16. Energy methods

Objectives:
To develop expressions for the strain energy for loaded structural elements and to use these expressions for the determination of elastic deformations in the structural elements due to the loadings.

Background:

- *Work/energy equation*
  For a given system, the total work done on the system is equal to the change in total energy:
  $$W^{(nc)} = \Delta T + \Delta U$$
  where $W^{(nc)}$ is the work done on the system by non-conservative forces, ΔT is the change in kinetic energy and ΔU is the change in potential energy.

- *Equilibrium*
  For a system in equilibrium, the work/energy equation reduces to:
  $$W^{(nc)} = \Delta U$$
  which says that the change in potential energy is equal to the work done on the system.

- *Strain energy in springs*
  Recall that the potential energy in a spring is given by:
  $$U = \frac{1}{2} k \Delta^2$$
  where k is the stiffness of the spring and Δ is the stretch/compression in the spring. Since this potential energy results from the change in strain in the spring, this is often times referred to as the "strain energy" in the spring.

Lecture topics:

a) Expressions for strain energy in a structural element.

b) Using the work-energy principle for determining deflections.

c) Castigliano's second theorem for determinate structures.

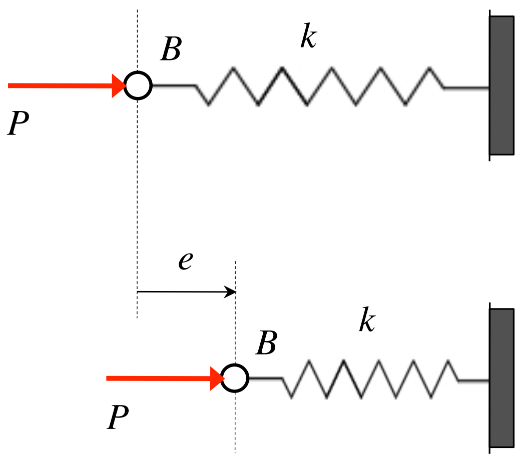
d) Castigliano's second theorem for indeterminate structures.

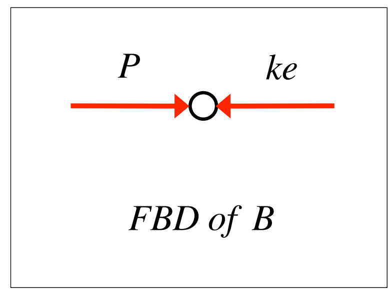
Lecture notes

a) Strain energy expressions

Motivating example #1

Consider a spring of stiffness k acted upon by a constant magnitude force P. Assume that the spring is uncompressed before the application of the force. Let e represent the compression in the spring resulting from the application of the force P. Write down the equilibrium form of the work energy equation for the system.





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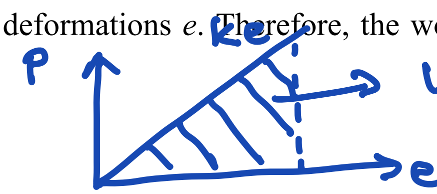
For equilibrium, we know that $P = ke$ for all deformations e. Therefore, the work done by P under equilibrium conditions is:



$W^{(P)} = \frac{1}{2}ke^2 = \frac{1}{2}Pe$

$$W^{(P)} = \int_0^e P\,de = k\int_0^e e\,de = \frac{1}{2}ke^2 = \frac{1}{2}Pe$$

We know that the potential (strain) energy in the spring can be written down directly as: $U = \frac{1}{2}ke^2$. However, for practice, let's derive this expression. To do so, recall that the change in potential of a conservative force is equal to the negative of the work done by the force:

$$W^{(sp)} = -\int_0^e ke\,de \quad ; \quad \text{"-"}\ \textit{since spring force opposes motion}$$

$$= -\frac{1}{2}ke^2$$

Therefore,

$$U = -W^{(sp)} = \frac{1}{2}ke^2 \quad \text{(which agrees with what we already knew above)}$$

From this, the work/energy equation for equilibrium is:

$$W^{(P)} = U \quad \Rightarrow \quad \frac{1}{2}Pe = \frac{1}{2}ke^2$$

Alternate representation:

Since we are considering equilibrium, $P = ke$, we could have written the strain energy in the spring as:

$$U = \frac{1}{2}ke^2 = \frac{1}{2}k\left(\frac{P}{k}\right)^2 = \frac{P^2}{2k}$$

This representation directly shows the dependence of the strain energy on both the applied load and the stiffness of the spring. For this expression, the work energy equation $W^{(P)} = U$ takes on the form of:

$$\frac{1}{2}Pe = \frac{P^2}{2k}$$

from which we can recover the expected expression for the static elongation of the spring:

$$e = \frac{P}{k}$$

Newton's

$F = m \cdot \underline{a}$

$\downarrow \qquad \underline{a} = 0$

$\Sigma F = 0$

$P = ke$

$e = \frac{P}{k}$

Energy

$U = \frac{1}{2}ke^2$

$W^{(P)} = \int_0^e P \, de$

$W^{(P)} = \frac{Pe}{2}$

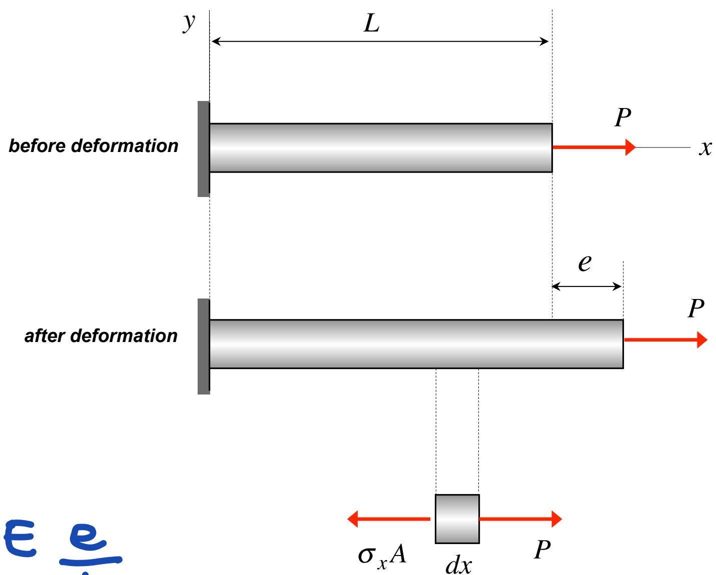
$U = W^{(P)}$

$\frac{1}{2}ke^2 = \frac{Pe}{2}$

$e = \frac{P}{k}$

Motivating example #2

Consider a straight rod (with constant cross-sectional area A) under the action of an extensive axial load *P* and fixed to ground on its left end. Determine an expression for the strain energy in the rod as a result of the axial load *P*.



$e = \frac{P \cdot L}{EA}$

$P = \sigma_x \cdot A$

$\sigma_x = E\,\varepsilon_x = E\,\frac{e}{L}$

The axial load *P* is related to the axial stress through:

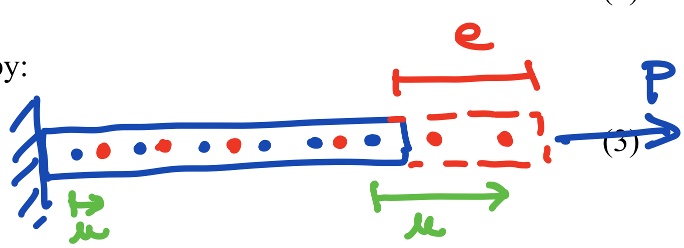
$$P = \sigma_x A \tag{1}$$

For a linearly elastic material:

$$\sigma_x = E\varepsilon_x = E\frac{du}{dx} \tag{2}$$

And, the elongation in the rod is given by:

$$e = L\frac{du}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{e}{L} \tag{3}$$



Combining (1)-(3) gives:

$$P = \frac{EA}{L}e \tag{4}$$

$u(x) \rightarrow \frac{du}{dx} = \varepsilon$

The work done by the axial load P is given by:

$$W^{(P)} = \int_0^e P\,de = \int_0^e \frac{EA}{L}e\,de = \frac{1}{2}\frac{EA}{L}e^2 = \frac{1}{2}Pe \tag{5}$$

Since $U = W^{(P)}$, the strain energy in the rod is given by:

$$U = \frac{1}{2}Pe \tag{6}$$

Alternate representation:

From equation (1),

$$e = \frac{PL}{EA}$$

we can write the strain energy in the rod as:

$$U = \frac{1}{2}\frac{P^2 L}{EA} \tag{7}$$

This representation directly shows the dependence of the strain energy on both the applied load and the material and properties of the rod. For this expression, the work energy equation $W^{(P)} = U$ takes on the form of:

$$\frac{1}{2}Pe = \frac{1}{2}\frac{P^2 L}{EA}$$

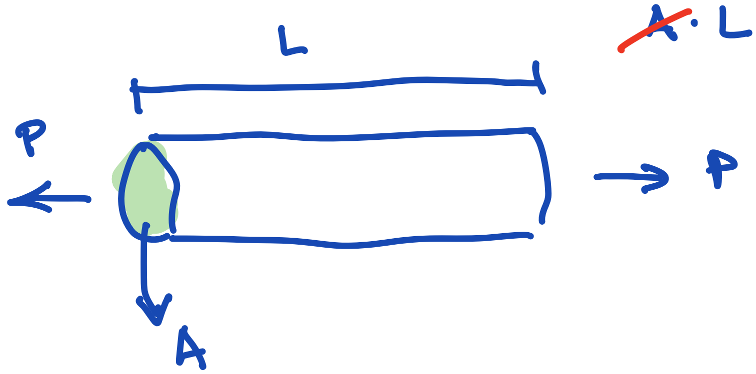
from which we can recover the expected expression for the static elongation of the spring:

$$e = \frac{PL}{EA}$$

Define strain energy density

$$U = \frac{1}{2}Pe \qquad \bar{U} = \frac{U}{\text{Volume}} = \frac{\frac{1}{2}P\cdot e}{A\cdot L}$$

$$= \frac{1}{2}\frac{\sigma_x \cdot \cancel{A} \cdot e}{\cancel{A} \cdot L} = \frac{1}{2}\sigma_x \cdot \epsilon_x$$



$\Delta T = 0$

General expressions for strain energy and work

The total strain energy for a linearly elastic body can be written as:

$$U = \int_{vol} \bar{u}\, dV \tag{8}$$

where:

$$\bar{u} = \textit{strain energy density function}$$
$$= \frac{1}{2}\left[\sigma_x\left(\varepsilon_x - \alpha\Delta T\right) + \sigma_y\left(\varepsilon_y - \alpha\Delta T\right) + \sigma_z\left(\varepsilon_z - \alpha\Delta T\right) + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right] \tag{9}$$

From this general expression above, we will derive strain energy functions for a number of types of components, including: rods, shafts and bending beams.

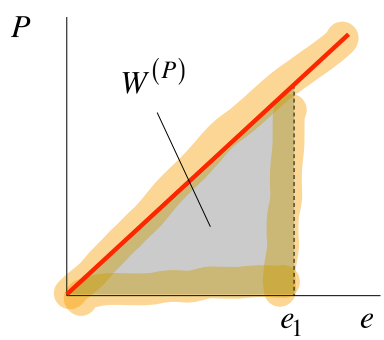
Also, recall that the work due a force P acting through a distance e_1 can be written as:

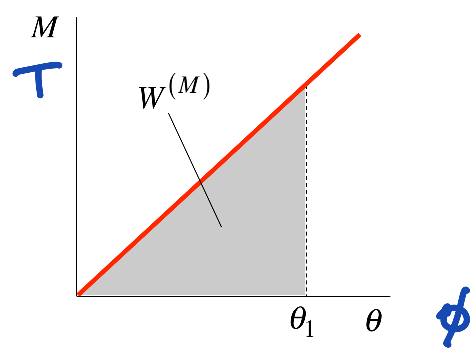
$$W^{(P)} = \int_0^{e_1} P\, de$$

And, the work due to a couple M acting through ~~an~~ a rotation angle θ_1 can be written as:

$$W^{(M)} = \int_0^{\theta_1} M\, d\theta$$

Suppose that these forces and moments act slowly (such that dynamic effects are not significant) and with *linear* relationships between P and e, and between M and θ, as indicated by the plots below.





In this case, the work due to P and and the work due to M (the areas under the respective P vs. e and M vs. θ curves) can be written as:

$$W^{(P)} = \frac{1}{2} P\left(e_1\right) e_1$$

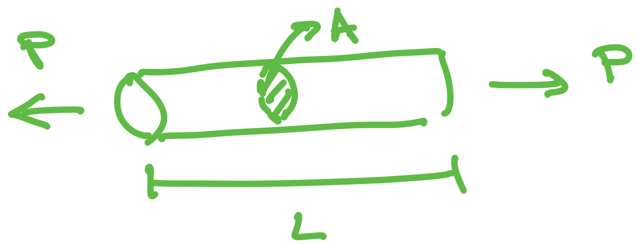
$$W^{(M)} = \frac{1}{2} M\left(e_1\right) \theta_1$$

$$W^{(T)} = \frac{1}{2} T \cdot \phi$$

Note that the second expression above applies to both a torque T acting through a twist angle of ϕ and to a bending moment M acting through an angle of beam rotation θ.

Component: rod carrying axial load P

Here we consider a rod of length L, cross-sectional area A and Young's modulus E carrying an axial load of P. For axially-loaded rods, we have the following stress and strain functions:



$$\sigma_x = E(x)\frac{du(x)}{dx} \overset{OR}{=} \frac{P(x)}{A(x)}$$

$$\varepsilon_x = \frac{du(x)}{dx} \overset{OR}{=} \frac{P(x)}{A(x)E(x)}$$

and, in addition, $dV = A(x)\,dx$.

Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \sigma_x \varepsilon_x A\,dx = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx \qquad (10a)$$

OR

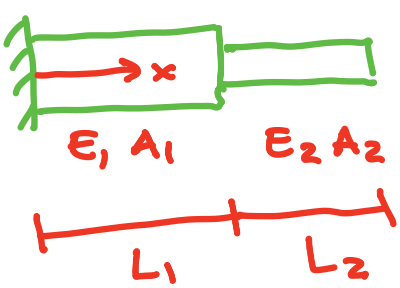
$$\boxed{U = \frac{1}{2}\int_0^L \sigma_x \varepsilon_x A\,dx = \frac{1}{2}\int_0^L \frac{P^2}{EA} dx} \qquad (10b)$$

where, in general, E, A, P and u may all be functions of x.

For the special case where E, A, P and u are all constants in x, expression (10b) reduces to:

$$U = \frac{1}{2}\frac{P^2 L}{EA} \qquad (10c)$$

as we derived earlier.



P $U = \frac{1}{2}\int_0^L \frac{F(x)^2\,dx}{A(x)\,E(x)}$

$= \frac{1}{2}\frac{F_1^2 L_1}{A_1 E_1} + \frac{1}{2}\frac{F_2^2 L_2}{A_2 E_2}$

Component: circular shaft carrying torque T

Here we consider a circular cross section shaft of length L, polar area moment I_P and shear modulus G carrying a torque of T. For a circular shaft carrying a torque T along the x-axis, we have the following stress and strain functions:

$$\tau = G(x)\rho \frac{d\phi(x)}{dx} \overset{OR}{=} \frac{T(x)\rho}{I_P(x)}$$

$$\gamma = \rho \frac{d\phi(x)}{dx} \overset{OR}{=} \frac{T\rho}{GI_P}$$

$$U = \int_{vol} \bar{U}\, dV = \int_{Area}\int_0^L \bar{U}\, dx\, dA$$

where ρ is the radial distance from the centerline of the shaft cross section, ϕ is the angle of twist and, in addition, $dV = dA\,dx$.

Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \int_{area} \tau\gamma\, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} \rho^2\, dA \right) G \left(\frac{d\phi}{dx} \right)^2 dx = \frac{1}{2}\int_0^L GI_P \left(\frac{d\phi}{dx} \right)^2 dx \qquad (11a)$$

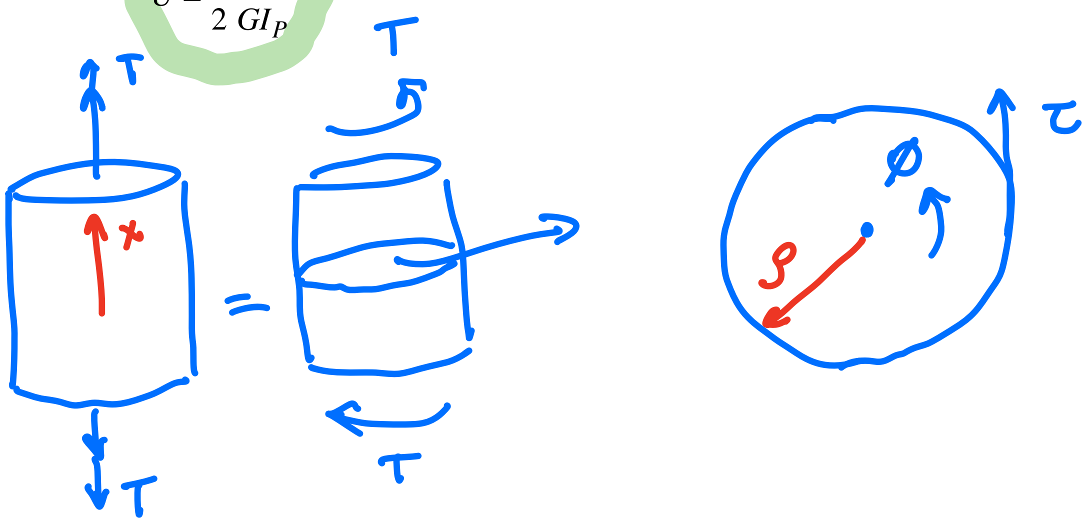
OR

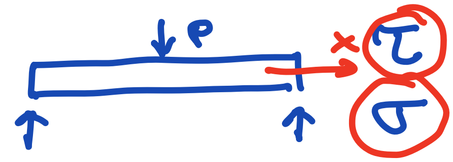
$$U = \frac{1}{2}\int_0^L \int_{area} \tau\gamma\, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} \rho^2\, dA \right) \frac{T^2}{GI_P{}^2} dx = \frac{1}{2}\int_0^L \frac{T^2}{GI_P} dx \qquad (11b)$$

where $I_P = \int_{area} \rho^2\, dA$. Here G, I_P, T and ϕ may all be functions of x.

For the special case where G, I_P, T and ϕ are all constants in x, expression (11b) reduces to:

$$U = \frac{1}{2}\frac{T^2 L}{GI_P} \qquad (11c)$$





Component: bending beam – flexural strain energy

Here we consider a thin beam of length L, second area moment I and Young's modulus modulus E. The transverse deflection of the beam is $v(x)$, the bending moment in the beam is $M(x)$ and with y being the cross sectional coordinate in the direction transverse to the beam. For a thin Euler-Bernoulli beam we have the following stress and strain components corresponding to the normal (flexural) stress:

$$\sigma_x = -E(x)y\frac{d^2v(x)}{dx^2} = -\frac{M(x)y}{I(x)}$$

$$U = \int_0^L \iint_{Area} \bar{\sigma} \, dA \, dx$$

$$\varepsilon_x = -y\frac{d^2v(x)}{dx^2} = \frac{M(x)y}{E(x)I(x)}$$

and, in addition, $dV = dA\,dx$. Here we will assume that the Young's modulus does not vary across the beam's cross-section. Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \int_{area} \sigma_x \varepsilon_x \, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} y^2\, dA \right) E \left(\frac{d^2 v}{dx^2} \right)^2 dx = \frac{1}{2}\int_0^L EI \left(\frac{d^2 v}{dx^2} \right)^2 dx \qquad (12a)$$

OR

$$U = \frac{1}{2}\int_0^L \int_{area} \sigma_x \varepsilon_x \, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} y^2\, dA \right) \frac{M^2}{EI^2} dx = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx \qquad (12b)$$

$$\rightarrow \frac{1}{2}\frac{M^2 L}{EI}$$

where $I = \int_{area} y^2\, dA$. Here E, I, M and v may all be functions of x.

Component: bending beam – shear strain energy

For a bending beam, we also have energy that is attributed to shear stress/strain. For the same notation as before, we can write for the shear stress and shear strain:

$$\tau_{xy} = \frac{V(x)Q(x,y)}{I(x)t(y)}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{1}{G}\frac{V(x)Q(x,y)}{I(x)t(y)}$$

Substituting into the general strain energy expression (8) gives:

$$U = \frac{1}{2}\int_0^L \int_{area} \tau_{xy}\gamma_{xy}\, dA\, dx$$

$$= \frac{1}{2}\int_0^L \left(\int_{area} \frac{Q^2(x,y)}{t^2(y)} dA \right) \frac{V^2}{GI^2} dx = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$$

where:

$$f_s(x) = \frac{A(x)}{I^2(x)} \int_{area} \frac{Q^2(x,y)}{t^2(y)} dA = \text{"}\textit{form factor}\text{" } \textit{for the beam cross section}$$

Note that the form factor expression above has been calculated for some common cross-sections, as presented to the right.

rectangle	$f_s = \frac{6}{5}$
circle	$f_s = \frac{10}{9}$
thin-walled tube	$f_s = 2$

Summary

The strain energy functions for the three types of members investigated here (axially-loaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$\boxed{U=\frac{1}{2}\int_0^L \frac{F^2 dx}{EA}}$ $\frac{1}{2}\frac{F^2 L}{EA}$	$U=\frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U=\frac{1}{2}\int_0^L \frac{T^2}{GI_p}dx$ $=\frac{1}{2}\frac{T^2 L}{G I_p}$	$U=\frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$\boxed{U_\sigma=\frac{1}{2}\int_0^L \frac{M^2}{EI}dx}$	$U_\sigma \frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$\boxed{U_\tau=\frac{1}{2}\int_0^L \frac{f_s V^2}{GA}dx}$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.

c) Work-energy equation

Recall that the work-energy equation for a system can be written as:

$$W = T + U$$

For static equilibrium, the change in kinetic energy T is zero. Therefore, the above reduces to:

$$W = U$$

The usage of the work-energy equation above is very limited in its usefulness in displacement analysis. For simple systems of having an applied load acting at only a single point, the work-energy equation can be used to determine the static deflection of the structure at the point at which the load is applied. For more complicated loads, we will still have only a single work-energy equation for loads at multiple points; however, we will need multiple equations to solve for displacements. In that case, we need to appeal to more advanced methods, such as Castigliano's methods that follow.

d) Castigliano's Second Theorem – applied to determinate structures

Consider a determinate linearly elastic deformable body or system acting upon by N forces P_i ; $i = 1,2,...,N$. Among all possible equilibrium configurations of the system, the actual configuration is the one for which:

$$\boxed{\Delta_i = \frac{\partial U}{\partial P_i}} \quad ; \quad i = 1,2,...,N$$

$U = U(P, P_2 P_3 ... P_N)$

where Δ_i is the displacement corresponding to and in the direction of the force P_i, and U is the strain energy for the system.

e) Castigliano's Second Theorem – applied to indeterminate structures

Consider an indeterminate linearly elastic deformable body or system acting upon by N forces P_i ; $i = 1,2,...,N$. Since the system is indeterminate, there will be a number (N_R) of redundant forces in the strain energy function: R_i ; $i = 1,2,...,N_R$. Among all possible equilibrium configurations of the system, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \quad ; \quad i = 1,2,...,N$$

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1,2,...,N_R$$

where Δ_i is the displacement corresponding to and in the direction of the force P_i (or R_i), and U is the strain energy for the system.

Comments on the usage of Castigliano's theorem for deflection analysis

a) For *determinate structures*, one is able to solve for the external reactions directly through equilibrium equations. As a result, it is possible to also find the internal resultants (such as shear forces, axial forces and bending moments), and consequently, the strain energy for the structure can be written in terms of only the applied forces that are used for find deflections.

b) For *indeterminate structures*, one has too few equilibrium equations for determining the external reactions on the structure; therefore, it is not possible to find internal resultants, and, consequently, the strain energy will include many of these unknown reactions. Suppose that the structure of interest has an indeterminacy of order N_R; that is, there are N_R too few equations available for finding reactions. Therefore, we have N_R redundant forces/couples. For these problems, one needs to first choose which reactions that will be considered redundant, and write the equilibrium equations so that the remaining reactions are in terms of these redundant forces/couples. The additional N_R equations needed for determining the reactions are found from the second Castigliano equation above: $0 = \partial U / \partial R_i \quad ; \quad i = 1,2,...,N_R$. Once these reactions are found, then the first set of Castigliano equations are used to find the desired deflections.

c) Note that Castigliano's theorem allows us to determine components of displacements only at points where loadings are applied and only components of displacements that are aligned with the loadings. If the structure is not acted upon by a force at a point and/or along a line of action for which deflections are needed, we simply need to apply a "dummy" force/couple to the structure, treating as a regular applied load. After applying Castigliano's theorem, then set the dummy force/couple to zero.

Example 16.1

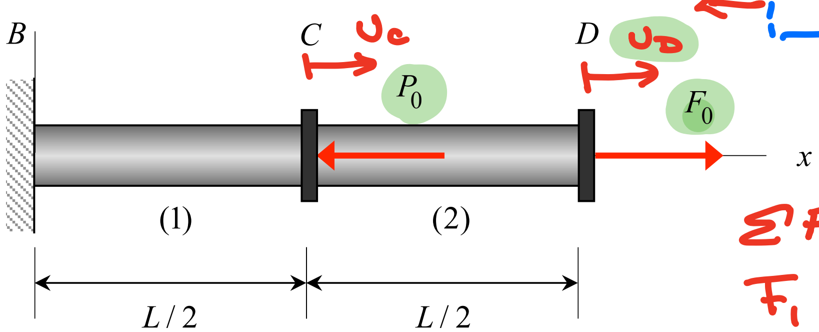
A rod having a solid cross section of area A and made up of a material with a Young's modulus of E is made up of components (1) and (2). Components (1) and (2) are joined by rigid connector C, with component (1) being attached to rigid wall at end B and with a second connector at end D of (2). Loads P_0 and F_0 act on connectors C and D.

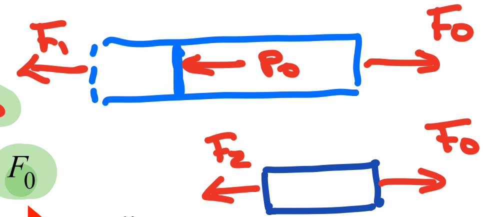
a) Determine the strain energy stored in the rod in terms of the applied loads and the work done by the applied loads under static equilibrium conditions.

b) Write down the work-energy equation for the system under static equilibrium conditions. Explain why the *work-energy method* cannot be used directly to determine the static displacements of either C or D.

c) Use *Castigliano's theorem* to determine the static displacements of C and D.

$$\boxed{U = \frac{1}{2}\int_0^L \frac{F^2 dx}{EA}} \quad \frac{1}{2}\frac{F^2 L}{EA}$$





$$\Sigma F = 0$$

$$F_1 = F_0 - P_0$$

$$F_2 = F_0$$

$$U = U_{(1)} + U_{(2)}$$

$$= \frac{F_1^2 \, L/2}{2EA} + \frac{F_2^2 \, L/2}{2EA}$$

$$U = \frac{L/2}{2EA}\left((F_0 - P_0)^2 + F_0^2 \right)$$

$$W^{(F_0)} = \frac{1}{2} F_0 u_D \qquad W^{(P_0)} = -\frac{1}{2} P_0 u_C$$

$$W = W^{(P_0)} + W^{(F_0)}$$

$$W = \frac{1}{2} F_0 u_D - \frac{1}{2} P_0 u_C$$

$$U = W$$

$$\frac{1}{2} F_0 u_D - \frac{1}{2} P_0 u_C = \frac{L/2}{2EA}\left((F_0 - P_0)^2 + P_0^2\right)$$

1 equation 2 unknowns

u_D u_C

c) Castigliano's

$$u_D = \frac{\partial U}{\partial F_0}$$

opposite direction

$$u_C = - \frac{\partial U}{\partial P_0}$$

$$U = \frac{L/2}{2EA}\left((F_0 - P_0)^2 + F_0^2\right)$$

$$\frac{\partial U}{\partial F_0} = u_D = \frac{L/2}{2EA}\left(2(F_0 - P_0) + 2F_0\right)$$

$$-\frac{\partial U}{\partial P_0} = u_C = \cancel{-}\frac{L/2}{2EA}\left(2(F_0 - P_0)\cancel{(-1)}\right)$$

$F_0 > P_0$

$u_C > 0$

$$U = \frac{1}{2}\int_0^L \frac{T^2}{GI_p}\,dx = \frac{1}{2}\frac{T^2 L}{G I_p}$$

Example 16.2

A shaft (made up of a material with a ~~Young's modulus of E~~ G) is composed of elements (1) and (2), where (1) is a hollow circular tube and (2) has a solid circular cross section. Elements (1) and (2) are joined by a rigid connector C, with (1) attached to fixed wall at B and (2) joined to a rigid connector at D. A torque T_0 is applied to connector D.

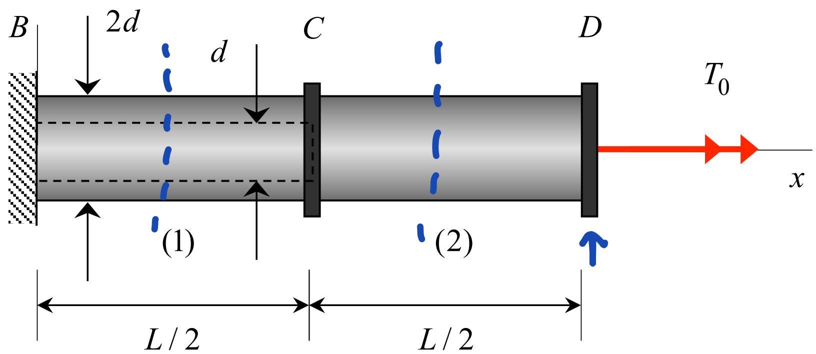
a) Determine the strain energy stored in the shaft in terms of the applied torque T_0 and the work done by the applied torque under static equilibrium conditions.

b) Write down the work-energy equation for the system under static equilibrium conditions. Use the *work-energy method* to determine the static rotation of connector D.

c) Use *Castigliano's theorem* to determine the static rotation of D.

$U(T_0)$

$$\phi_D = \frac{\partial U}{\partial \phi_D}$$



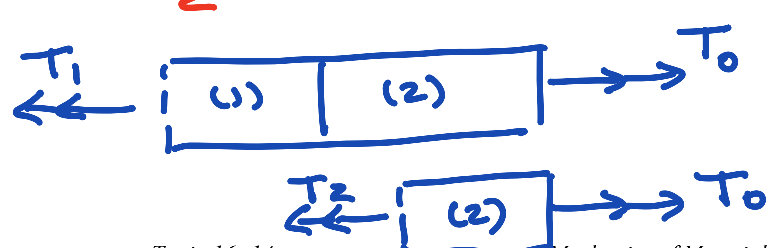
a) $U = U_{(1)} + U_{(2)}$

$$= \frac{1}{2}\frac{T_1^2 L/2}{G I_{p1}} + \frac{1}{2}\frac{T_2^2 L/2}{G I_{p2}}$$

$$I_{p1} = \frac{\pi}{2}\left[\left(\frac{2d}{2}\right)^4 - \left(\frac{d}{2}\right)^4\right] = \frac{15}{\cancel{16}\ 32}\pi d^4$$

$$I_{p2} = \frac{\pi}{2}\left(\frac{2d}{2}\right)^4 = \frac{\pi}{2} d^4$$

Find T_1, T_2



$$T_1 = T_2 = T_0$$

$$U = \frac{1}{2}\,\frac{L/2}{G}\,T_0^2\left(\frac{1}{I_{p1}} + \frac{1}{I_{p2}}\right)$$

$$W^{(T_0)} = \frac{1}{2}\,T_0\,\phi_D$$

b) $$U = W^{(T_0)}$$

$$2\,\frac{L}{\cancel{4}}\,\frac{T_0^{\cancel{2}}}{G}\left(\frac{1}{I_{p1}} + \frac{1}{I_{p2}}\right) = \frac{1}{\cancel{2}}\,\cancel{T_0}\,\phi_D$$

$$\phi_D = \frac{L\,T_0}{2G}\left(\frac{1}{I_{p1}} + \frac{1}{I_{p2}}\right)$$

c) $$\phi_D = \frac{\partial U}{\partial T_0} = \frac{L}{2G}\,T_0\left(\frac{1}{I_{p1}} + \frac{1}{I_{p2}}\right)$$

$$U_\sigma = \frac{1}{2}\int_0 \frac{M^2}{EI}\,dx \qquad U_\tau = \frac{1}{2}\int_0 \frac{V^2 f_s}{GA}\,dx$$

$$f_s = \frac{6}{5}$$

Example 16.3

A load P is applied at the midspan of the beam of length L. The beam has a rectangular cross section, with the cross-sectional dimensions shown. The beam is made up of a material with a Young's modulus E and Poisson's ratio of ν.

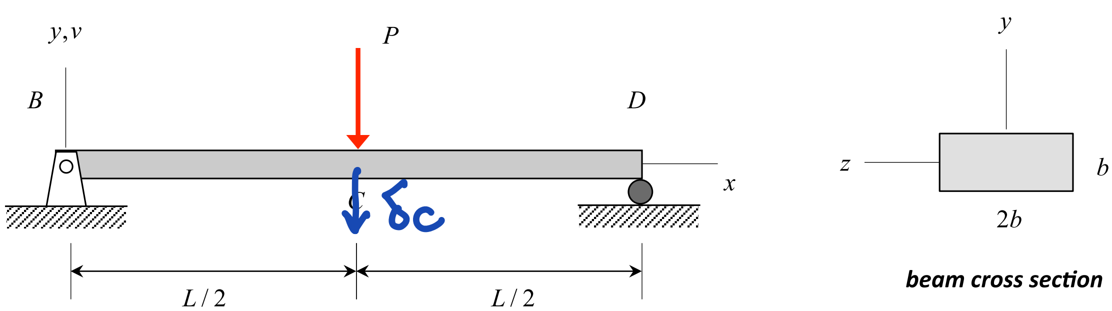
$$G = \frac{E}{2(1+\nu)}$$

a) Determine the strain energy stored in the beam in terms of the load P and the work done by the applied load P under static equilibrium conditions.

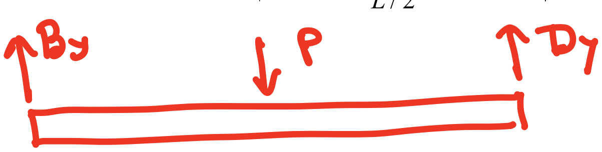
b) Write down the work-energy equation for the system under static equilibrium conditions. Use the *work-energy method* to determine the static deflection of ppoint C of the beam.

c) Use *Castigliano's theorem* to determine the static deflection of C.

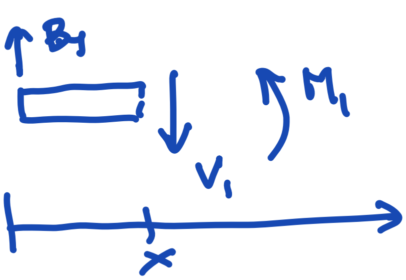
d) Identify the contributions to your solution in c) above that come from flexure stresses and those contributions that come from shear stresses. Compare the sizes of these contributions for $b/L = 0.05$ and $\nu = 0.4$. Are the contributions from shear effects signficant?



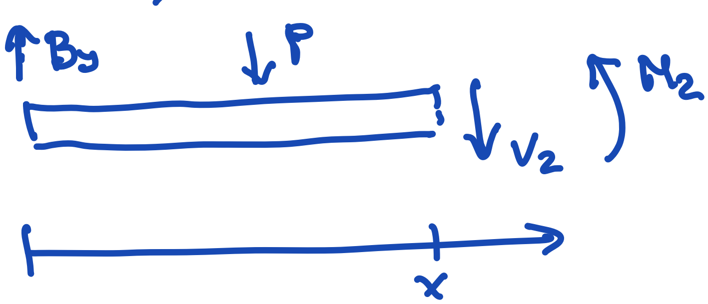
beam cross section



$$B_y = D_y = P/2$$



$$V_1 = P/2 \qquad M_1 = \frac{P}{2}x$$



$$V_2 = -\frac{P}{2}$$

$$M_2 = \frac{P}{2}(L-x)$$

$$U = \underbrace{\int_0^L \frac{M^2(x)}{2EI}\,dx}_{U_\sigma} + \underbrace{\int_0^L \frac{V^2(x)}{2GA}\cdot f_s\,dx}_{U_\tau}$$

$$U_\sigma = \int_0^{L/2} \frac{M_1^2(x)\,dx}{2EI} + \int_{L/2}^{L} \frac{M_2^2(x)\,dx}{2EI} = \frac{P^2L^3}{96EI}$$

$$U_\tau = \int_0^{L/2} \frac{V_1^2(x)}{2GA}\,\frac{6}{5}\,dx + \int_{L/2}^{L} \frac{V_2^2(x)}{2GA}\,\frac{6}{5}\,dx$$

$$= \frac{P^2L}{GA}\cdot f_s$$

$$U_\sigma \gg U_\tau$$

$$\frac{\cancel{P^2}\,L^{\cancel{3}\,2}}{96EI} \qquad \frac{\cancel{P^2}\,\cancel{L}}{GA}\cdot f_s$$

$$I = \frac{2b^4}{12} \qquad \frac{L^2}{96\cancel{E}\,\frac{2b^{\cancel{4}\,2}}{12}} \qquad \frac{f_s}{\frac{\cancel{E}}{\cancel{2}(1+\nu)}\,\cancel{2b^2}}$$

$$\frac{L^2}{b^2} \gg 16\cdot\overbrace{f_s(1+\nu)}^{\sim 1}$$

$L \gg b \rightarrow$ beam

We can neglect shear effects

$$U_\tau \ll U_\sigma$$

$$U = \frac{P^2 L^3}{96EI}$$

$$\delta_c = \frac{\partial U}{\partial P} = \frac{2PL^3}{96EI} \qquad (*)$$

$$\boxed{\delta_c = \frac{PL^3}{Eb^4\, 8}}$$

Work done by force P

$$W^{(P)} = \frac{1}{2} P \delta_c$$

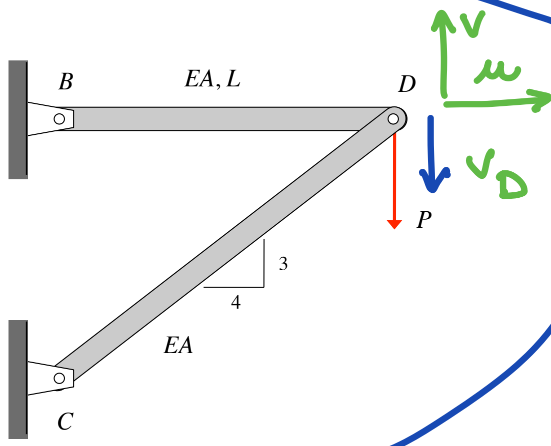
$$U = W^{(P)} \qquad \frac{P^{\cancel{2}} L^3}{96EI} = \frac{1}{2}\cancel{P}\delta_c$$

$$\frac{2}{96}\frac{PL^3}{EI} = \delta_c$$

Same result as (*)

Example 16.4

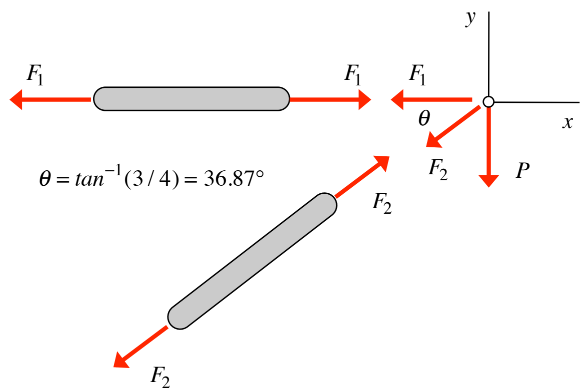
Determine the deflection of node D on the truss.



U

$\Delta_D = \frac{\partial U}{\partial P}$

Equilibrium analysis



$$\sum F_y = -P - F_2 sin\theta = 0 \quad \Rightarrow \quad F_2 = -\frac{P}{sin\theta} = -\frac{5P}{3}$$

$$\sum F_x = -F_1 - F_2 cos\theta = 0 \quad \Rightarrow \quad F_1 = -F_2 cos\theta = \frac{P}{tan\theta} = \frac{4P}{3}$$

Strain energy in truss

$$U = U_1 + U_2 = \frac{1}{2}\frac{F_1^2 L_1}{EA} + \frac{1}{2}\frac{F_2^2 L_2}{EA} \quad ; \qquad L_2 = \frac{L}{cos\theta} = \frac{5L}{4}$$

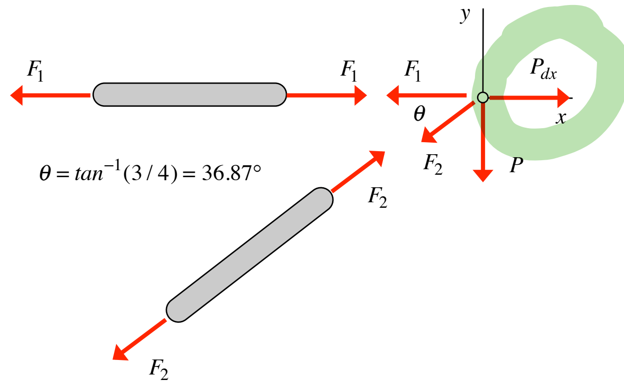
$$= \frac{1}{2EA}\left[\left(\frac{4P}{3}\right)^2 L + \left(-\frac{5P}{3}\right)^2\left(\frac{5L}{4}\right)\right] = \frac{21}{8}\frac{P^2 L}{EA}$$

Castigliano's theorem

$$v_D = \frac{\partial U}{\partial P} = \frac{21}{4}\frac{PL}{EA} \qquad (in\ the\ direction\ of\ P - DOWN)$$

To find horizontal component of deflection of D, add horizontal dummy force at D and apply Castigliano.

Equilibrium analysis



$$\sum F_y = -P - F_2 sin\theta = 0 \quad \Rightarrow \quad F_2 = -\frac{P}{sin\theta} = -\frac{5P}{3}$$

$$\sum F_x = -F_1 - F_2 cos\theta + P_{dx} = 0 \quad \Rightarrow \quad F_1 = -F_2 cos\theta + P_{dx} = \frac{4P}{3} + P_{dx}$$

Strain energy in truss

$$U = U_1 + U_2 = \frac{1}{2}\frac{F_1^2 L_1}{EA} + \frac{1}{2}\frac{F_2^2 L_2}{EA} \quad ; \quad L_2 = \frac{L}{cos\theta} = \frac{5L}{4}$$

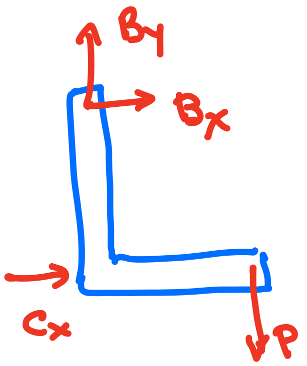
$$= \frac{1}{2EA}\left[\left(\frac{4P}{3} + P_{dx}\right)^2 L + \left(-\frac{5P}{3}\right)^2\left(\frac{5L}{4}\right)\right]$$

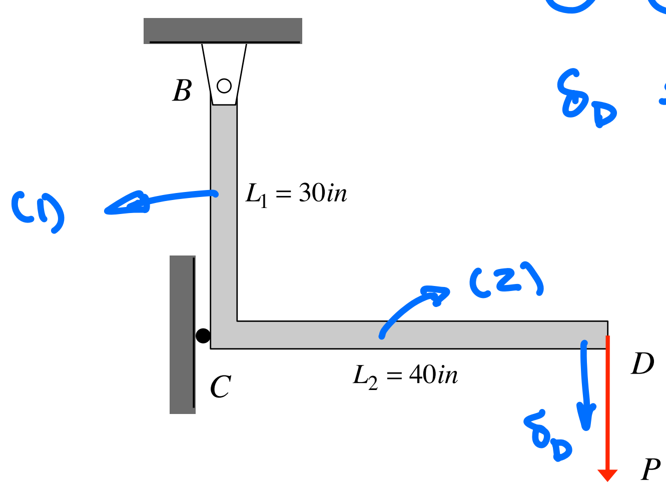
Castigliano's theorem

$$u_D = \frac{\partial U}{\partial P_{dx}} = \frac{1}{EA}\left[\left(\frac{4P}{3} + P_{dx}\right)L\right]_{P_{Dx}=0} = \frac{4PL}{3EA} \quad (same\ direction\ as\ P_{dx} - to\ RIGHT)$$

Example 16.5

Determine the vertical deflection of point D ofnthe structural member shown. The cross section of the member is rectangular and constant throughout. Use $A = 2in^2$, $I = 1.3in^4$, $E = 30\times10^6\, psi$ and $\nu = 0.3$.





$U(P)$

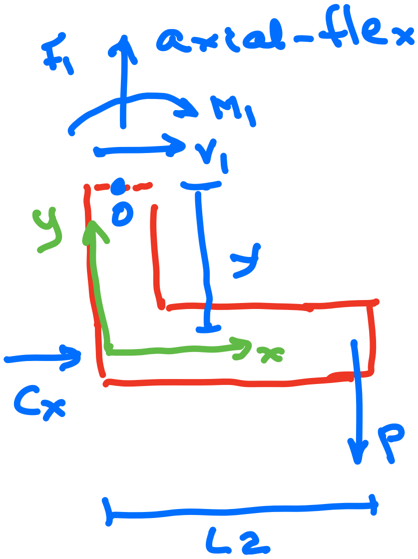
$$\delta_D = \frac{\partial U}{\partial P}$$

$$\Sigma M_B = 0$$

$$C_x \cdot L_1 - P L_2 = 0$$

$$C_x = \frac{4}{3} P$$

$$U = \underbrace{U_{(1)}}_{\text{axial-flex}} + \underbrace{U_{(2)}}_{\text{flex}}$$



$$\Sigma F_y = F_1 - P = 0$$

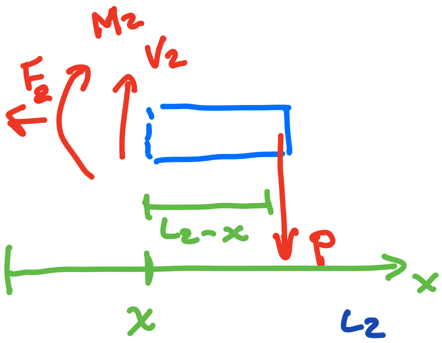
$$\Sigma F_x = V_1 + C_x = 0$$

$$\Sigma M_o = M_1 + L_2 P - C_x \cdot y = 0$$

$$\boxed{M_1(y) = -L_2 P + \frac{4}{3} P \cdot y}$$

$$U_{(1)} = \int_0^{L_1} \frac{F_1^2}{2AE}\, dy + \int_0^{L_1} \frac{M_1^2(y)\, dy}{2EI}$$

$$+ \int \cancel{\text{shear}}$$ small



$F_2 = 0$

$M_2(x) = -P(L_2 - x)$

$$U_{(2)} = \int_0^{L_2} \frac{(-P(L_2-x))^2}{2EI}\,dx$$

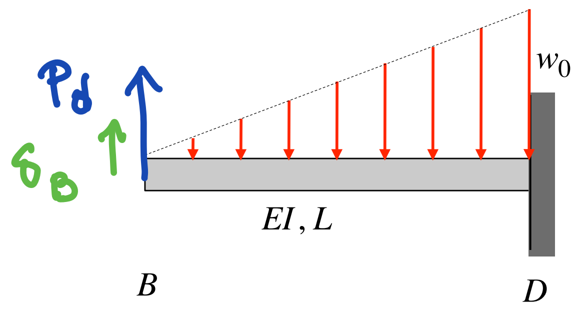
$$U = \underbrace{P^2 \frac{16000}{2EI} + P^2 \frac{30}{2EA}}_{(1)} + \underbrace{P^2 \frac{64000}{6IE}}_{(2)}$$

$$= P^2\, 2\cdot 10^{-4} + P^2\, 2.5\, 10^{-7} + P^2\, 2.7\, 10^{-4}$$

$$\frac{\partial U}{\partial P} = \delta_D = 2P \cdot 4\cdot 10^{-4} \quad \text{in}$$

Example 16.6

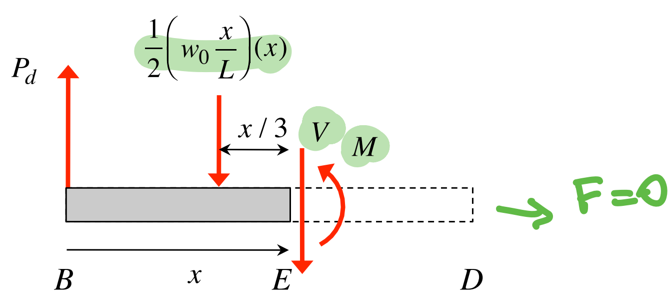
Determine the vertical deflection and beam rotation at end B of the cantilevered beam shown.



$U(P_d, w_0)$

$$\delta_B = \left.\frac{\partial U}{\partial P_d}\right|_{P_d=0}$$

Vertical deflection at B: apply "dummy" load P_d at end B



Determining internal bending moment

$$\sum M_E = M - P_d x + \left(\frac{w_0 x^2}{2L}\right)\left(\frac{x}{3}\right) = 0 \quad \Rightarrow \quad M(x) = -\frac{w_0 x^3}{6L} + P_d x$$

Strain energy in beam (ignoring contributions from shear stress/strain)

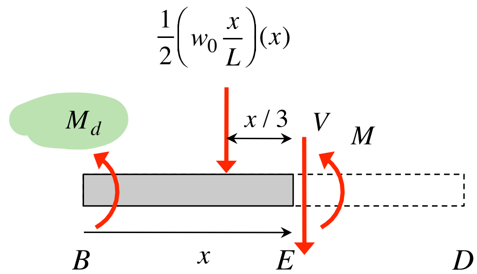
$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx = \frac{1}{2EI}\int_0^L \left(-\frac{w_0 x^3}{6L} + P_d x\right)^2 dx$$

Castigliano's theorem

$$v_B = \left.\frac{\partial U}{\partial P_d}\right|_{P_d=0} = \left[\frac{1}{2EI}\int_0^L 2\left(-\frac{w_0 x^3}{6L} + P_d x\right)(x)\,dx\right]_{P_d=0} = \left[\frac{1}{EI}\int_0^L \left(-\frac{w_0 x^4}{6L} + P_d x^2\right)dx\right]_{P_d=0}$$

$$= \left\{\frac{1}{EI}\left[-\frac{w_0 x^5}{30L} + \frac{P_d}{3}x^3\right]_{x=0}^{x=L}\right\}_{P_d=0} = \frac{1}{EI}\left[-\frac{w_0 L^4}{30} + \frac{P_d L^3}{3}\right]_{P_d=0} = -\frac{w_0 L^4}{30EI} \;\; (DOWN)$$

Beam rotation at B: need to apply "dummy" couple M_d at end B



Determining internal bending moment

$$\sum M_E = M + M_d + \left(\frac{w_0 x^2}{2L}\right)\left(\frac{x}{3}\right) = 0 \quad \Rightarrow \quad M(x) = -\frac{w_0 x^3}{6} - M_d$$

Strain energy in beam (ignoring contributions from shear stress/strain)

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx = \frac{1}{2EI}\int_0^L \left(-\frac{w_0 x^3}{6L} - M_d\right)^2 dx$$

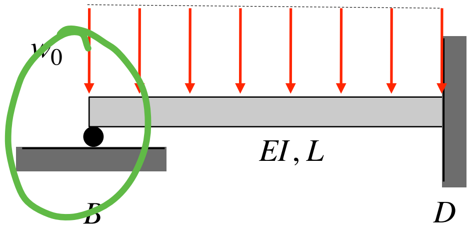
Castigliano's theorem

$$\theta_B = \left.\frac{\partial U}{\partial M_d}\right|_{M_d=0} = \left[\frac{1}{2EI}\int_0^L 2\left(-\frac{w_0 x^3}{6L} - M_d\right)(-1)\,dx\right]_{M_d=0} = \left[\frac{1}{EI}\int_0^L \left(\frac{w_0 x^3}{6L} + M_d\right)dx\right]_{M_d=0}$$

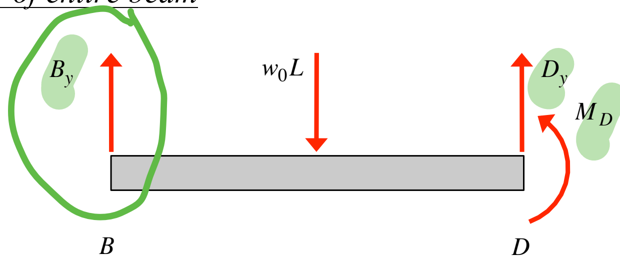
$$= \left\{\frac{1}{EI}\left[\frac{w_0 x^4}{24L} + M_d x\right]_{x=0}^{x=L}\right\}_{M_d=0} = \frac{1}{EI}\left[\frac{w_0 L^3}{24} + M_d L\right]_{M_d=0} = \frac{w_0 L^3}{24EI} \quad (CCW)$$

Example 16.7

Determine the reaction at end B of the beam shown.



<u>*Equilibrium – FBD of entire beam*</u>



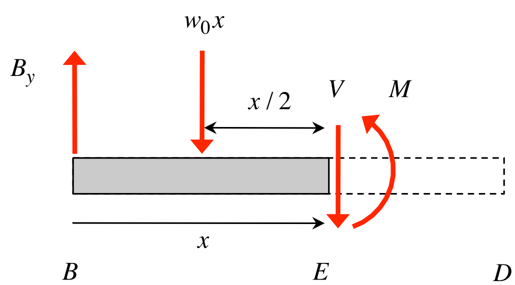
From here, we see that the problem is statically indeterminate: 3 unknowns (B_y , D_y and M_D) and only two equations. Here, we will choose B_y to be our redundant reaction:

#1 $$\sum F_y = B_y - w_0 L + D_y = 0 \;\Rightarrow\; D_y = w_0 L - B_y$$

#2 $$\sum M_B = -\left(w_0 L\right)\left(\frac{L}{2}\right) + D_y L + M_D = 0 \;\Rightarrow\; M_D = -\left(w_0 L - B_y\right)L + \frac{1}{4} w_0 L^2$$

3 unknowns
2 equation
1 redundant force

<u>*Determining internal bending moment*</u>



$$\sum M_E = M - B_y x + \left(w_0 x\right)\left(\frac{x}{2}\right) = 0 \;\Rightarrow\; M(x) = -\frac{w_0 x^2}{2} + B_y x$$

<u>*Strain energy in beam (ignoring contributions from shear stress/strain)*</u>

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx = \frac{1}{2EI}\int_0^L \left(-\frac{w_0 x^2}{2} + B_y x\right)^2 dx$$

Castigliano's theorem

With B_y being our choice for the redundant reaction:

equation #3

$$0 = \frac{\partial U}{\partial B_y} = \frac{1}{2EI}\int_0^L 2\left(-\frac{w_0 x^2}{2} + B_y x\right)(x)\,dx = \frac{1}{EI}\int_0^L \left(-\frac{w_0 x^3}{2} + B_y x^2\right)dx$$

$$= \frac{1}{EI}\left[-\frac{w_0 x^4}{8} + \frac{B_y x^3}{3}\right]_{x=0}^{x=L} = \frac{1}{EI}\left[-\frac{w_0 L^4}{8} + \frac{B_y L^3}{3}\right] \Rightarrow B_y = \frac{3}{8} w_0 L$$

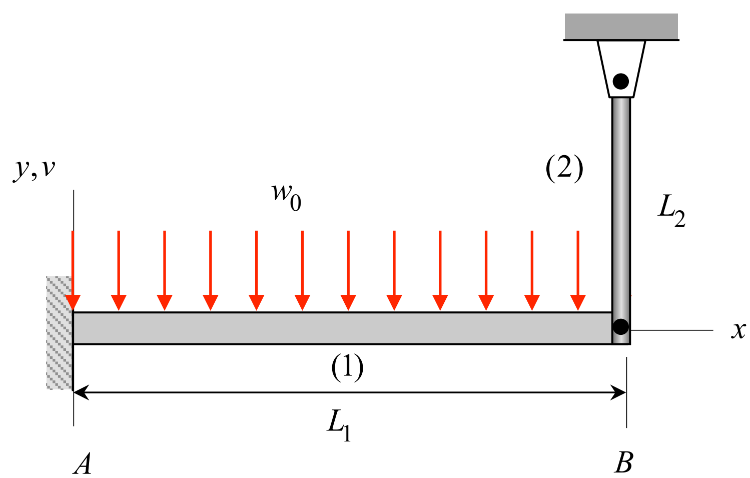
Example 16.8

For the following examples, ***set up*** the problem for determining the requested deflections using Castigliano's method:

- draw appropriate FBDs;
- determine internal results for each section;
- set up the integrals for calculating the required deflections;
- explain how Castigliano's method is used to solve. Discuss the application of dummy forces (when needed) and how to handle redundant forces for indeterminate structures.

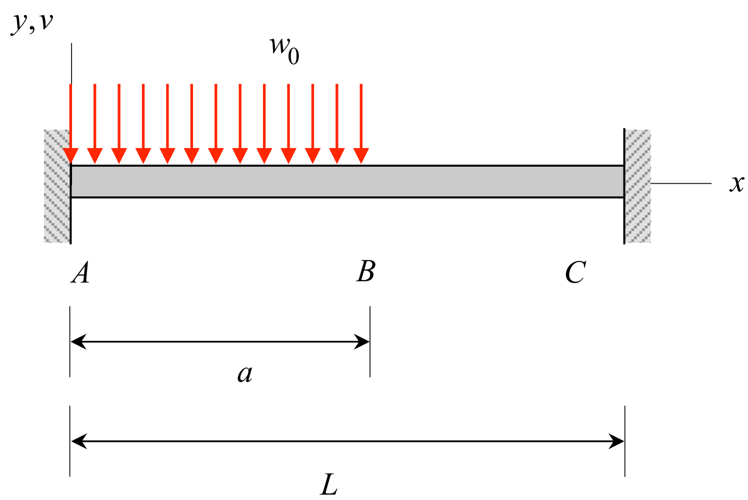
Problem A

Find the load carried by member (2) of the structure below. Let E and A be the Young's modulus and cross-sectional area, respectively, of member (2), whereas E and I are the Young's modulus and second area moment of the cross section of (1), respectively.



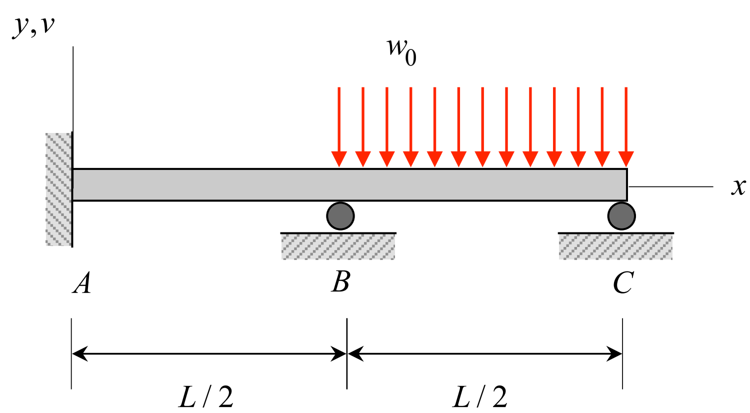
Problem B

Find the vertical deflection of the beam at point B and the angle of rotation of the beam at B. Let E and I be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.



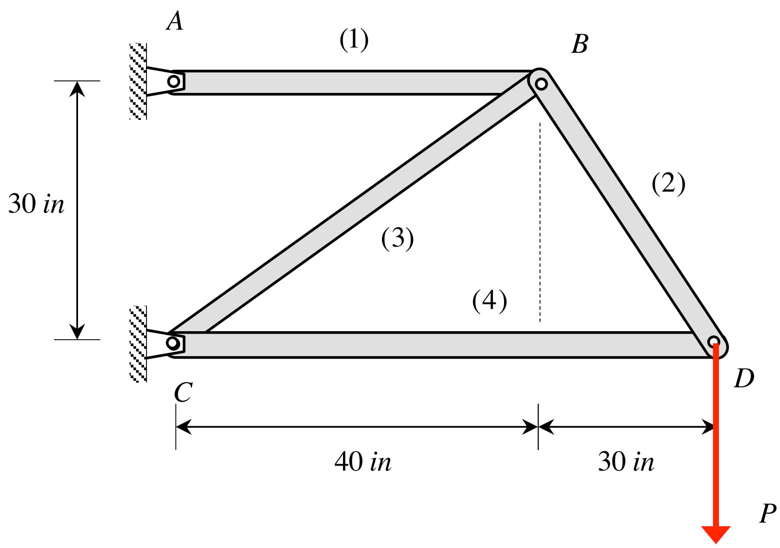
Problem C

Determine the reactions at rollers B and C on the beam below. Let E and I be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.



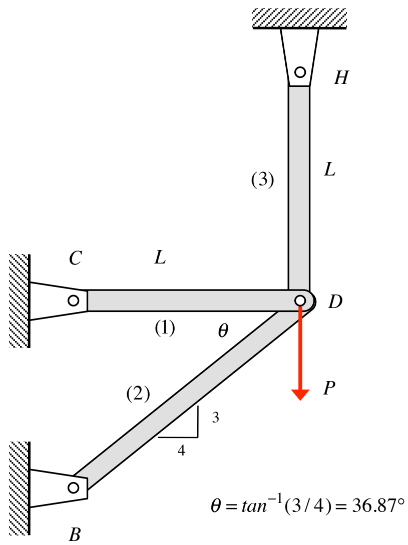
Problem D

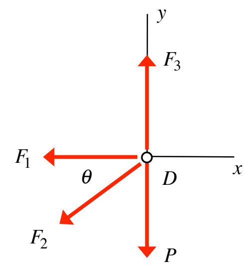
Determine the vertical and horizontal deflection of the truss at joint D. All members of the truss have a cross-sectional area of A and are made of a material with a Young's modulus of E.



Example 16.9

Determine the vertical component of deflection of node D on the truss. All members have the same cross-sectional area A and are made of the same material having a Young's modulus of E.





Equilibrium analysis of joint D

$$\sum F_y = -P - F_2 sin\theta + F_3 = 0 \quad \Rightarrow \quad F_3 = P + F_2 sin\theta \qquad (1)$$

$$\sum F_x = -F_1 - F_2 cos\theta = 0 \quad \Rightarrow \quad F_1 = -F_2 cos\theta \qquad (2)$$

From this we see that the problem is INDETERMINATE (two equilibrium equations and three unknowns). We will consider the force F_2 to be the "redundant" force (this was an arbitrary choice).

Strain energy in truss

$$U = U_1 + U_2 + U_2 = \frac{1}{2}\frac{F_1^2 L_1}{EA} + \frac{1}{2}\frac{F_2^2 L_2}{EA} + \frac{1}{2}\frac{F_3^2 L_3}{EA} \quad ; \quad L_2 = \frac{L}{cos\theta} = \frac{5L}{4}$$

$$= \frac{1}{2EA}\left[F_1^2 L + F_2^2\left(\frac{5L}{4}\right) + F_3^2 L\right]$$

$$= \frac{L}{2EA}\left[\left(-F_2 cos\theta\right)^2 + \frac{5}{4}F_2^2 + \left(P + F_2 sin\theta\right)^2\right] \quad ; \quad using\ (1)\ and\ (2)$$

Castigliano's theorem as applied to indeterminate structures

Since we chose F_2 as our redundant force:

$$0 = \frac{\partial U}{\partial F_2} = \frac{L}{EA}\left[F_2 cos^2\theta + \frac{5}{4}F_2 + \left(P + F_2 sin\theta\right) sin\theta \right] \Rightarrow$$

$$0 = F_2 cos^2\theta + \frac{5}{4}F_2 + \left(P + F_2 sin\theta\right) sin\theta \Rightarrow$$

$$\left(cos^2\theta + \frac{5}{4} + sin^2\theta \right) F_2 = -Psin\theta \Rightarrow F_2 = -\frac{4}{9}Psin\theta$$

Therefore, the strain energy function becomes:

$$U = \frac{L}{2EA}\left\{ \left[-\left(-\frac{4}{9}Psin\theta \right) cos\theta \right]^2 + \frac{5}{4}\left(-\frac{4}{9}Psin\theta \right)^2 + \left(\frac{5}{9}Psin\theta \right)^2 \right\}$$

$$= \frac{P^2 Lsin^2\theta}{162EA}\left(\frac{5}{9} + \frac{16}{81}cos^2\theta \right)$$

Using Castigliano's theorem gives:

$$v_D = \frac{\partial U}{\partial P} = \frac{PLsin^2\theta}{81EA}\left(\frac{5}{9} + \frac{16}{81}cos^2\theta \right)$$ (since "+", in same direction as P - DOWN)

Deflection analysis – Castigliano's method

The procedure for deflection analysis using Castigliano's method:

i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). *Add in ALL of the needed dummy loads from the start*; this can save you a lot of time down the road.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.

- If *DETERMINATE*, solve these equations for the external reactions.
- If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of *n* redundant reactions (R_i ; $i = 1,2,...,N_R$). *Write the remaining reactions in terms of these N_R redundant reactions*.

iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).

iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI}\int_{x_i}^{x_{i+1}} M_i^2\, dx + \frac{f_s}{2GA}\int_{x_i}^{x_{i+1}} V_i^2\, dx + \frac{1}{2EA}\int_{x_i}^{x_{i+1}} F_{Ni}^2\, dx$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \dots$. It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1,2,...,N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.

Additional notes: