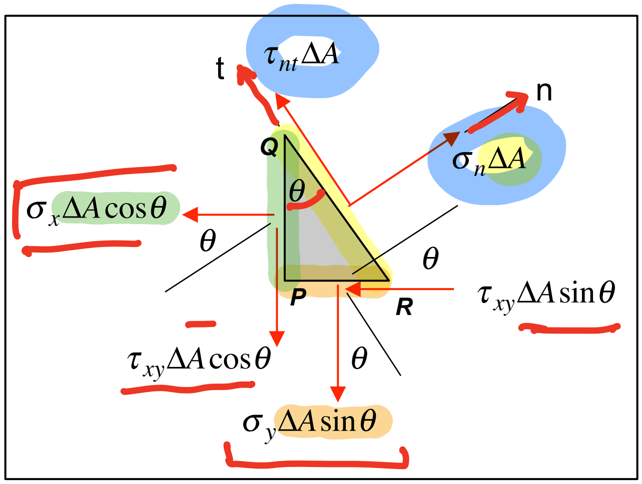
Here we start with a state of plane stress with normal stresses σ_x and σ_y acting on faces perpendicular to the *x*- and *y*-axes, respectively, and shear stress τ_{xy} acting on the four faces. Our goal here is to determine the normal and shear components of stress acting on a face, QR, whose normal "*n*" is at an angle of θ measured CCW from the *x*-axis, as indicated in the figure below.

Let ΔA be the area of the cut face QR. Therefore, the area of faces PQ and PS are $\Delta A\cos\theta$ and $\Delta A\sin\theta$, respectively.



cut through stress element at an angle of θ w.r.t. the y-axis

At this point, we will perform an equilibrium analysis of the cut section PQR (left side of the cut) to determine the stress components σ_n and τ_{nt}.



Summing forces in the n-direction on the cut section gives:

$$\sum F_n = \sigma_n \Delta A - \left(\sigma_x \Delta A \cos\theta\right)\cos\theta - \left(\tau_{xy}\Delta A \cos\theta\right)\sin\theta$$
$$- \left(\sigma_y \Delta A \sin\theta\right)\sin\theta - \left(\tau_{xy}\Delta A \sin\theta\right)\cos\theta$$
$$0 = \left(\sigma_n - \sigma_x \cos^2\theta - \sigma_y \sin^2\theta - 2\tau_{xy}\cos\theta\sin\theta\right)\Delta A$$

$$\boxed{\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\cos\theta\sin\theta} \qquad (1)$$

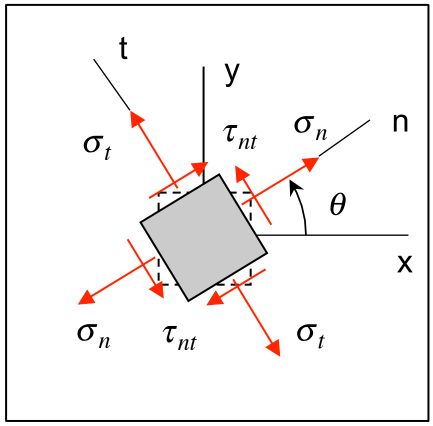
Similarly, summing forces in the t-direction on the cut section gives:

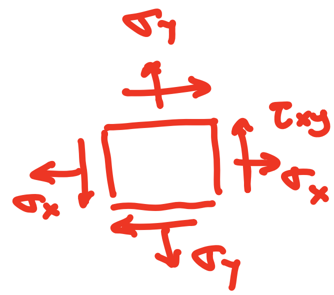
$$\sum F_t = \tau_{nt}\Delta A + \left(\sigma_x \Delta A \cos\theta\right)\sin\theta - \left(\tau_{xy}\Delta A \cos\theta\right)\cos\theta$$
$$- \left(\sigma_y \Delta A \sin\theta\right)\cos\theta + \left(\tau_{xy}\Delta A \sin\theta\right)\sin\theta$$
$$0 = \left[\tau_{nt} + \left(\sigma_x - \sigma_y\right)\cos\theta\sin\theta - \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)\right]\Delta A \quad \Rightarrow$$

$$\boxed{\tau_{nt} = -\left(\sigma_x - \sigma_y\right)\cos\theta\sin\theta - \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)} \qquad (2)$$

Equations (1) and (2) provide us with equations for determining two of the three components of stress, σ_n and τ_{nt}, on the rotated stress element. The remaining state of stress, σ_t, can be found from equation (1) by substituting $\theta + 90°$ in for θ (since the "t" face of the stress element is a $90°$ CCW rotation from the "n" face); that is,

$$\sigma_t = \sigma_x \cos^2(\theta+90°) + \sigma_y \sin^2(\theta+90°) + 2\tau_{xy}\cos(\theta+90°)\sin(\theta+90°)$$
$$= \boxed{\sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta} \qquad (3)$$





With the use of some trigonometric identities[1], equations (1) and (2) can be written in a slightly modified form, a form that we will find useful later on in interpreting the results of stress transformations:

$$\sigma_n = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1a)$$

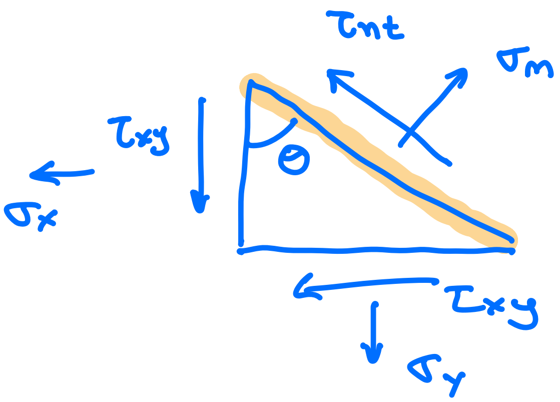
$$\tau_{nt} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2a)$$

$$\sigma_t = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

We can find θ $\tau_{nt}=0$

If $\sigma_x = -\sigma_y$ we can find an angle θ such that $\sigma_n = 0$ (pure shear)

[1] Here, we use the trig identities: $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos^2\theta = (1+\cos 2\theta)/2$ and $\sin^2\theta = (1-\cos 2\theta)/2$.



We can find an angle θ for which $\tau_{nt} = 0$

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \rightarrow \text{Matrix}$$

Eigenvalues of $\underline{\underline{\sigma}}$ are

principal stresses

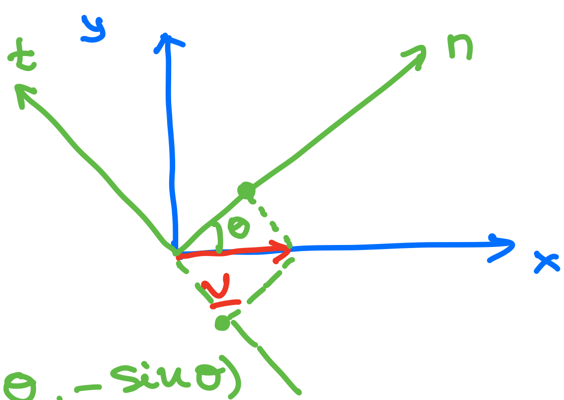
$$\sigma_1 = \sigma_a + R$$

$$\sigma_2 = \sigma_a - R$$

$$\sigma_a = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\tau_{xy}^2 + \frac{(\sigma_x - \sigma_y)^2}{4}}$$

$\underline{V} = (1\,0)$



$$\underline{v}' = (\cos\theta, -\sin\theta)$$

$$\underline{v}' = R^T \cdot \underline{v} \qquad \underline{\underline{R}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

For a matrix

$$\underline{\underline{\sigma}}' = R^T \cdot \underline{\underline{\sigma}} \cdot R$$

$$\sigma' = \begin{bmatrix} \sigma_n & \tau_{nt} \\ \tau_{nt} & \sigma_t \end{bmatrix}$$

c) Principal normal stresses and maximum shear stress

Equations (1a) and (2a) show how the normal and shear stresses on the n-axis face of a 2D stress element varies with the rotation angle θ of the stress element:

$$\sigma_n = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta \tag{1a}$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \tag{2a}$$

where:

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

The question for us in this section is to determine the maximum and minimum values of these normal and shear stress components as the stress element is rotated. The maximum and minimum values of the normal stress are known as the "principal" stresses.

Principal stresses

To determine the rotations that correspond to the principal stresses we need to set $\frac{d\sigma_n}{d\theta} = 0$ and solve for the rotation angle. To this end, we write from equation (1a):

$\tau_{nt} = 0$

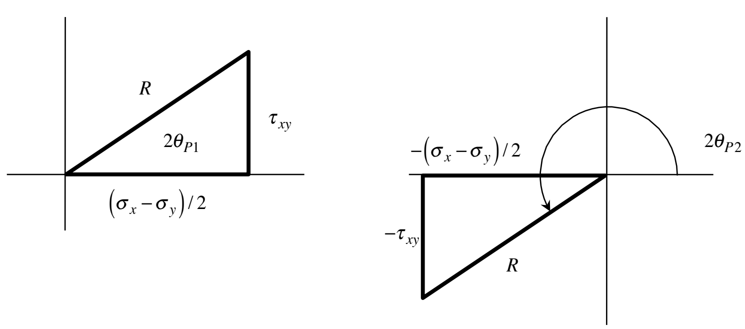
$$\frac{d\sigma_n}{d\theta} = -2\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0 \quad \Rightarrow$$

$$\tan 2\theta_p = \frac{\sin 2\theta_p}{\cos 2\theta_p} = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \quad \Rightarrow \quad \theta_P = \frac{1}{2}\tan^{-1}\left[\frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}\right] \tag{5}$$

- We see that there are two values of $2\theta_P$ ($2\theta_{P1}$ and $2\theta_{P2}$) separated by 180° that satisfy equation (5): $2\theta_{P2} = 2\theta_{P1} \pm 180°$, or:

  $$\theta_{P2} = \theta_{P1} \pm 90°$$

- Substitution of these two angles back into equation (1a) gives the two values for the principal stresses, σ_{P1} and σ_{P2}. For given numerical values for the stress state, this process of calculating principal stresses is straightforward. However, we desire to develop general expressions for these principal stresses. To this end, consider a right triangle having τ_{xy} and $(\sigma_x - \sigma_y)/2$ as opposite and adjacent sides, respectively, for the angle $2\theta_{P1}$ shown below left:



From this figure we see that:

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{R} \tag{6}$$

$$\cos 2\theta_{p1} = \frac{(\sigma_x - \sigma_y)/2}{R} \tag{7}$$

where $R = \sqrt{\tau_{xy}^2 + (\sigma_x - \sigma_y)^2 / 4}$. Substituting (6) and (7) into equation (1a) gives:

$$\sigma_{P1} = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\left[\frac{(\sigma_x - \sigma_y)/2}{R}\right] + \tau_{xy}\left[\frac{\tau_{xy}}{R}\right] = \sigma_{ave} + R \tag{8}$$

For the triangle corresponding to the angle θ_{P2} we have:

$$\sin 2\theta_{p2} = -\frac{\tau_{xy}}{R} \tag{6a}$$

$$\cos 2\theta_{p2} = -\frac{(\sigma_x - \sigma_y)/2}{R} \tag{7a}$$

Substituting (6a) and (7a) into equation (1a) gives:

$$\sigma_{P2} = \sigma_{ave} - \left(\frac{\sigma_x - \sigma_y}{2}\right)\left[\frac{(\sigma_x - \sigma_y)/2}{R}\right] - \tau_{xy}\left[\frac{\tau_{xy}}{R}\right] = \sigma_{ave} - R \tag{9}$$

- If we substitute either (6) and (7), or (6a) and (7a), into equation (2a) we see that:

$$\tau_{nt}(\theta_{P1}) = \tau_{nt}(\theta_{P2}) = 0 \tag{10}$$

In summary:

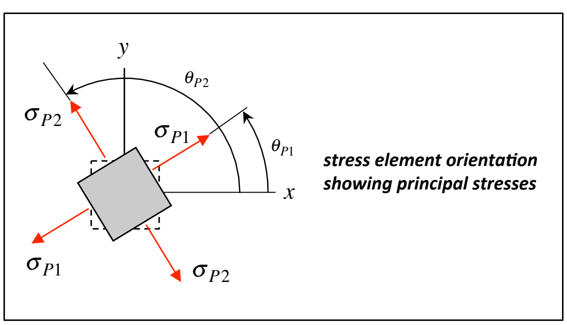
a) The two "principal" stress components σ_{P1} and σ_{P2} are given by:

$$\sigma_{P1} = \sigma_{ave} \pm R$$

b) These stress states occur on faces whose rotations are separated by $90°$:

$$\theta_{P2} = \theta_{P1} \pm 90°$$

c) Equation (10) shows that the shear stress on the faces corresponding to principal stresses is ZERO.



Maximum in-plane shear stress

To determine the rotations that correspond to the maximum shear stress in the plane we need to set $\frac{d\tau_{nt}}{d\theta} = 0$ and solve for the rotation angle. To this end, we write from equation (1b):

$$\frac{d\tau_{nt}}{d\theta} = -2\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \quad \Rightarrow$$

$$\tan 2\theta_s = \frac{\sin 2\theta_s}{\cos 2\theta_s} = -\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \quad \Rightarrow \quad \theta_s = \frac{1}{2}\tan^{-1}\left[-\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}}\right] \qquad (11)$$

Using a procedure similar to that above for principal stresses (and detailed in the textbook), we can show that there are two orientations 90° apart producing maximum shear stresses of:

$$\tau_{s1,s2} = \pm R$$

where, as before, $R = \sqrt{\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2 / 4}$.

In summary:

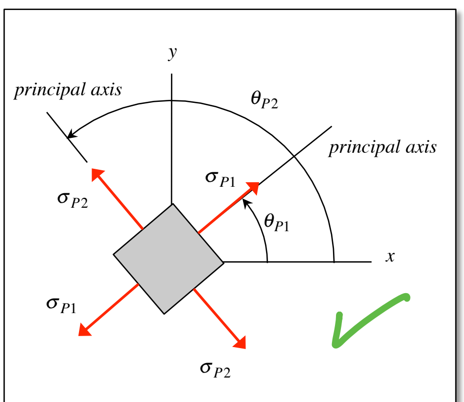
a) The two maximum shear stress values τ_{s1} and τ_{s2} are given by:

$$\tau_{s1,s2} = \pm R$$

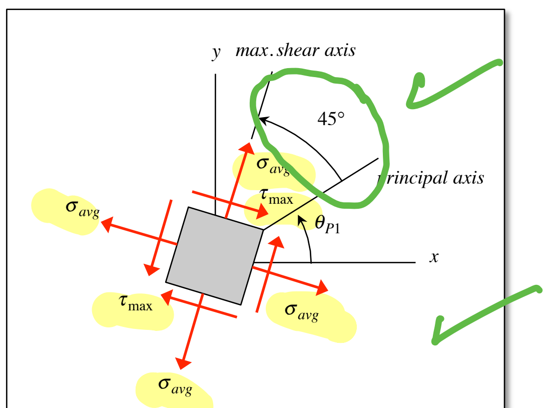
b) These stress states occur on faces whose rotations are separated by 90° : $\theta_{s2} = \theta_{s1} \pm 90°$

These orientations are 45° rotations from the principal stress axes.

c) The normal stress on the faces corresponding to maximum shear stress is NOT zero, rather they are given by: $\sigma_n\left(\theta_{s1}\right) = \sigma_n\left(\theta_{s2}\right) = \sigma_{avg}$, where, as before, $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$.



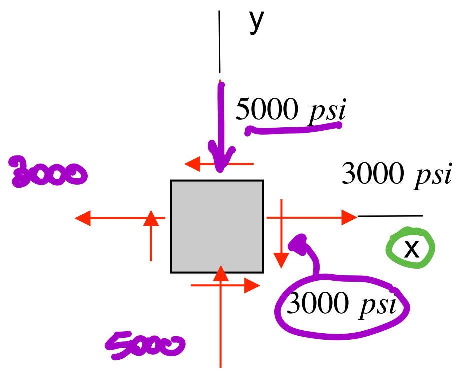
stress element orientation showing principal axes

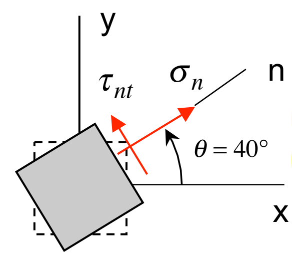


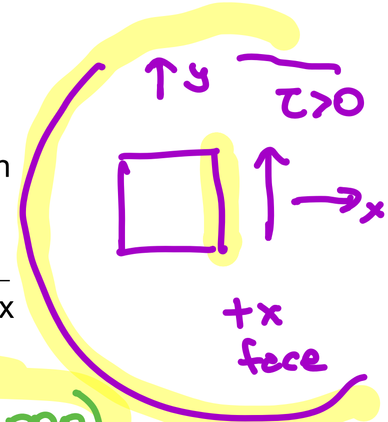
stress element orientation showing axes of max. shear stress

Example 13.1

Consider the state of plane stress shown below. ~~Determine the stresses on the plane shown below whose orientation is a 40° CCW rotation from the x-axis.~~







X (3000, −3000)

Y (−5000, 3000)

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{3000 - 5000}{2}$$

$$= -1000 \text{ psi}$$

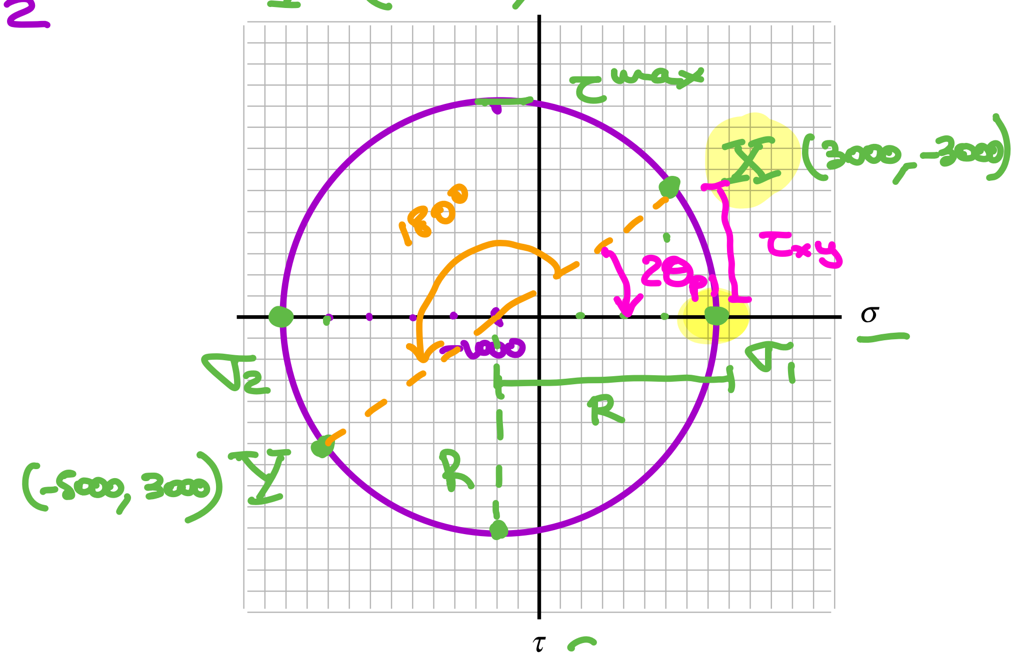
$$R = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$$

$$= 5000 \text{ psi}$$

$$\sigma_1 = 4000 \text{ psi}$$

$$\sigma_2 = -6000 \text{ psi}$$

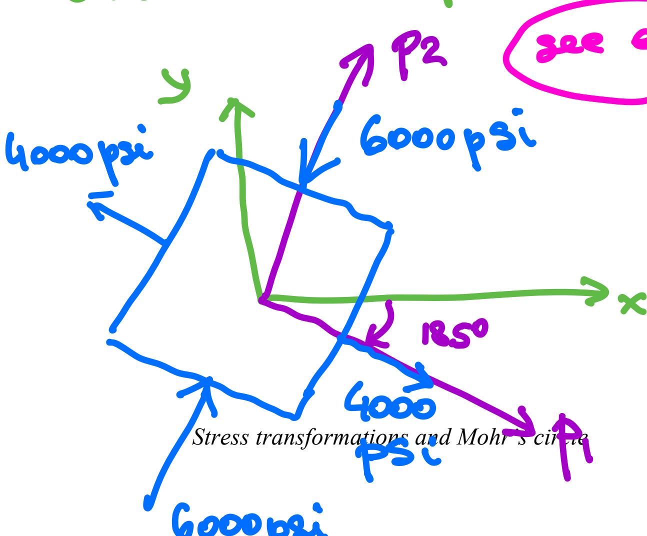
$$\tau_{max} = 5000 \text{ psi}$$



see eq 6 ← $\sin 2\theta_{p1} = \frac{\tau_{xy}}{R}$

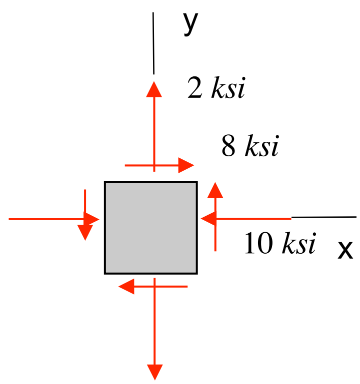
$$2\theta_{p1} = 37°$$

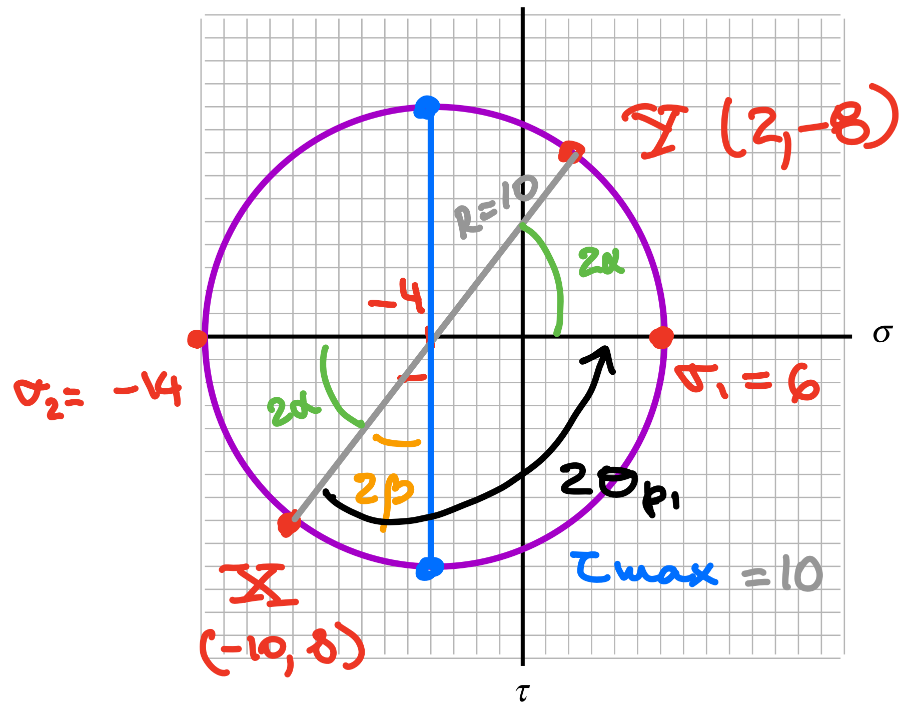
$$\theta_{p1} = 18.5°$$



Example 13.2

For the given state of stress shown, determine the principal stresses, the maximum in-plane shear stress and the stress element rotation angles corresponding to these stresses.





$\sigma_{ave} = -4$ ksi

$R = 10$ ksi

$\tau_{max} = 10$ ksi

$\sigma_1 = 6$ ksi

$\sigma_2 = -14$ ksi

$\sin 2\alpha = \dfrac{8}{10}$ $\quad 2\alpha = 53^\circ$

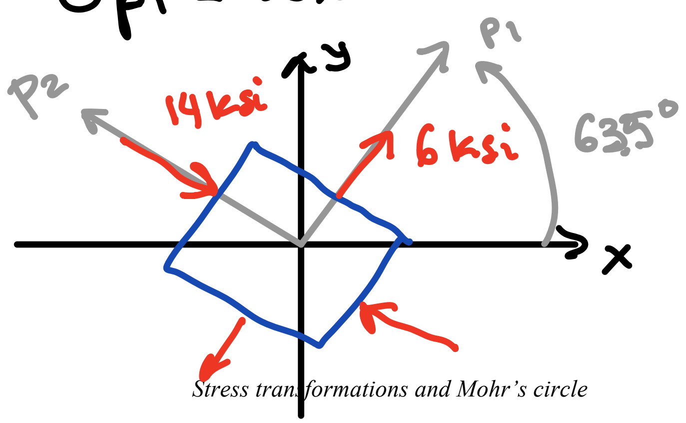
$\alpha = 26.5^\circ$

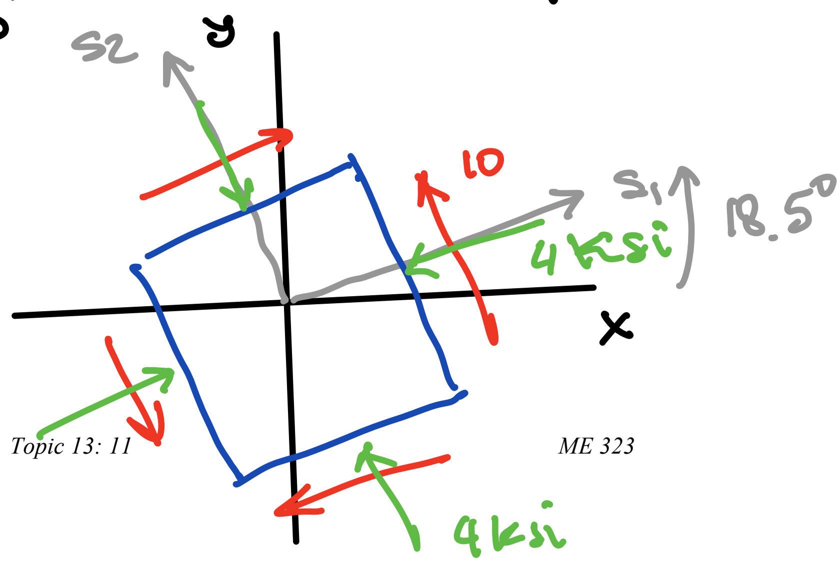
$2\beta = 90 - 2\alpha$ $\quad \beta = 18.5^\circ$

$2\theta_{p1} = 2\beta + 90$

$\theta_{p1} = 18.5 + 45 = 63.5^\circ$

$\theta_{s1} = \beta$

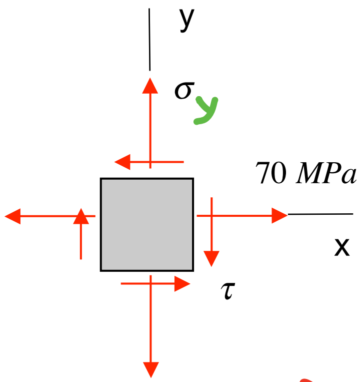


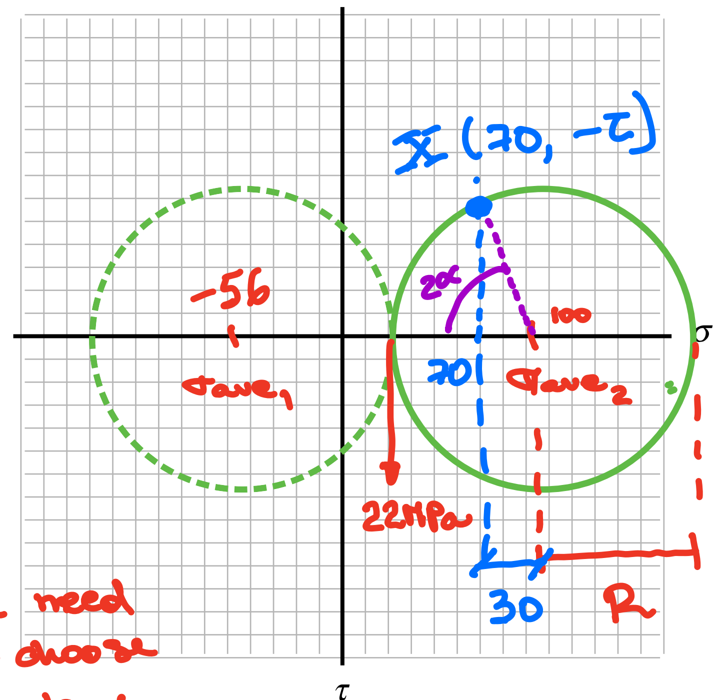


Example 13.3

The state of plane stress at a point shown below can be described by a known tensile stress $\sigma_x = 70\,MPa$, and unknown tensile stress σ_y and an unknown shear stress τ. At this point, the maximum in-plane shear stress is known to be 78 *MPa*, and one of the two in-plane principal stresses is 22 *MPa* (in tension).

Determine the values of σ and τ, as well as the other in-plane principal stress.





$R = 78$ MPa

$\sigma_1 = 22$ MPa

$\sigma_x = 70$ MPa

$\sigma_{ave_1} = 22 - R = -56$

$\sigma_{ave2} = 22 + R = 100$

we need to choose circle 1 (dashed) or circle 2 (solid)

check if $\sigma_x = 70$ is in the circle

choose circle 2

Then $\dfrac{\sigma_y + \sigma_x}{2} = 100 \rightarrow \boxed{\sigma_y = 130\ \text{MPa}}$

$\cos 2\alpha = \dfrac{30}{78} \rightarrow 2\alpha = 67.4$

$\sin 2\alpha = \dfrac{\tau}{R} \rightarrow \boxed{\tau = 72\ \text{MPa}}$

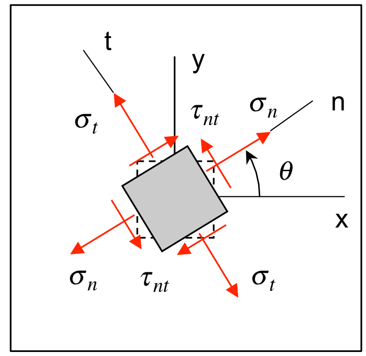
d) Mohr's circle: visualizing the stress transformation

As seen in equations (1a) and (2a), for a given state of plane stress at a point (σ_x, σ_y, τ_{xy}), we have the following stress transformation equations:

(1) $\longrightarrow$ $$\sigma_n - \sigma_{ave} = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

(2) $\longrightarrow$ $$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$.



Suppose we take the square of both sides of the above two equations and add together the results:

$(\sigma_n - \sigma_{ave})^2 + \tau_{nt}^2 = R^2$
$\downarrow$ x $\downarrow$ y

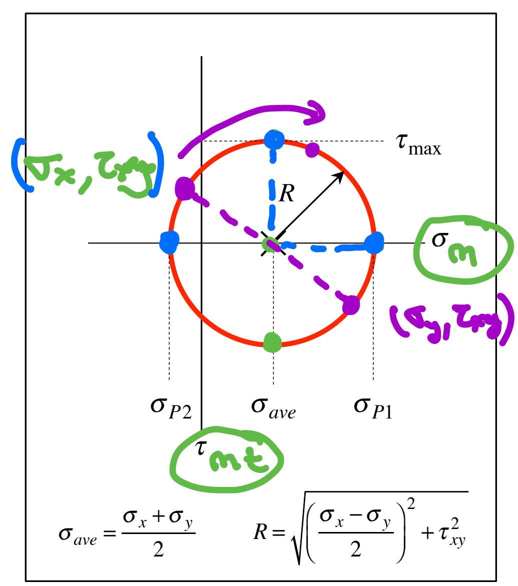
$\longrightarrow$ $$\boxed{\left(\sigma_n - \sigma_{ave}\right)^2 + \tau_{nt}^2} = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta\right]^2$$
$$+\left[-\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\right]^2$$
$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2\left(\cos^2 2\theta + \sin^2 2\theta\right) + \tau_{xy}^2\left(\sin^2 2\theta + \cos^2 2\theta\right)$$
$$\boxed{= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \triangleq R^2}$$

$R^2 = \tau_{xy}^2 + \frac{(\sigma_x - \sigma_y)^2}{4}$

The above shows us that if the results of the stress transformation equations (1a) and (2a) are plotted in the (σ, τ) space, the result is a circle:

- whose center is located at $(\sigma_{ave}, 0)$, where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$, and
- whose radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

as shown in the figure to the right. This representation is known as "Mohr's circle" for a given state of plane stress, $\left(\sigma_x, \sigma_y, \tau_{xy}\right)$.



What can we learn from Mohr's circle?

a) The principal stresses, σ_{P1} and σ_{P2}, are given by:

$$\sigma_{P1} = \sigma_{ave} + R$$
$$\sigma_{P2} = \sigma_{ave} - R$$

b) The principal stresses occur at stress element orientations at which the shear stress is zero, $\tau = 0$.

c) The maximum shear stress is given by:

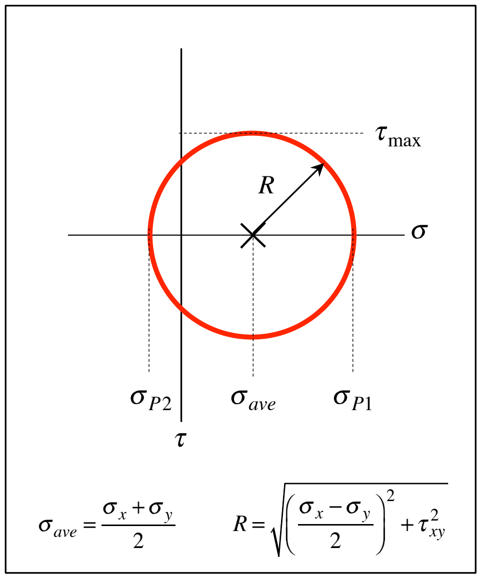
$$\tau_{\max} = R$$

d) The maximum shear stress occurs at stress element orientations at which the normal stress is $\sigma = \sigma_{ave}$.

These are all things that we discovered from analysis earlier when considering principal stresses and maximum shear stress. The Mohr's circle simply allows us to visualize these results and will help us to remember these important relations.

Using Mohr's circle to locate planes of principal stresses and in-plane maximum shear stress

Up to this point we have seen that Mohr's circle in the $\sigma - \tau$ plane provides us with information on the description of the state of plane stress: the stress states lie on a circle of radius $R = \sqrt{\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2 / 4}$ and centered on $\left(\sigma, \tau\right) = \left(\sigma_{ave}, 0\right)$, where $\sigma_{ave} = \left(\sigma_x + \sigma_y\right)/2$, and where σ_x, σ_y and τ_{xy} are the two normal components of stress and shear component of stress, respectively, corresponding to a set of *x-y* coordinate axes.



What we have *not* done at this point is discussed how to relate a transformed stress state through a rotation angle of θ to its location on the Mohr's circle in the $\sigma - \tau$ plane.

Before attempting this, let's review a couple points related to what we already know about stress states and Mohr's circle.

- Direction of positive shear stress in Mohr's circle. We have defined a positive shear stress on the x-face of the stress cube as being in the positive y-direction. Once we rotate this stress cube, this notation is equivalently stated as being positive in the n-face pointing in the t-direction. We will continue that here. However, here we will point the positive τ direction DOWNWARD in the $\sigma - \tau$ plane when constructing our Mohr's circle diagram. The reasoning behind this somewhat odd choice of positive direction is to maintain an equivalence in the direction of rotation of the element in the physical space with the direction of rotation (e.g., to insure that a CCW rotation in the physical space corresponds to a CCW rotation in Mohr's circle plane).

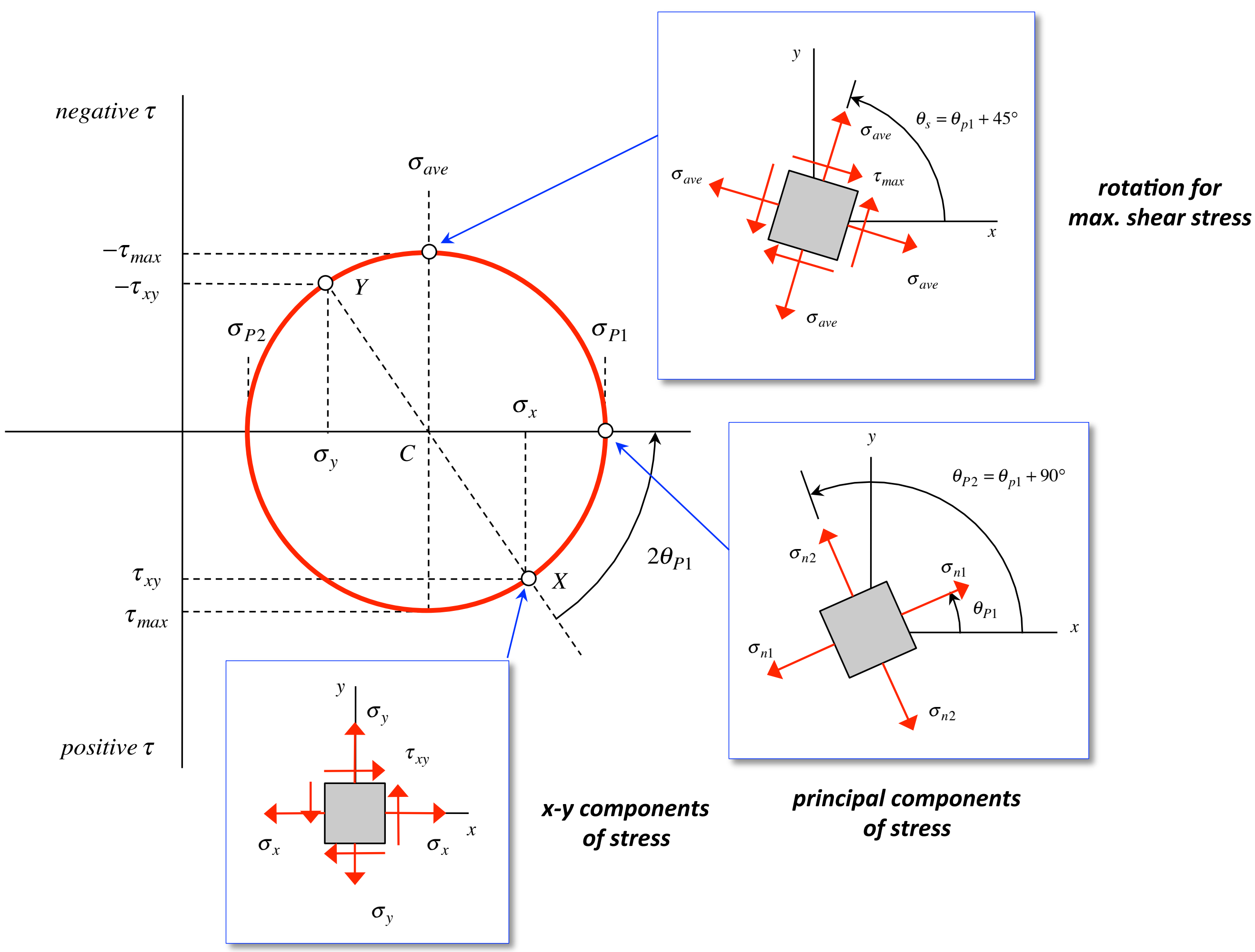
- Angle of rotation vs. angle in the $\sigma - \tau$ plane. Note that the stress transformation equations are all written in terms of the angle 2θ, where θ is the physical angle of rotation in the *x-y* plane:

$$\sigma = \sigma_{ave} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

As a result, a physical angle of rotation of θ corresponds to an angle of rotation of 2θ in the $\sigma - \tau$ plane.

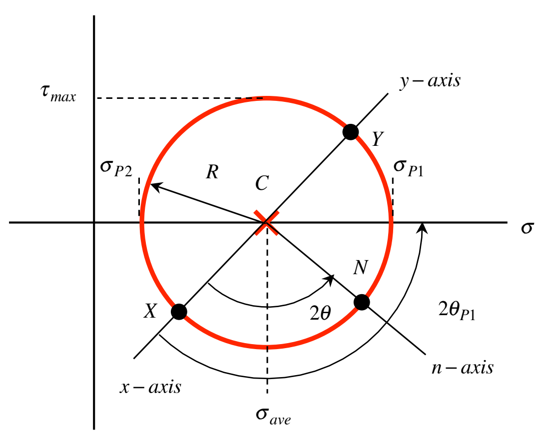
Both of these are demonstrated in the following figure. This figure contains much detailed information concerning the construction of Mohr's circle and the relationship of rotations in the physical $x - y$ plane to rotations in the $\sigma - \tau$ plane. In addition, we can readily see the locations of the principal stresses and maximum in-plane shear stress. Study this figure, and then move onto the next page where we have listed a series of steps that are convenient for constructing Mohr's circle from a state of stress $\left(\sigma_x, \sigma_y, \tau_{xy}\right)$.



Construction of Mohr's circle for a general state of plane stress

For a given state of stress (σ_x, σ_y, τ_{xy}) for a point:

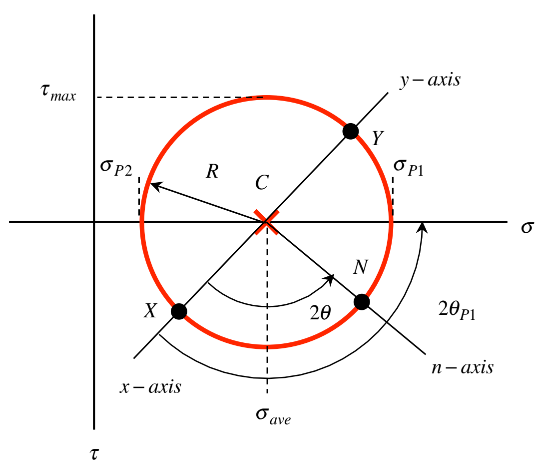
1) Establish a set of σ-τ axes (be sure to use the same scale on each axis)::
   - +σ points to *right*
   - +τ points *down*
2) Calculate the two parameters that define the location and size of Mohr's circle:
   - $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2} = \textit{average normal stress}$
   - $R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$
3) Draw a circle in the $\sigma - \tau$ plane with its center C at $(\sigma, \tau) = (\sigma_{ave}, 0)$ and having a radius of R.
4) Show the point X given by the coordinates $(\sigma, \tau) = (\sigma_x, \tau_{xy})$ on the Mohr's circle. Line OX is the x-axis. (Note that the y-axis is at a $180°$ from the x-axis in the $\sigma - \tau$ plane.)
5) The components of stress on the face of a stress element rotated through an angle of θ corresponds to a point N on Mohr's circle found through a rotation of 2θ on the circle.
6) The angle from the x-axis to the $\sigma -$axis in the Mohr's circle plane is $2\theta_{P1}$, where θ_{P1} the rotation angle for the stress element that produces the largest principal stress σ_{P1}. It is readily seen from the figure that the principal stresses are given by: $\sigma_{P1,P2} = \sigma_{ave} \pm R$.



Alternate (graphical) construction of Mohr's circle for a general state of plane stress

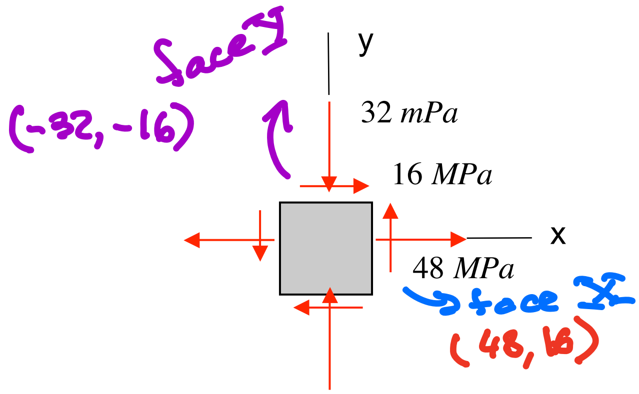
For a given state of stress (σ_x, σ_y, τ_{xy}) for a point:

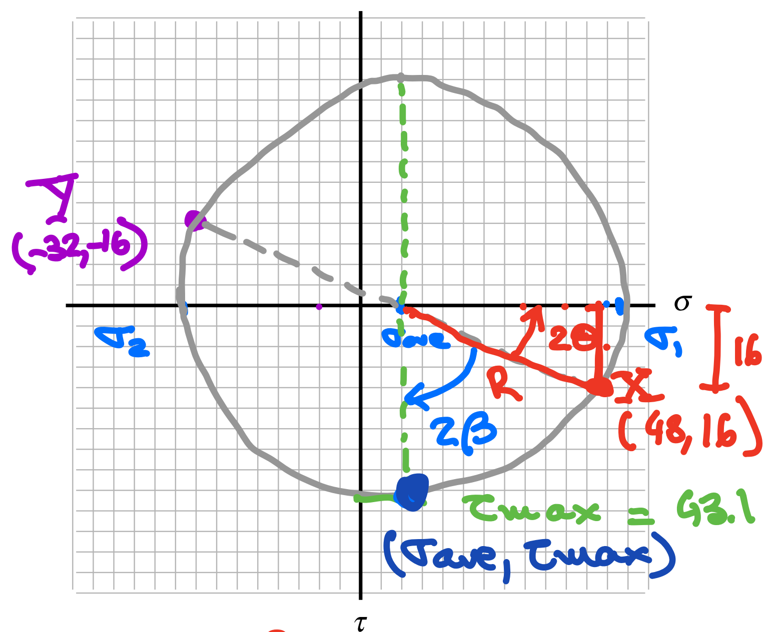
1) Establish a set of σ-τ axes (be sure to use the same scale on each axis):
   - +σ points to *right*
   - +τ points *down*
2) Locate points X and Y at locations $\left(\sigma_x, \tau_{xy}\right)$ and $\left(\sigma_y, -\tau_{xy}\right)$ on your set of axes.
3) Connect points X and Y with a straight line, and locate the center of the Mohr's circle at location C where this line crosses the $\sigma - axis$. This intersection occurs at $\left(\sigma_{ave}, 0\right)$. Note that the x- and y-axes correspond to lines CX and CY, respectively.
4) Draw a circle with its center at $\left(\sigma_{ave}, 0\right)$ and passing through points X and Y.
5) Calculate the radius of the circle using $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.
6) The components of stress on the face of a stress element rotated through an angle of θ corresponds to a point N on Mohr's circle found through a rotation of 2θ on the circle.
7) The angle from the x-axis to the $\sigma -$axis in the Mohr's circle plane is $2\theta_{P1}$, where θ_{P1} the rotation angle for the stress element that produces the largest principal stress σ_{P1}. It is readily seen from the figure that the principal stresses are given by: $\sigma_{P1,P2} = \sigma_{ave} \pm R$.



Example 13.4

Draw the in-plane Mohr's circle for the plane stress state shown below. What are the principal stresses and the maximum in-plane shear stress? Draw them in p.o.s.e.





$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{48-32}{2}$$

$$= 8 \text{ MPa}$$

$$R = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \underbrace{\tau_{xy}^2}_{=16^2}} = 43.1 \text{ MPa}$$

$$\sigma_1 = \sigma_{ave} + R = 51.1 \text{ MPa}$$

$$\sigma_2 = \sigma_{ave} - R = -35.1 \text{ MPa}$$

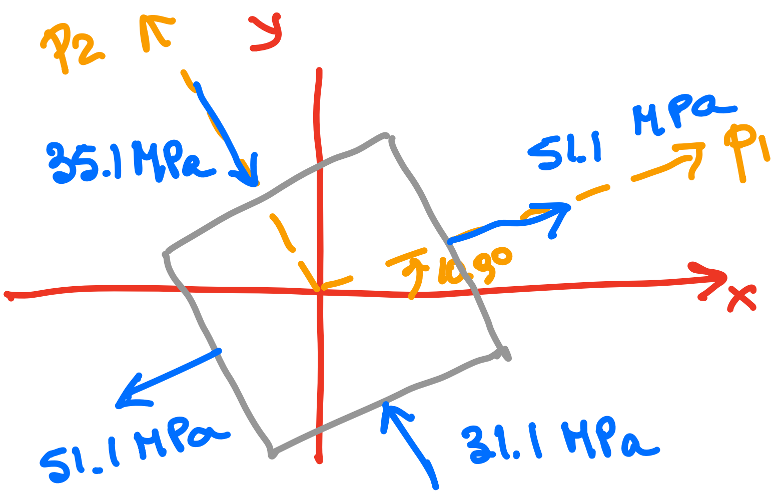
2θ? $\quad \sin 2\theta = \frac{16}{R}$

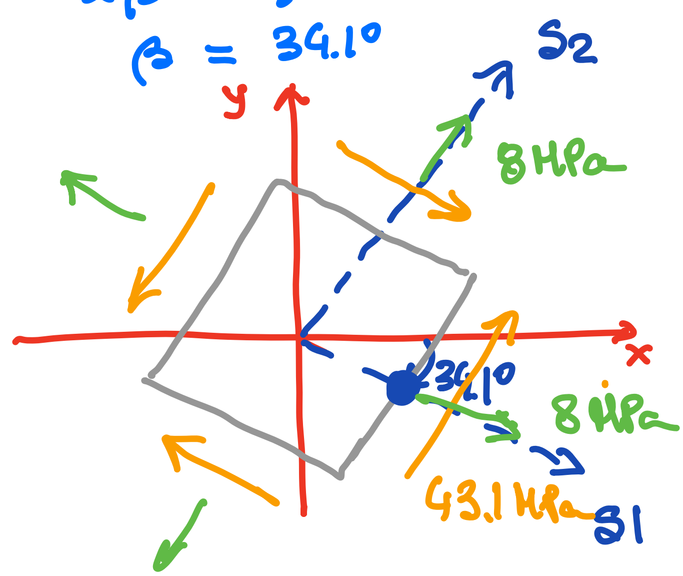
$$2\theta = 21.8°$$

$$\theta = 10.9$$

$$2\beta = 90 - 2\theta$$

$$\beta = 34.1°$$



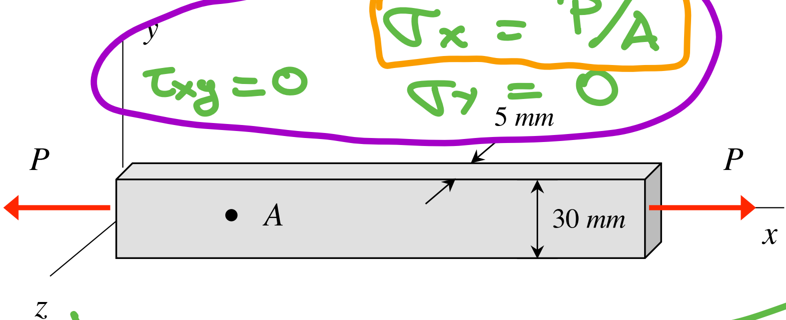


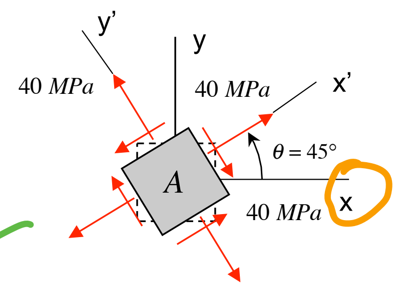
Example 13.5

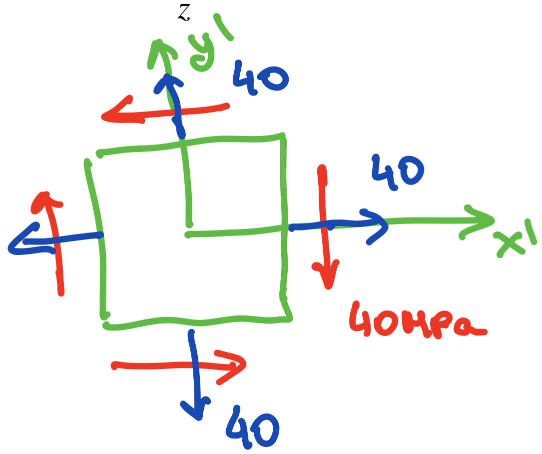
Draw the in-plane Mohr's circle for the plane stress state shown below. Determine the axial load P acting on the bar.

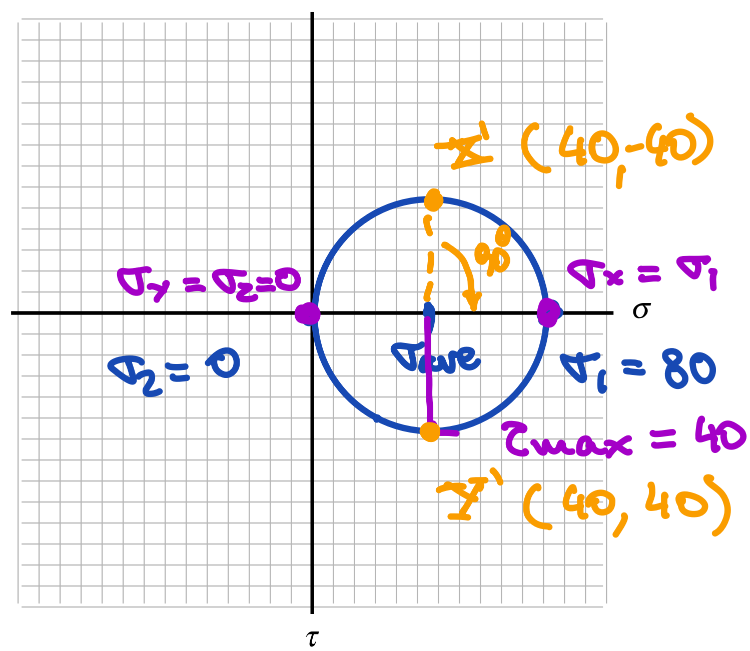
$\tau_{xy} = 0$ $\boxed{\sigma_x = P/A}$ $\sigma_y = 0$

$A = 30 \cdot 5 = 150\ mm^2$









$\sigma_{ave} = 40$

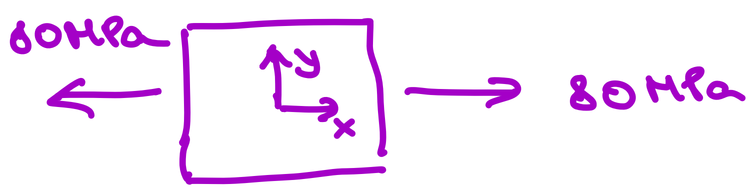
$R = 40$

$\sigma_1 = 80$

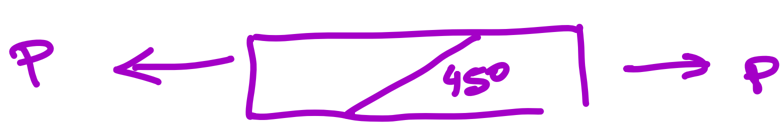
$\sigma_2 = 0$

$\sigma_x = 80\ MPa$

$\sigma_y = 0$



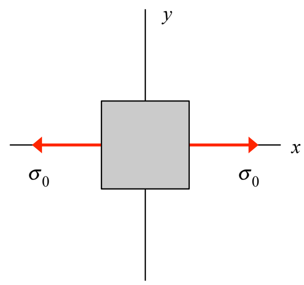
Note Max shear is 45° from axis of loading.



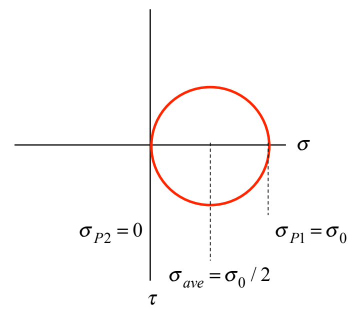
Mohr's circle examples – some special stress states

Uniaxial stress *(e.g., uniaxial load on member)*

$$\left(\sigma_x, \sigma_y, \tau_{xy}\right) = \left(\sigma_0, 0, 0\right)$$



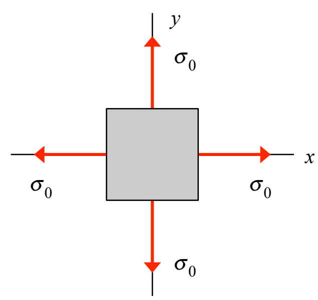
stress element



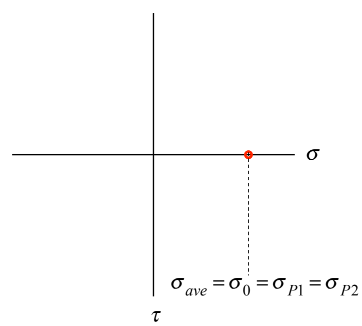
Mohr's circle

Hydrostatic stress *(e.g., thin-walled <u>spherical</u> pressure vessel)*

$$\left(\sigma_x, \sigma_y, \tau_{xy}\right) = \left(\sigma_0, \sigma_0, 0\right)$$



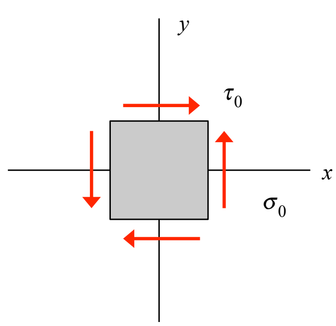
stress element



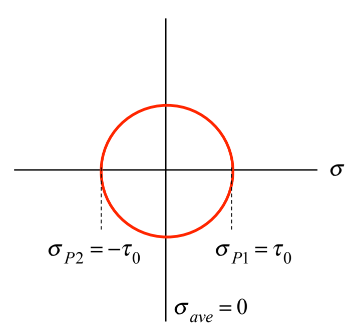
Mohr's circle

Pure shear stress *(e.g., axial torque on member)*

$$\left(\sigma_x, \sigma_y, \tau_{xy}\right) = \left(0, 0, \tau_0\right)$$

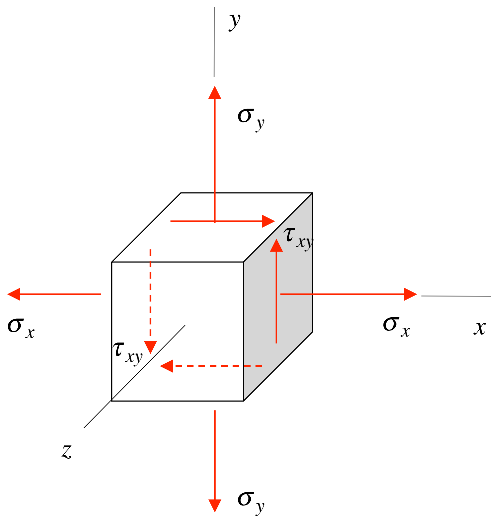


stress element

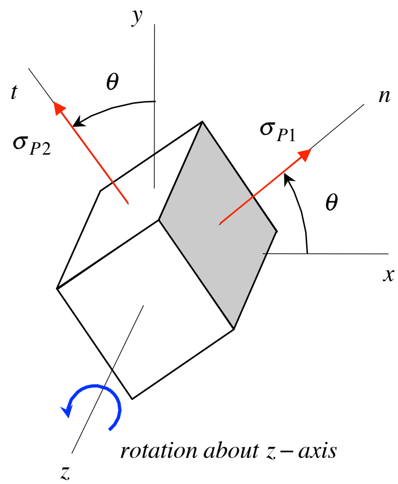


Mohr's circle

e) Absolute maximum shear stress for plane stress



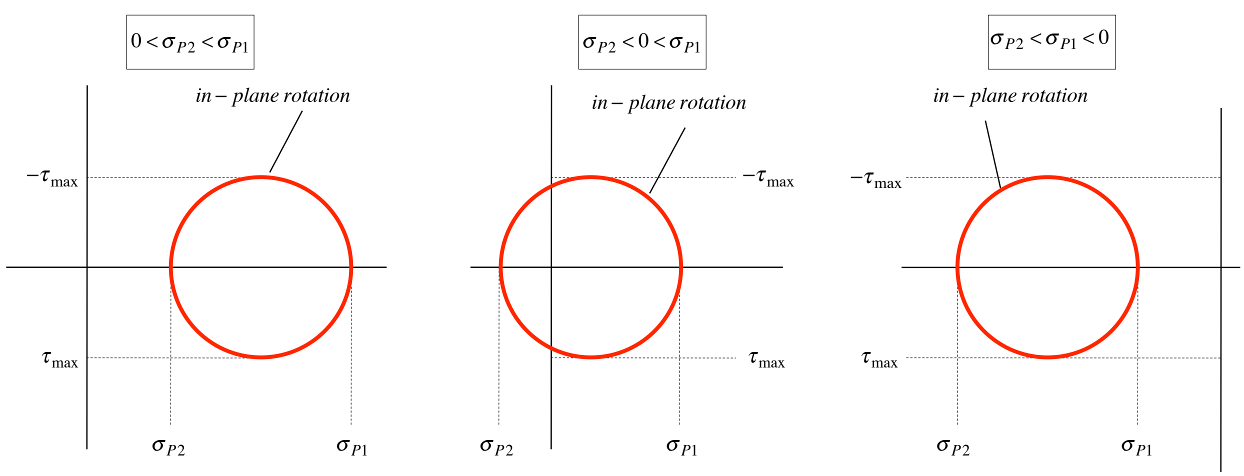
stress element in plane stress



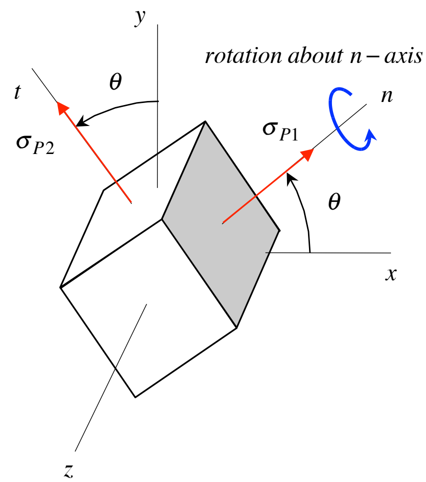
stress element in rotated to orientation for principal stresses

Recall for a state of plane stress, the stress element shows zero normal and shear stresses on the *z*-face. As we rotate the stress element about the *z*-axis, we observe the normal and shear stresses on the other four faces change according to our stress transformation equations. When rotated to the orientation showing the principal stresses, only the normal stresses appear on these four faces, as shown in the figure above right.

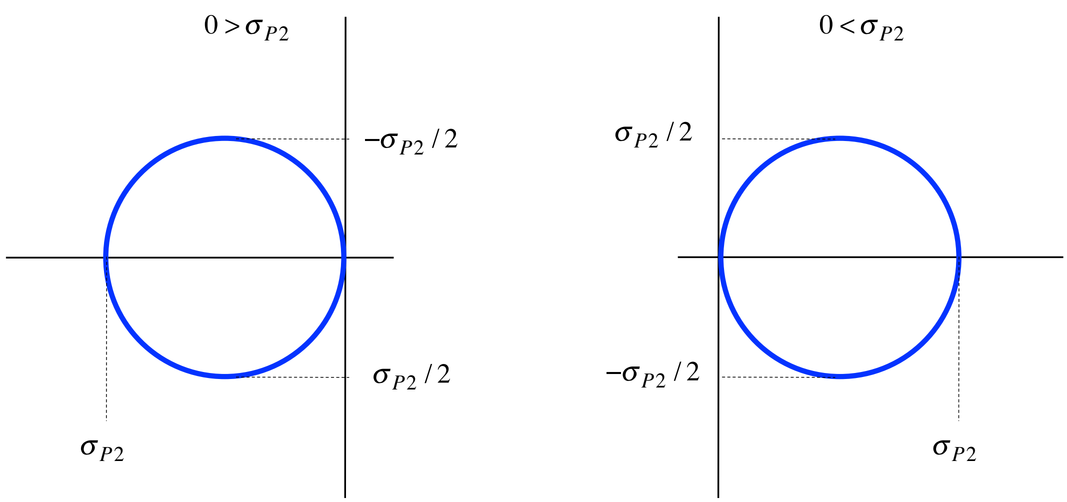
The Mohr's circle for the stress element rotation described above is shown in the following figures, considering the three possibilities on the signs of the principal stresses σ_{P1} and σ_{P2} .



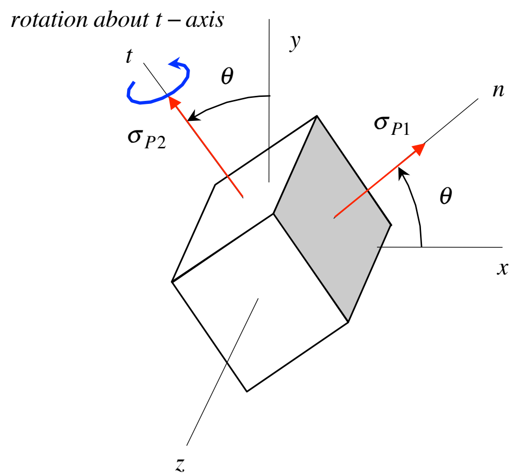
Now, starting with the orientation of the stress element corresponding to the principal stresses, say we rotate the stress element about the "n" axis, as shown below.

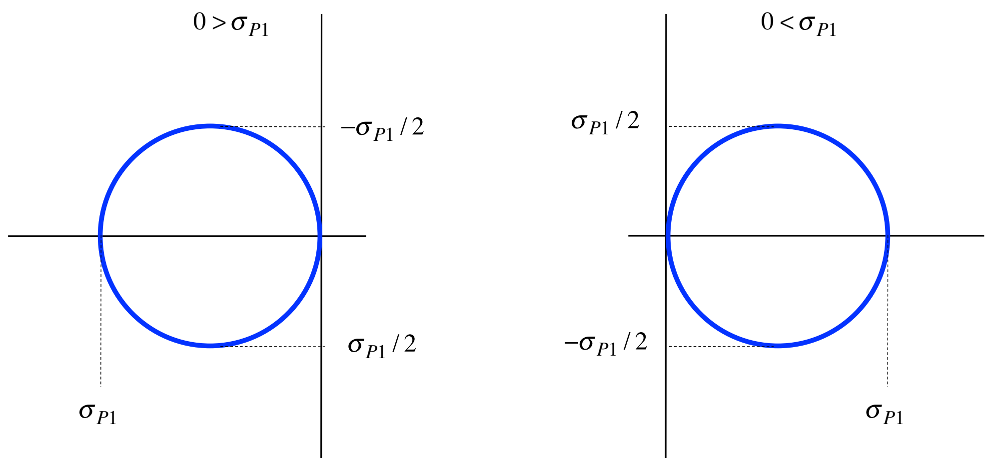


Since the z-face of the stress element is stress free, a 90° rotation produces zero normal and zero shear stress on the t-face. The Mohr's circle for this rotation is shown below, considering the two possibilities on the sign for σ_{P2}. Note that $\sigma = 0$ is an out-of-plane principal stress.

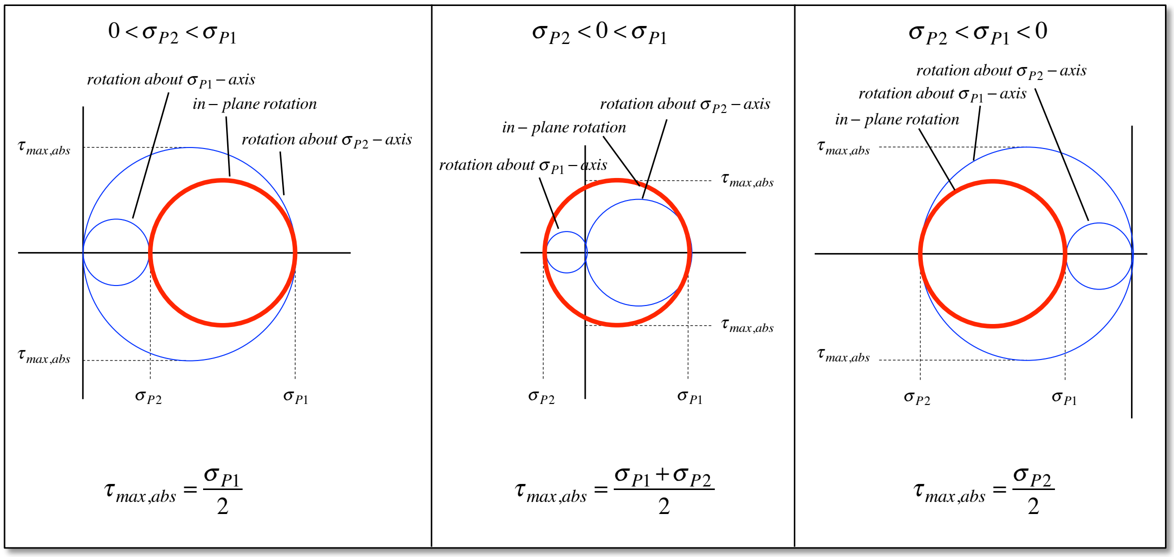


Similarly, a rotation of the stress element about the "t" axis, starting with the principal stresses orientation, produces the Mohr's circle shown below, depending on the sign of σ_1. Note that $\sigma = 0$ an the out-of-plane principal stress.





Suppose that we superimpose the Mohr's circle for these three stress element rotations. In the end, we will have three sets of Mohr's circles, depending on the signs of the principal stresses σ_{P1} and σ_{P2}, as shown in the following.



In conclusion, we see that the "absolute maximum shear stress", $\tau_{max,abs}$, (the largest shear stress observed for both in-plane and out-of-plane rotations) depends on the signs of the in-plane principal stresses, σ_{P1} and σ_{P2}: if they are of the same sign, then $\tau_{max,abs}$ is half of the larger of the two, and if they are of opposite signs, then $\tau_{max,abs}$ is the average value of the two in-plane principal stresses.

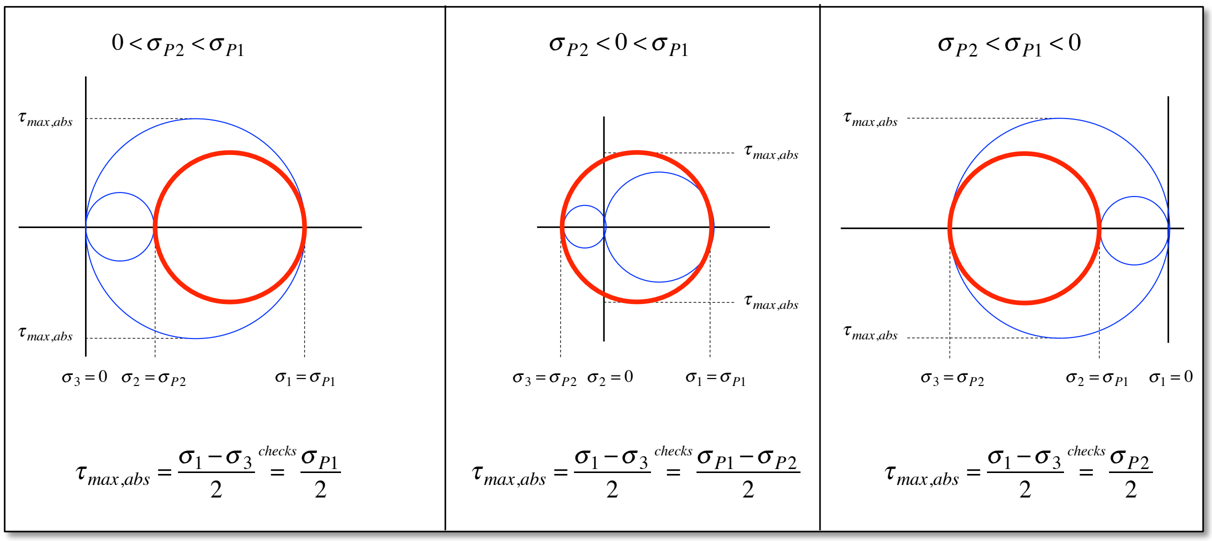
An alternate way (and easier to remember way) to express this result is to first introduce the following notation for principal stresses: consider the two in-plane principal stresses σ_{P1} and σ_{P2}, and the out-of-plane principal stress (which is zero) and rename these in the following way:

- σ_1 is the *largest* of the three
- σ_3 is the *smallest* of the three
- σ_2 is the *intermediate* of the three

That is, $\sigma_3 \le \sigma_2 \le \sigma_1$. With this notation, we can simply write:

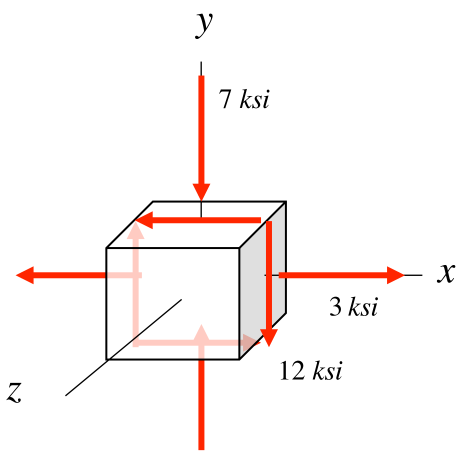
$$\tau_{max,abs}=\frac{\sigma_1-\sigma_3}{2}$$

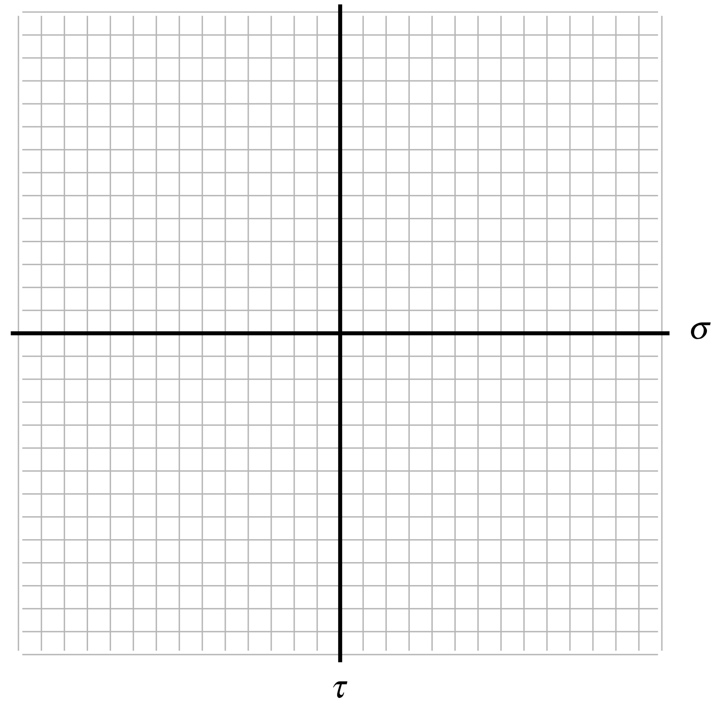
as summarized in the following.


$0 < \sigma_{P2} < \sigma_{P1}$
$\tau_{max,abs}$
$\tau_{max,abs}$
$\sigma_3 = 0$ $\sigma_2 = \sigma_{P2}$ $\sigma_1 = \sigma_{P1}$
$\tau_{max,abs} = \frac{\sigma_1 - \sigma_3}{2} \overset{checks}{=} \frac{\sigma_{P1}}{2}$
$\sigma_{P2} < 0 < \sigma_{P1}$
$\tau_{max,abs}$
$\tau_{max,abs}$
$\sigma_3 = \sigma_{P2}$ $\sigma_2 = 0$ $\sigma_1 = \sigma_{P1}$
$\tau_{max,abs} = \frac{\sigma_1 - \sigma_3}{2} \overset{checks}{=} \frac{\sigma_{P1} - \sigma_{P2}}{2}$
$\sigma_{P2} < \sigma_{P1} < 0$
$\tau_{max,abs}$
$\tau_{max,abs}$
$\sigma_3 = \sigma_{P2}$ $\sigma_2 = \sigma_{P1}$ $\sigma_1 = 0$
$\tau_{max,abs} = \frac{\sigma_1 - \sigma_3}{2} \overset{checks}{=} \frac{\sigma_{P2}}{2}$

Example 13.6

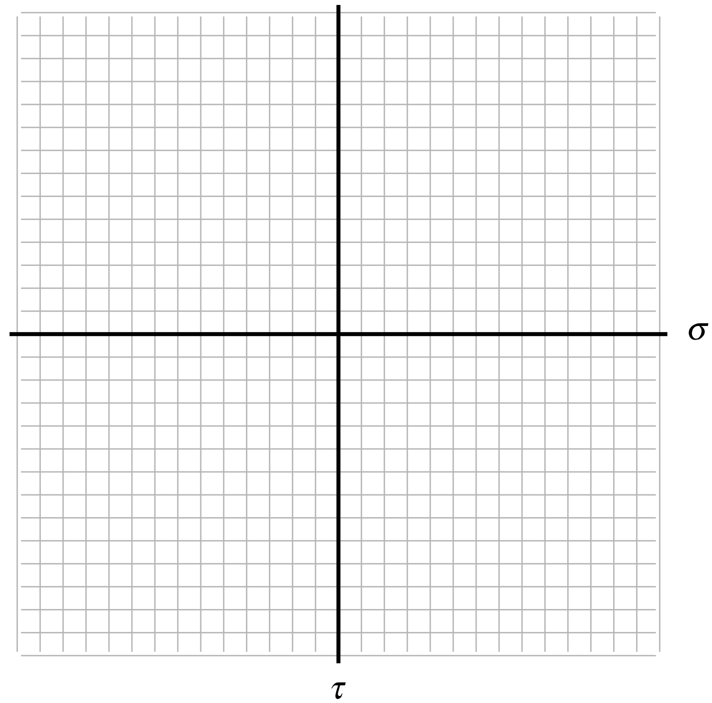
Determine the principal stresses and the absolute maximum shear stress for the plane stress element shown.



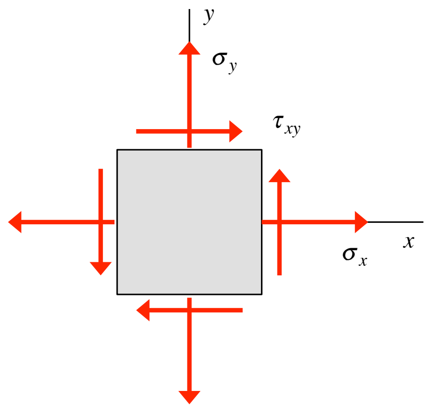


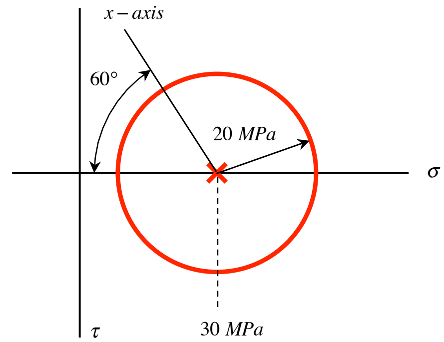
Example 13.7

A state of plane stress is given by: $\sigma_x = 12\ ksi$, $\sigma_y = 12\ ksi$ and $\tau_{xy} = 4\ ksi$. Determine the principal stresses and the absolute maximum shear stress for this state of stress.



Example 13.8





The Mohr's circle for a stress state is presented above.

a) Show the locations of the y-axis in the Mohr's circle above.

b) Determine the principal stresses and the absolute maximum shear stress for this state.

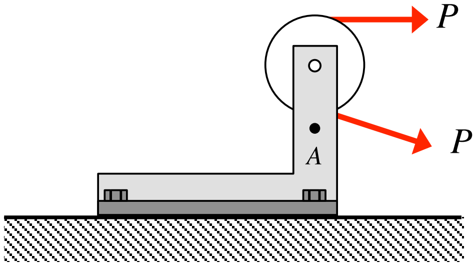
c) Determine the values for σ_x, σ_y and τ_{xy} of this stress state.

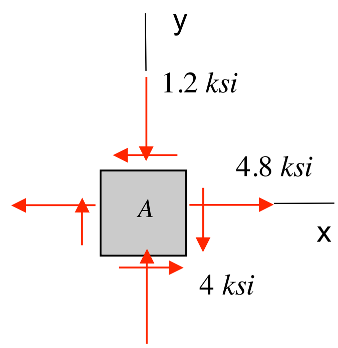
Example 13.9

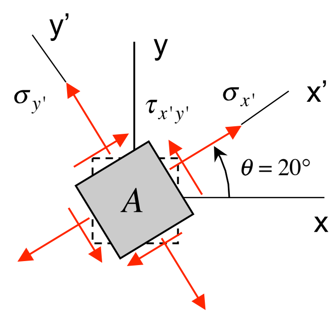
Consider the loaded pulley bracket shown below.

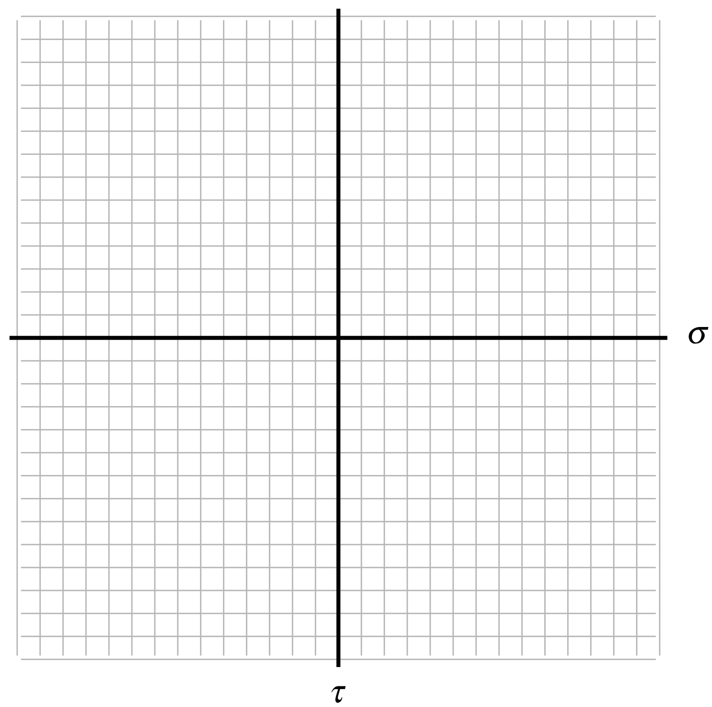
a) Determine $\sigma_{x'}$, $\sigma_{y'}$ and $\tau_{x'y'}$ corresponding to $\theta = 20°$.

b) Determine the principal stresses and maximum in-plane shear stress, along with the corresponding rotation angles.

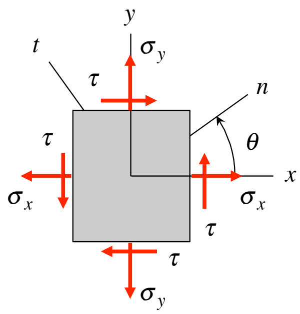


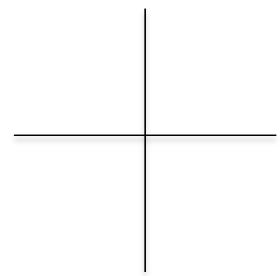






Reflection: Stress transformations and Mohr's circle (for a state of plane stress)





a) *What* is a *stress element*?

b) *Why* are we interested in *stress transformations*?

c) There are the *two important parameters* that we need to represent a state of stress – what are they?

d) What are *principal stresses*? How are these related to the two parameters mentioned in c) above?

e) What is the *maximum in-plane shear stress*? How is this related to the two parameters mentioned in c) above?

f) Where is the *center* of Mohr's circle? What is the *radius* of Mohr's circle?

g) *Why* do we choose the "*positive*" direction of τ as *downward*?

h) *How* do we know that a rotation of θ in the physical world correspond to a rotation of 2θ in Mohr's circle?

i) *How* can we use Mohr's circle to find the rotations of the stress element that correspond to the principal components of stress and the maximum in-plane shear stress?

Additional notes: